POLYNOMIALS OF DEGREE 12 FOR A GIVEN GALOIS GROUP

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Abstract. This paper deals with the inverse Galois problem : computing a polynomials of degree m associated with a given subgroup of $\mathfrak{S}_m$.
We shall describe a method builded on recent developments in Galois constructive theory [17], which uses the results obtained for the direct problem concerning polynomials of n-th degree ($4 \leq n \leq 11$).
Using computer algebra systems, this method has enabled us to compute a lot of 12-th degree irreducible polynomials associated with given subgroups of $\mathfrak{S}_{12}$.

Introduction

The Galois theory consists of two problems : the direct problem and the inverse one. The direct problem is dedicated to the search of the Galois group of a given polynomial whereas the inverse one consists in computing a polynomial whose Galois group is given. The direct problem always has a solution, but it has never been proven that the inverse problem always does.

In this paper, some results on the inverse problem are given using the method of resolvent based on the groups matrix (see [17]).

In the first section, some notations and results are recalled. Then the second section describes a method using the partitions and groups matrices for the direct problem. And finally, the third section gives an algorithm using these matrices for the inverse problem and presents the polynomials we have computed using this algorithm.

1. The Galois group and resolvent

Let k be a field of characteristic zero and $f \in k[x]$ be a squarefree polynomial of degree n.

Assume $\Theta \in k[x_1, \ldots, x_n]$, $\widetilde{\Theta}$ is an evaluation of Θ in the n roots (distinct) of f.
The action of $\mathfrak{S}_n$, the symmetric group of degree n, over Θ is defined by $\sigma.\Theta = \Theta(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$.
The *orbit of* Θ under the action of $\mathfrak{S}_n$ is the following set :

$$\mathfrak{S}_n.\Theta = \{\sigma.\Theta \mid \sigma \in \mathfrak{S}_n\} \qquad .$$

The *ideal of the relationships* between the roots of f, denoted by $\mathcal{I}$ is given by :

$$\mathcal{I} = \{r \in k[x_1, \ldots, x_n] \mid \tilde{r} = 0\} \qquad .$$

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The Galois group of f over k, denoted by $\mathrm{Gal}_k(f)$ is the set of permutations which leave the relationships between the roots of f invariant:

$$\mathrm{Gal}_k(f) = \{\sigma \in \mathfrak{S}_n \mid \forall r \in \mathcal{I}\ \widetilde{\sigma.r} = 0\} \qquad .$$

The basic tool of the Galois theory is the resolvent introduced by Lagrange (see [11]). Let us recall that, the *resolvent of f by* Θ is the following univariate polynomial :

$$\mathcal{L}_{\Theta,f}(x) = \prod_{\Psi \in \mathfrak{S}_n . \Theta} (x - \widetilde{\Psi}) \qquad .$$

By the fundamental theorem of symmetric functions we know that the resolvent $\mathcal{L}_{\Theta,f}$ has its coefficients in the field k. There are many methods to compute these resolvents. The module `SYM` of `MACSYMA` ([18] and [13]) computes many of them.

In the following section we shall recall how these resolvents are used to compute the Galois group of a polynomial.

2. The matrix of partitions and the matrix of groups

The *partition* of a polynomial is the integer partition obtained from the degree of the irreducible factors of this polynomial.
The groups associated to a polynomial are the Galois groups of its irreducible factors.
We recall that a Galois group is defined up to a conjugacy class.

Let H be a subgroup of $\mathfrak{S}_n$. We say that Θ is a *primitive invariant of H relative to* $\mathfrak{S}_n$ if $H = \{\sigma \in \mathfrak{S}_n \mid \sigma.\Theta = \Theta\}$.
If Θ is a primitive invariant of H (relatively to $\mathfrak{S}_n$) then the resolvent $\mathcal{L}_{\Theta,f}$ is called an *H-resolvent of f.*

There are fundamental results :

(1) The partition of an H-resolvent of f depends only on the conjugacy classes of H and $\mathrm{Gal}_k(f)$ in $\mathfrak{S}_n$ (see [1]). This partition is denoted by $[\mathrm{Gal}_k(f), H]$.
(2) The groups associated with an H-resolvent of f can be computed up to an isomorphism and a conjugacy class only with the conjugacy classes of H and $\mathrm{Gal}_k(f)$ in $\mathfrak{S}_n$ (see [17]). The list of these groups is denoted by $\{\mathrm{Gal}_k(f), H\}$.

Let $\mathcal{H}_1, \ldots, \mathcal{H}_r$ be the conjugacy classes of subgroups in $\mathfrak{S}_n$. We note H_i a representative group of $\mathcal{H}_i$.

Hence we can compute two matrices : $\mathcal{A}$, *the partition matrix* and $\mathcal{B}$ *the groups matrix.* These matrices are indexed by the conjugacy classes of subgroups of $\mathfrak{S}_n$ for the lines and the columns. Hence we have

$$\mathcal{A} = ([H_i, H_j])_{1 \le i,j \le r}$$
$$\text{and } \mathcal{B} = (\{H_i, H_j\})_{1 \le i,j \le r} \qquad .$$

Since the lines of $\mathcal{A}$ are all distinct, it is always possible to determine the Galois group of any polynomial of degree n (see [1]).
The matrix $\mathcal{B}$ is used to speed up this computation. For example, let F be an irreducible factor of f, the Galois group of F can be used to determine the Galois group of f.
If the degree of F is less than the degree of f, the computation of its Galois group is clearly simpler than that of f. We can see [17] for some applications in degree 8 and 10.

In the next section we shall explain how the matrix $\mathcal{B}$ is used for the inverse problem.

3. The inverse problem

Given G a subgroup of the symmetric group $\mathfrak{S}_m$ we want to compute a polynomial $h(x)$ of degree m whose Galois group is G. The main idea behind our process is to use the matrices $\mathcal{B}$ associated with $\mathfrak{S}_n$ $3 \leq n \leq m$ and the resolvents.

First of all, let us state some notation :

(1) $f(x)$ denotes a squarefree polynomial of degree n whose coefficients belong to a field k of characteristic zero.
(2) G_f denotes the Galois group of f.
(3) H is a subgroup of $\mathfrak{S}_n$ whose Θ is a primitive invariant.
(4) If $\gamma_1, \gamma_2, ..., \gamma_e$ is a left transversal of H in $\mathfrak{S}_n$, $\Theta_l = \gamma_l\Theta$ and $K_l = \gamma_l H \gamma_l{}^{-1}$ for $1 \leq l \leq e$.
(5) $h(x)$ is an irreducible and simple factor of $\mathcal{L}_{\Theta,f}$ whose roots are $\widetilde{\Theta}_1, \widetilde{\Theta}_2, ..., \widetilde{\Theta}_m$.

Using these notation the following theorem of [17] can be proved:

Theorem 1. The Galois group of the polynomial $h(x)$ is isomorphic to the quotient group $G_f/G_f \bigcap_{l=1}^m K_l$. The order of that group is clearly $[G_f : G_f \bigcap_{l=1}^m K_l]$.

Hence if the matrix of partitions of $\mathfrak{S}_n$ contains a part equal to m, it is easy to find i and j such that H_j is a *test group* and H_j is an *applicant group* as denoted in [1].

That is to say that, if we know $f(x)$ whose Galois group is H_i, then the factorization of a squarefree H_j-resolvent of $f(x)$ contains an irreducible factor $h(x)$ of degree m whose Galois group G^* is defined, up to isomorphism, by theorem 1 and can be found in $\mathcal{B}$.

Hence we know that G^*, which is a subgroup of $\mathfrak{S}_n$, is the Galois group of the polynomial $h(x)$ of degree m, then we need to identify G^* with a subgroup of $\mathfrak{S}_m$.

Remark 1. Clearly, since $h(x)$ is an irreducible polynomial, we can find polynomials only for transitive subgroups. But our algorithm can easily be generalized for non transitive polynomials.

Theorem 2. Let h be a simple irreducible factor of a f H-resolvent, whose roots are $\widetilde{\gamma_1\Theta}, ..., \widetilde{\gamma_m\Theta}$. Let $G_f(h) = \{\gamma_1 H, ..., \gamma_m H\}$ be the G-orbit for the action of G_f on the $\mathfrak{S}_n$ *mod* H left cosets. Then, $Gal_k(h)$ is isomorphic to the quotient group $G_f / \bigcap_{K \in G_f(h)} Stab_{G_f}(K)$.

Indeed, $G_f(h) = \{\gamma_1 H, ..., \gamma_m H\}$ is a G-orbit and $Stab_{G_f}(\gamma H) = G_f \cap \gamma H \gamma^{-1}$ (see [1] p. 7 and 8) . Theorem 2 is a variant of theorem 1.

Remark 2. In theorem 2 the factor h must be irreducible, but it can be iterated in order to deal with reducible factors.

As noted by C. Quitté [15], theorem 2 has an effective form in the GAP system since the action of G_f on $G_f(h)$, is the same as the action of $G_f / \bigcap_{K \in G_f(h)} Stab_{G_f}(K)$ on $[1, ..., m]$. Indeed :

$$\{g \in G : g\gamma_i H = \gamma_i H \ for \ i = 1, ..., m\} = \bigcap_{i=1..m} Stab_{G_f}(\gamma_i H) = \bigcap_{K \in G_f(h)} K$$

At this time we have not used this method for our computations, but it could beused in order to conclude for the groups we can not distinguish using our algorithm (see 3.1).

In the next sections we shall first explain how to identify a quotient of subgroups of $\mathfrak{S}_n$ with a subgroup of $\mathfrak{S}_m$, and then we shall give a method in order to compute the polynomial $h(x)$.

3.1. Identifying a subgroup of $\mathfrak{S}_n$ with a subgroup of $\mathfrak{S}_m$. Let H be a subgroup of $\mathfrak{S}_n$. All the transitive subgroups of $\mathfrak{S}_m$ for $m \leq 15$ are known and tabulate in the systems `GAP` [9] and `MAGMA` [12]. Therefore we just have to compare some invariants of these subgroups to conclude. We have focused our attention on three invariants :

(1) the order of the subgroup
(2) the parity of the subgroup
(3) the Burnside table of marks of the subgroup [8]

3.1.1. *Order.* In our problem the order of G^* is immediately obtained by using the matrix of groups $\mathcal{B}$ (i.e. by application of theorem 1) as explained in section 2.

Remark 3. If the order of G^* is equal to m, identifying G^* in $\mathfrak{S}_m$ simply consists in computing the regular representation of G^*, which can easily be done using GAP.

3.1.2. *Parity of the subgroup.* The parity of G^* is given by the computation of $h(x)$ discriminant. Indeed, it is a well known result that the Galois group of a polynomial is even if and only if its discriminant is a square.

3.1.3. *Table of marks.* The Burnside table of marks of a subgroup H with n conjugacy classes of subgroups with representatives M_i is an n by n matrix each of whose coefficients $m_{i,j}$ is the number of right cosets of H_i that are fixed under the operation of H_j by multiplication from the right.
n is called the dimension of the table of marks.
Note that this table is only unique up to simultaneous permutations of rows and columns.

3.1.4. *Conclusion.* Therefore our method is firstly to compute the order of the subgroup H, secondly to look at the dimension of the table of marks and possibly the marks of this table, and finally to check the parity of the subgroup.

3.2. Computing $h(x)$. Let G^* be a finite subgroup of $\mathfrak{S}_m$, and $\mathcal{H}G_n$ be a set of pairs $(H_i, H_j) \in \mathfrak{S}_n \times \mathfrak{S}_n$ such that the partition $[H_i, H_j]$ contains a part equal to m. For each n the following pairs are omitted :

(1) the pairs for whose the computation of the H_j-resolvent is impossible
(2) the pairs for whose no polynomial of $n-th$ degree having H_i as Galois group is known
(3) the pairs for whose $[\mathfrak{S}_n : H_j]$ (the degree of the H_j-resolvent) is too large to make possible a factorization.

Among the remaining pairs, we take the pair (H_i, H_j) whose the m-th degree factor will be compute the most fastly. This choice involves the computation time of the H-resolvents and their factorizations. The most efficient resolvents are described in [4]. When (H_i, H_j) is chosen, we just have to compute the irreducible simple factor of degree m of the chosen resolvent.

3.3. Examples. In this section two examples are developped, in order to show how our method works. In these two cases (and in all the cases we have studied) we look for a polynomial of 12-th degree. As explained before, the first thing to do is to search, for each $\mathfrak{S}_n$ $4 \leq n \leq 11$, for all the parts equal to 12 in the matrix of partitions of $\mathfrak{S}_n$.

3.3.1. *In* $\mathfrak{S}_4$. In the matrix of partitions of $\mathfrak{S}_4$ [5] there is only a part equal to 12, which is obtained with H_2 as applicant group, and H_{10} as test group. Let us recall that, in our notations, H_2 is the alternating subgroup of $\mathfrak{S}_4$, and H_{10} is the subgroup of $\mathfrak{S}_4$ generated by [(3,4)].
Therefore we know that, if we assume that the galois group of $f(x)$ is H_2, then the factorization of a each H_{10}-resolvent of $f(x)$ provides a factor of the 12-th degree.

- On the one hand, in [14] we find that $f(x)=x^4 + 8x + 12$ is a polynomial whose Galois group is precisely H_2, and on the other hand we know that there are at least two primitive invariants of $H_{10} : x_1 + 2x_2$ or $x_1 - x_2$.
  Then we will easily compute the H_{10}-resolvent of $f(x)$ and by factorization we will find our polynomial $h(x)$.
  Indeed using the second invariant we find $h(x) = x^{12} + 96x^8 + 1664x^6 - 16128x^4 + 165888x^2 + 3331776$
- By application of the theorem 1 we can compute the Galois group G^* of $h(x)$, in this case $G^* = H_2/K$ where K is the subgroup of $\mathfrak{S}_4$ generated by [()], that is to say that G=H_2 and $|G| = 12$.
- The last point is to identify H_2 among all the transitive subgroups of $\mathfrak{S}_{12}$. Since $|G| =$ 12, we look at the transitive subgroups of order 12. Using the notation of the system GAP where Subgroup(S,[x,y,z]) denotes the subgroup of S generated by x, y and z, there are 5 transitive subgroups of order 12 in $\mathfrak{S}_{12}$:
  G_1=Subgroup($\mathfrak{S}_{12}$,[(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12)])
  G_2=Subgroup($\mathfrak{S}_{12}$,[(1, 2, 3, 4, 5, 6)(7, 8, 9, 10, 11, 12),(1, 7)(2, 8)(3, 9) (4, 10)(5, 11)(6, 12)])
  G_3=Subgroup($\mathfrak{S}_{12}$,[(1, 7)(2, 8)(3, 9)(4, 10)(5, 11)(6, 12),(1, 2, 3, 4, 5, 6)(7, 12, 11, 10, 9, 8)])
  G_4=Subgroup($\mathfrak{S}_{12}$,[(1, 2)(3, 4)(5, 8)(6, 7)(9, 11)(10, 12),(1, 5, 9)(2, 6, 10)(3, 7, 11)(4, 8, 12)])
  G_5=Subgroup($\mathfrak{S}_{12}$,[(1, 2, 3, 4, 5, 6)(7, 12, 11, 10, 9, 8),(1, 7, 4, 10)(2, 8, 5, 11)(3, 9, 6, 12)])
  G_1 and G_5 are both odd, whereas G_2, G_3 and G_4 are even.
  Looking at the dimension of the table of marks of these groups, we can distinguish some of them with the other :

Dimension	Names
5	G_4
6	G_1 G_5
10	G_2 G_3

- Now we must decide between G_1 and G_5 on the one hand and, G_2 and G_3 on the other. The table of marks of G_2 contains 3 times the line [6,6] whereas the table of marks of G_3 does not. Hence, we obtain the following table :

Dimension	Name	Parity	Particularity
5	G_4	Even	
6	G_1 G_5	Odd	
10	G_2	Even	3 lines [6,6]
	G_3	Even	

- In order to conclude, simply look at the regular representation of H_2, the dimension of the table of marks of this representation is 5, then the subgroup which we are looking for is G_4.

3.3.2. *In* $\mathfrak{S}_5$. This time, searching the parts equal to 12 in the matrix of partitions of $\mathfrak{S}_5$ gives a lot of possibilities.
If [j,i] denotes : "there is a part equal to 12 in the factorization of a H_j-resolvent for a polynomial whose Galois group is H_i", or "there is a part equal to 12 in the factorization of a resolvent with H_j as test group and H_i as applicant group" we find 28 pairs [i,j].
But it does not mean that we have 28 subgroups to identify. Indeed, the application of theorem 1 can lead to the same subgroup for distinct pairs.
This only means that we can choose the more efficient resolvent to compute our polynomial.

Subgroup($\mathfrak{S}_5$, [(4,5), (2,3), (1,3)]) of order 12 obtained with all the following pairs : [9, 7], [10, 7], [11, 7], [13, 7], [14, 7], [15, 7], [16, 7], [17, 7], [18, 7]
Subgroup($\mathfrak{S}_5$, [(2,3)(4,5), (2,4)(3,5), (3,4,5)]) of order 12 obtained with all the following pairs : [6, 4], [9, 4], [10, 4], [12, 4], [13, 4], [14, 4], [15, 4], [16, 4], [17, 4], [18, 4]
Subgroup($\mathfrak{S}_5$, [(2,3)(4,5), (2,4)(3,5), (4,5), (3,4)]) of order 24 obtained with all the following pairs : [6, 2], [8, 2], [11, 2], [12, 2], [13, 2], [14, 2], [16, 2], [18, 2]
Subgroup($\mathfrak{S}_5$, [(1,2,5), (1,3,5), (1,4,5)]) of order 60 obtained with the pair [15, 3]

Therefore we have only 4 distinct subgroups to identify :

(1) For the first two groups, we repeat the method of 3.3.1, and we conclude that :
Subgroup($\mathfrak{S}_5$, [(4,5), (2,3), (1,3)]) is G_4, whereas
Subgroup($\mathfrak{S}_5$, [(2,3)(4,5), (2,4)(3,5), (3,4,5)]) is G_3.
The last point is to compute the polynomials $h(x)$ for G_4 and G_3.
   - Concerning G_4 :
   Using the fact that $H_4 = \mathfrak{S}_1 \times A_4$ (where A_4 is the alternating subgroup of $\mathfrak{S}_4$) and the results of [14] we choose $f(x) = (x-1)(x^4+8x+12)$.
   Taking $x_1 - x_2$ as H_{12}-primitive invariant we compute $h(x) = x^{12} + 96x^8 + 1664x^6 - 16128x^4 + 165888x^2 + 331776$
   - Concerning G_3:
   using the same method we choose $f(x) = (x^2+1)(x^3+2)$ and with an H_{13}-primitive invariant we compute $h(x) = x^{12} + 72x^{10} + 3240x^8 - 9288x^6 + 50544x^4 + 1749600x^2 + 7290000$

(2) For the third subgroup (say H) we need to characterize the transitive subgroups of order 24 of $\mathfrak{S}_{12}$.
Using the same process as in 3.3.1 we can state the following table :

Dimension	Name	Parity	Particularity
11	G_8	Odd	
	G_9	Even	
12	G_6 G_7	Even	
16	G_{11}		3 lines [4,4,4,4]
	G_{12}		
	G_{13} G_{15}	Odd	1 line[6,6,6,6]
	G_{14}		1 line [3,3,3,3,3,3,3,3]
32	G_{10}		

Computing the dimension of the table of marks of H, gives 11. The only way to decide between G_8 and G_9 is to check the parity of the subgroup. Then we need a polynomial of 5-th degree whose Galois group is H_2 in $\mathfrak{S}_5$.

Since $H_2 = \mathfrak{S}_1 \times \mathfrak{S}_4$ and according to the results of [14], such a polynomial can be $f(x) = (x-1)(x^4+x+1)$
Using H_{12} as test group and $x_1 - x_2$ as H_{12}-primitive invariant we prove that the group must be odd, (i.e. H is G_8), and we compute $h(x) = x^{12}+8x^8+26x^6-112x^4+216x^2+229$.
Using H_6 as test group we cannot reach conclusions about the parity of the group, but we obtain a new $h(x) = x^{12}+x^{10}+5x^9+x^8-7x^7+29x^6+12x^5+51x^4+16x^3-22x^2-25x+47$ using $x_1x_2 + x_4x_5$ as H_6-primitive invariant.

(3) For the last group that is Subgroup($\mathfrak{S}_5$, [(1,2,5), (1,3,5), (1,4,5)]), we have no problem to identify the group in $\mathfrak{S}_{12}$, indeed there is only one transitive group of order 60 in $\mathfrak{S}_{12}$ which is :
G_{33}=Subgroup($\mathfrak{S}_{12}$,[(1, 2, 4, 6, 7)(5, 8, 9, 10, 11),(1, 3, 2, 5, 9)(4, 7,10, 12, 11)]).
However, we have not found any polynomial $h(x)$.

3.4. Practical results. In this last section we list a part of the polynomials we have obtained for some of the transitive subgroups of $\mathfrak{S}_{12}$.Indeed, as said before, several $\mathfrak{S}_n$ can be used to compute those polynomials. In this article we hase chosen to give only a polynomial associated with a subgroup.
To do this we first of all have to consider all the distinct transitive subgroups of $\mathfrak{S}_{12}$ and to caracterize them as explained in section 3.1.

- All these subgroups, which are denoted G_s, can be founded in the MAGMA system library or in the GAP system last version. The index s given in our table, is exactly the one given by the MAGMA system. Just note that our table do not list all them. Indeed, a subgroup does not appear in the table, when we have no polynomial associated to it.

For each of this subgroup G_s, a polynomial $h_s(x)$ is computed using $(H_i, H_j) \in \mathcal{H}G_n$ (see section 3.2), Θ as primitive invariant of H_j in $\mathfrak{S}_n$ and $f(x)$ as first polynomial of degree n.

- The H_i and H_j are subgroups of $\mathfrak{S}_n$ and are described in [2], [3], [5], [6] and [7].
- The polynomials $f(x)$, used in this work, have been computed and described in [10, 14, 16].

Moreover, there are still some problems to study :

- There are several cases where, at this time, we can not decide between two (or more) G_s, in these cases the polynomial given is denoted by $h_{s_1,s_2,\ldots,s_l}$ meaning that its Galois group in $\mathfrak{S}_{12}$ is G_{s_1} or G_{s_2} or ... or G_{s_l}.
- There are other cases where we do not have any polynomial for the applicant group, that is the reason why we can not conclude for the subgroups :

$$G_{12}, G_{13}, G_{15}, G_{14}, G_{27}, G_{30}, G_{32}, G_{44}, G_{49}, G_{50}, G_{51}, G_{52}, G_{51}, G_{127}$$

- We do not know a resolvent which has a factor whose Galois group is G_{33}, G_{85}.
The computation of a resolvent is possible, but not realized at this time for G_{16}, G_{32}.
- Considering on the one hand the polynomials :

$$y^{12} + 16y^{10} - 4y^9 + 90y^8 - 12y^7 + 220y^6 + 108y^5 + 121y^4 + 244y^3 + 252y^2 + 80y + 8 \ \textit{and}$$
$$y^{12} + 16y^{10} + 4y^9 + 90y^8 + 12y^7 + 220y^6 - 108y^5 + 121y^4 - 244y^3 + 252y^2 - 80y + 8$$

and on the other hand the Galois groups G_{34} and G_{77}, we can not decide which one is associated with the other.

- Using $\mathfrak{S}_9$ and $\mathfrak{S}_{10}$ we could find polynomials for the following subgroups :

$$G_{16}, G_{34}, G_{46}, G_{47}, G_{84}, G_{122}, G_{132}, G_{133}, G_{177}, G_{178}$$

H_i	$f(x)$	H_j	Θ	$h(x)$
H_{223}	$x^8+2x^7-x^5-x^4-2x^3-3x^2-x$	H_3	x_1x_2	$h^1_{1,5}$ $h^2_{1,5}$
H_{249}	$x^8+8x^7+20x^6+9x^5-23x^4-11x^3+12x^2-2x$	H_3	x_1x_2	h_2
H_{33}	$(x-1)(x^2+2)(x^3+2)$	H_{16}	$x_1x_2+x_2x_4$	h_3
H_{20}	x^6-3x^2-1	H_7	x_1+x_2	h_4
H_{14}	x^6-3x^2+1	H_{30}	$x_1x_2-x_3x_4$	$h_{6,7}$
H_2	$(x-1)(x^4+x+1)$	H_{12}	x_1-x_2	h_8
H_{23}	$(x^4+x+1)(x-1)(x-2)(x+1)$	H_9	$\delta_7x_1x_2$	h_9
H_{29}	$(x^3+2)(x^4+1)$	H_4	$x_1x_2x_3$	h_{10}
H_{28}	$(x^3+2)(x^4+x^3+x^2+x+1)$	H_4	$x_1x_2x_3$	h_{11}
H_{12}	$x^6+6x^4+2x^3+9x^2+6x-4$	H_{11}	x_1-x_2	h_{17}
H_{24}	$(x^3+x^2-2x-1)(x^4+8x+12)$	H_4	$x_1x_2x_3$	h_{20}
H_{14}	$x-1)(x^2+2)(x^4+x+1)$	H_4	$x_1x_2x_3$	$h_{21,23,24}$
H_6	$x^6+6x^5+15x^4+20x^3+17x^2+10x+5$	H_{30}	$x_1x_2-x_3x_4$	h_{22}
H_{87}	$(x^2+1)(x^6-3x^2+1)$	H_{37}	$\delta_8(x_1x_2x_3)$	$h_{25,26}$
H_{17}	$(x^3+2)(x^4-2)$	H_5	x_1x_2	h_{28}
H_{67}	$(x^4+x^3+x^2+x+1)(x^4+8x+12)$	H_3	b_8	h_{29}
H_5	$x^6+2x^4+2x^3+x^2+2x+2$	H_{11}	δ_2	$h_{35,36}$
H_{108}	$(x^2+1)(x^6+2x^3-2)$	H_{16}	$\delta_8(x_1x_2)$	h_{37}
H_{82}	$(x^2+1)(x^6+6x^4+2x^3+9x^2+6x-4)$	H_9	x_1x_2	h_{40}
H_8	$(x^3+2)(x^4+8x+12)$	H_5	x_1x_2	h_{43}
H_7	$(x^3+x^2-2x-1)(x^4+x+1)$	H_5	x_1x_2	h_{45}
H_{52}	$(x^2+1)(x^6+2x^2+2)$	H_{18}	x_1+2x_2	h_{48}
H_{33}	$(x^4+x^3+x^2+x+1)(x^4+x+1)$	H_3	b_8	h_{53}
H_{27}	$x^8-112x^6-1344x^5-4480x^4+53760x^3$ $+247296x^2-301056x+50176$	H_3	b_8	h_{58}
H_{26}	$x^8-6x^6-4x^5+24x^4-28x^2+18$	H_9	x_1x_2	$h_{67,68,69}$
H_3	$x^6+10x^5+55x^4+140x^3+175x^2-3019x+25$	H_{19}	δ_5	h_{74}
H_{91}	$x^8+10x^7+56x^6+150x^5+230x^4$ $+310x^3+200x^2+170x+25$	H_9	x_1x_2	$h_{75,76}$
H_{51}	$(x^2+2)(x^6+2x^4+2x^3+x^2+2x+2)$	H_{16}	$\delta_8x_1x_2$	h_{77}
H_4	$(x^3+2)(x^4+x+1)$	H_5	x_1x_2	h_{83}
H_{22}	$x^8+24x^7+252x^6+1513x^5+5684x^4$ $+13686x^3+20628x^2+17791x+6715$	H_3	b_8	h_{86}
H_{20}	$x^8+16x^4+16x^3+8$	H_3	b_8	$h_{108,109}$
H_{45}	$(x^2+1)(x^6+10x^5+55x^4+140x^3+175x^2-3019x+25)$	H_9	x_1x_2	h_{123}
H_{11}	$x^8+7x^4+8x^3+9$	H_9	x_1x_2	$h_{126,128}$
H_6	$x^8-8x^6-8x^5+8$	H_9	x_1x_2	$h_{161,163}$
H_4	$x^8-4x^6+x^4-4x^3+2x^2+4x+2$	H_9	x_1x_2	h_{162}
H_{17}	$(x^2+1)(x^6+24x-20$	H_9	x_1x_2	h_{180}
H_2	$(x-1)(x^6+x+1)$	H_6	δ_6	h_{183}
H_3	x^8+4x^5+8	H_9	x_1x_2	$h_{202,203}$
H_9	$(x^6+x+1)(x^2+1)$	H_9	x_1x_2	h_{219}

where the polynomials of the last column are :

$h^1_{1,5} = y^{12} - 2y^{11} + 5y^{10} - 11y^9 + 25y^8 + y^7 + 12y^6 + 2y^5 + 9y^4 - 4y^3 + 3y^2 - y + 1$

$h^2_{1,5} = y^{12} - y^{11} + 3y^{10} - 4y^9 + 9y^8 + 2y^7 + 12y^6 + y^5 + 25y^4 - 11y^3 + 5y^2 - 2y + 1$

$h_2 = y^{12} - 12y^{11} - 336y^{10} - 1552y^9 + 596y^8 + 10768y^7 - 4480y^6 - 24160y^5$
$+26912y^4 - 6592y^3 - 1792y^2 + 512y + 64$

$h_3 = y^{12} + 32y^9 + 688y^6 + 864y^3 + 432$

$h_4 = y^{12} + 6y^8 + 26y^6 - 63y^4 + 162y^2 + 81$

$h_{5,6} = y^{12} + 96y^8 - 1664y^6 - 16128y^4 - 165888y^2 + 331776$

$h_8 = y^{12} + 8y^8 + 26y^6 - 112y^4 + 216y^2 + 229$

$h_9 = y^{12} - 53569512y^{10} - 717423153979536y^8 + 57648012384276902259648y^6$
$-514695981865945021153506775296y^4 - 27572012576919524202023035041300375552y^2$
$+369254814650860343693640849980340240934342656$

$h_{10} = y^{12} + 256$

$h_{11} = y^{12} - 2y^9 + 4y^6 - 8y^3 + 16$

$h_{17} = y^{12} + 36y^{10} + 486y^8 + 3456y^6 + 16281y^4 + 43740y^2 + 58320$

$h_{20} = y^{12} - 88y^9 + 312y^8 - 256y^6 + 288y^5 + 1872y^4 + 512y^3 - 1536y^2 - 1152y + 1728$

$h_{21,23,24} = y^{12} + 4y^{10} - 4y^8 - 24y^6 - 16y^4 + 64y^2 + 64$

$h_{22} = y^{12} - 64y^8 - 3328y^6 - 7168y^4 + 221184y^2 - 573440$

$h_{25,26} = y^{12} + 2728536192y^{10} + 5583682313290395648y^8 + 4890336557870633400256167936y^6$
$+3207562569521809059269946222261043200y^4$
$+175039011180895443030630427306658648857509888y^2$
$+8844449574425293986610171953382281592546947785293824$

$h_{28} = y^{12} - 128$

$h_{29} = y^{12} - 132y^{10} + 64y^9 + 6624y^8 - 9216y^7 - 153152y^6 + 442368y^5$
$+1390080y^4 - 7921664y^3 + 728064y^2 + 40501248y - 39071744$

$h_{35,36} = y^{12} + 12y^{10} + 54y^8 + 116y^6 + 129y^4 + 72y^2 + 2932$

$h_{37} = y^{12} + 2213206251144649721355857559552y^6$
$+24991234235233440490006672624594436498165867207521965441024$

$h_{40} = y^{12} - 12y^{10} + 54y^8 - 96y^6 + 9y^4 + 108y^2 + 16$

$h_{43} = y^{12} - 48y^9 + 768y^6 - 4096y^3 + 27648$

$h_{45} = y^{12} - 4y^9 + 13y^8 - 11y^6 + 5y^5 + 26y^4 + y^3 - y^2 - 2y + 1$

$h_{48} = y^{12} + 6y^{10} + 79y^8 + 20y^6 - 3441y^4 + 13830y^2 + 9025$

$h_{53} = y^{12} - 11y^{10} + y^9 + 46y^8 - 12y^7 - 90y^6 + 48y^5 + 78y^4 - 73y^3 - 19y^2 + 36y - 4$

$h_{58} = y^{12} - 12544y^{11} + 107978752y^{10} - 535741595648y^9 + 1909791416909824y^8$
$-4082957118880088064y^7 + 5378360428284577054720y^6 - 4669438823093108794720256y^5$
$+2753851931806180010794942464y^4 - 1086269647847504399246234222592y^3$
$+271148496090643049249988329603072y^2 - 38518485084379470049875377443045376y$
$+2406354062637598140504000290845884416$

$h_{67,68,69} = y^{12} + 6y^{11} + 12y^{10} - 36y^9 - 150y^8 - 48y^7 + 784y^6$
$+1008y^5 - 2700y^4 - 4968y^3 + 3888y^2 + 11664y + 5832$

$h_{74} = y^{12} - 301434118652343750y^{10} + 14146286008474067628383636474609375y^8$
$-208692510095936067318575747776776552200317382812500y^6$
$+1920691909491408027619787679305485283975940546952188014984130859375y^4$
$-10693375822710497850016369478140898191786609117137629709759494289755821228027343750y^2$
$+25725683711566327813155738120217582668009332954532535965054551585495$
$801191427744925022125244140625$

$h_{75,76} = y^{12} - 10y^{10} + 575y^8 + 3700y^6 - 14225y^4 + 20150y^2 + 625$

$h_{77} = y^{12} - 192151552y^{10} + 9230554734002176y^8 + 7094661671837861159108608y^6$

$+6816251255792798570416189818798 08y^4 + 12583556059461298785999239682367689032631284924416$

$h_{86} = y^{12} - 1944y^{11} + 1734728y^{10} - 939630780y^9 + 344124883032y^8 - 89790952719888y^7$
$+17120727662202570y^6 - 2404497922515606456y^5 + 246974028198987137720y^4$
$-18102362981475140906148y^3 + 899273987839372123582504y^2$
$-27202431280969584288872352y + 379176853257990089902121761$

$h_{108,109} = y^{12} + 8y^{11} - 192y^{10} - 3328y^9 - 17920y^8 + 28672y^7 + 884736y^6 + 4063232y^5$
$+4456448y^4 - 6291456y^3 - 67108864y - 268435456$

$h_{123} = y^{12} - 10y^{10} + 575y^8 - 60080y^6 + 878695y^4 + 9105611y^2 + 625$

$h_{126,128} = y^{12} + 2y^{11} + 11y^{10} + 22y^9 + 31y^8 + 8y^7 - 118y^6 - 216y^5 - 297y^4$
$-90y^3 + 891y^2 + 1458y + 729$

$h_{161,163} = y^{12} + 8y^{11} + 16y^{10} + 16y^9 - 40y^8 + 256y^7 - 320y^4$
$-512y^3 + 1024y^2 - 1024y + 512$

$h_{162} = y^{12} + 2y^{11} + y^{10} + 12y^9 + 6y^8 - 8y^7 - 16y^6 - 48y^5 + 60y^4 - 8y^3 - 28y^2 - 16y + 8$

$h_{180} = y^{12} + 40y^6 + 576y^2 + 400$

$h_{183} = y^{12} - 28199y^{10} + 483151275y^8 - 6016051760112y^6 + 46602104639500215y^4$
$-165610319898887770806y^2 + 212034789279955398557529$

$h_{202,203} = y^{12} + 4y^{11} + 8y^{10} + 8y^9 + 8y^8 + 32y^7 + 64y^6 + 128y^5$
$+320y^4 + 512y^3 + 512y^2 + 512y + 512$

$h_{219} = y^{12} - 2y^6 + y^2 + 1$

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