# Program Semantics and Properties

MPRI 2–6: Abstract Interpretation,
application to verification and static analysis

Antoine Miné

Year 2020–2021

Course 02
25 September 2020

# Introduction

# Language syntax

$$
\begin{array}{llll}
{}^{\ell}\mathtt{stat}^{\ell} & ::= & {}^{\ell}X \leftarrow \mathtt{exp}^{\ell} & \textit{(assignment)} \\
 & \mid & {}^{\ell}\textbf{if } \mathtt{exp} \bowtie 0 \textbf{ then } {}^{\ell}\mathtt{stat}^{\ell} & \textit{(conditional)} \\
 & \mid & {}^{\ell}\textbf{while } {}^{\ell}\mathtt{exp} \bowtie 0 \textbf{ do } {}^{\ell}\mathtt{stat}^{\ell} \textbf{ done}^{\ell} & \textit{(loop)} \\
 & \mid & {}^{\ell}\mathtt{stat};\, {}^{\ell}\mathtt{stat}^{\ell} & \textit{(sequence)} \\
\mathtt{exp} & ::= & X & \textit{(variable)} \\
 & \mid & -\mathtt{exp} & \textit{(negation)} \\
 & \mid & \mathtt{exp} \diamond \mathtt{exp} & \textit{(binary operation)} \\
 & \mid & c & \textit{(constant } c \in \mathbb{Z}\textit{)} \\
 & \mid & [c, c'] & \textit{(random input, } c, c' \in \mathbb{Z} \cup \{\pm\infty\}\textit{)}
\end{array}
$$

Simple structured, numeric language

- $X \in \mathbb{V}$, where $\mathbb{V}$ is a finite set of program variables
- $\ell \in \mathcal{L}$, where $\mathcal{L}$ is a finite set of control points
- numeric expressions: $\bowtie \in \{=, \leq, \ldots\}$, $\diamond \in \{+, -, \times, /\}$
- random inputs: $X \leftarrow [c, c']$
  model environment, parametric programs, unknown functions, ...

# Expression semantics

$\mathsf{E}[\![\, e \,]\!] : (\mathbb{V} \to \mathbb{Z}) \to \mathcal{P}(\mathbb{Z})$

- semantics of an expression in a memory state $\rho \in \mathcal{E} \stackrel{\text{def}}{=} \mathbb{V} \to \mathbb{Z}$
- outputs a set of values in $\mathcal{P}(\mathbb{Z})$
  - divisions by zero return no result (omit error states for simplicity)
  - random inputs lead to several values (non-determinism)
- defined by structural induction

$$
\begin{array}{lcl}
\mathsf{E}[\![\, [c,c'] \,]\!]\,\rho & \stackrel{\text{def}}{=} & \{\, x \in \mathbb{Z} \mid c \leq x \leq c' \,\} \\
\mathsf{E}[\![\, X \,]\!]\,\rho & \stackrel{\text{def}}{=} & \{\, \rho(X) \,\} \\
\mathsf{E}[\![\, -e \,]\!]\,\rho & \stackrel{\text{def}}{=} & \{\, -v \mid v \in \mathsf{E}[\![\, e \,]\!]\,\rho \,\} \\
\mathsf{E}[\![\, e_1 + e_2 \,]\!]\,\rho & \stackrel{\text{def}}{=} & \{\, v_1 + v_2 \mid v_1 \in \mathsf{E}[\![\, e_1 \,]\!]\,\rho,\, v_2 \in \mathsf{E}[\![\, e_2 \,]\!]\,\rho \,\} \\
\mathsf{E}[\![\, e_1 - e_2 \,]\!]\,\rho & \stackrel{\text{def}}{=} & \{\, v_1 - v_2 \mid v_1 \in \mathsf{E}[\![\, e_1 \,]\!]\,\rho,\, v_2 \in \mathsf{E}[\![\, e_2 \,]\!]\,\rho \,\} \\
\mathsf{E}[\![\, e_1 \times e_2 \,]\!]\,\rho & \stackrel{\text{def}}{=} & \{\, v_1 \times v_2 \mid v_1 \in \mathsf{E}[\![\, e_1 \,]\!]\,\rho,\, v_2 \in \mathsf{E}[\![\, e_2 \,]\!]\,\rho \,\} \\
\mathsf{E}[\![\, e_1 / e_2 \,]\!]\,\rho & \stackrel{\text{def}}{=} & \{\, v_1 / v_2 \mid v_1 \in \mathsf{E}[\![\, e_1 \,]\!]\,\rho,\, v_2 \in \mathsf{E}[\![\, e_2 \,]\!]\,\rho,\ v_2 \neq 0 \,\}
\end{array}
$$

# Invariant semantics and properties

Invariant property: true of all program executions.

$$
\begin{array}{l}
{}^{\ell 1}X \leftarrow [0,10];\ {}^{\ell 2}\\
\quad Y \leftarrow 100;\\
\quad \textbf{while}\ {}^{\ell 3}\ X \geq 0\ \textbf{do}\ {}^{\ell 4}\\
\qquad X \leftarrow X - 1;\ {}^{\ell 5}\\
\qquad Y \leftarrow Y + 10\\
\quad \textbf{done}\ {}^{\ell 6}
\end{array}
$$

$$
\begin{cases}
\mathcal{X}_1 = \mathcal{E}\\
\mathcal{X}_2 = \mathsf{C}[\![\, X \leftarrow [0,10] \,]\!]\, \mathcal{X}_1\\
\mathcal{X}_3 = \mathsf{C}[\![\, Y \leftarrow 100 \,]\!]\, \mathcal{X}_2 \cup \mathsf{C}[\![\, Y \leftarrow Y + 10 \,]\!]\, \mathcal{X}_5\\
\mathcal{X}_4 = \mathsf{C}[\![\, X \geq 0 \,]\!]\, \mathcal{X}_3\\
\mathcal{X}_5 = \mathsf{C}[\![\, X \leftarrow X - 1 \,]\!]\, \mathcal{X}_4\\
\mathcal{X}_6 = \mathsf{C}[\![\, X < 0 \,]\!]\, \mathcal{X}_3
\end{cases}
$$

(atomic command semantics $\mathsf{C}[\![\, \texttt{com} \,]\!]$ on next slide)

- $\mathcal{X}_i \in \mathcal{P}(\mathcal{E})$: set of memory states at program point $i \in \mathcal{L}$
  e.g.: $\mathcal{X}_3 = \{\, \rho \in \mathcal{E} \mid \rho(X) \in [0,10],\ 10\rho(X) + \rho(Y) \in [100,200] \cap 10\mathbb{Z} \,\}$
- we look for the smallest solution $(\mathcal{X}_i)_{i \in \mathcal{L}}$ of the system
- $I \subseteq \mathcal{E}$ is invariant at $i$ if $\mathcal{X}_i \subseteq I$
- state invariants $I$ can express absence of assertion failures, overflows, memory errors, non-termination, etc.

# From programs to equations

**Atomic commands:** $\mathsf{C}[\![\, \mathtt{com} \,]\!] : \mathcal{P}(\mathcal{E}) \to \mathcal{P}(\mathcal{E})$

$\mathtt{com} \stackrel{\text{def}}{=} \{\, X \leftarrow \mathtt{exp},\ exp \bowtie 0 \,\}$: assignments and tests.

- $\mathsf{C}[\![\, X \leftarrow e \,]\!]\, \mathcal{X} \stackrel{\text{def}}{=} \{\, \rho[X \mapsto v] \mid \rho \in \mathcal{X},\ v \in \mathsf{E}[\![\, e \,]\!]\, \rho \,\}$
- $\mathsf{C}[\![\, e \bowtie 0 \,]\!]\, \mathcal{X} \stackrel{\text{def}}{=} \{\, \rho \in \mathcal{X} \mid \exists v \in \mathsf{E}[\![\, \rho \,]\!]\, \rho\text{:}\ v \bowtie 0 \,\}$

$\mathsf{C}[\![\, \cdot \,]\!]$ are $\cup$−morphisms: $\mathsf{C}[\![\, s \,]\!]\, \mathcal{X} = \cup_{\rho \in \mathcal{X}} \mathsf{C}[\![\, s \,]\!]\, \{\rho\}$, monotonic, continuous

**Systematic derivation of the equation system:** $eq({}^{\ell}stat^{\ell'})$

by structural induction:

$eq({}^{\ell 1}X \leftarrow e^{\ell 2}) \stackrel{\text{def}}{=} \{\, \mathcal{X}_{\ell 2} = \mathsf{C}[\![\, X \leftarrow e \,]\!]\, \mathcal{X}_{\ell 1} \,\}$

$eq({}^{\ell 1}s_1; {}^{\ell 2}s_2{}^{\ell 3}) \stackrel{\text{def}}{=} eq({}^{\ell 1}s_1{}^{\ell 2}) \cup ({}^{\ell 2}s_2{}^{\ell 3})$

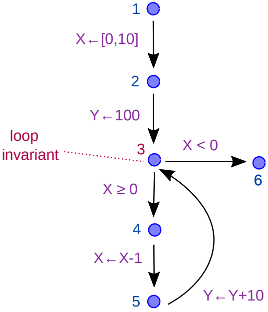
$eq({}^{\ell 1}\textbf{if}\ e \bowtie 0\ \textbf{then}\ {}^{\ell 2}s^{\ell 3}) \stackrel{\text{def}}{=}$
$\{\, \mathcal{X}_{\ell 2} = \mathsf{C}[\![\, e \bowtie 0 \,]\!]\, \mathcal{X}_{\ell 1} \,\} \cup eq({}^{\ell 2}s^{\ell 3'}) \cup \{\, \mathcal{X}_{\ell 3} = \mathcal{X}_{\ell 3'} \cup \mathsf{C}[\![\, e \not\bowtie 0 \,]\!]\, \mathcal{X}_{\ell 1} \,\}$

$eq({}^{\ell 1}\textbf{while}\ {}^{\ell 2}e \bowtie 0\ \textbf{do}\ {}^{\ell 3}s^{\ell 4}\ \textbf{done}^{\ell 5}) \stackrel{\text{def}}{=}$
$\{\, \mathcal{X}_{\ell 2} = \mathcal{X}_{\ell 1} \cup \mathcal{X}_{\ell 4},\ \mathcal{X}_{\ell 3} = \mathsf{C}[\![\, e \bowtie 0 \,]\!]\, \mathcal{X}_{\ell 2} \,\} \cup eq({}^{\ell 3}s^{\ell 4}) \cup \{\, \mathcal{X}_{\ell 5} = \mathsf{C}[\![\, e \not\bowtie 0 \,]\!]\, \mathcal{X}_{\ell 2} \,\}$

where: $\mathcal{X}^{\ell 3'}$ is a fresh variable storing intermediate results

# From control-flow graphs to equations

Programs can also be viewed as a control-flow graphs.




$$\rightarrow \quad \begin{cases} \mathcal{X}_1 = \mathcal{E} \\ \mathcal{X}_2 = \mathsf{C}[\![\, X \leftarrow [0,10] \,]\!]\, \mathcal{X}_1 \\ \mathcal{X}_3 = \mathsf{C}[\![\, Y \leftarrow 100 \,]\!]\, \mathcal{X}_2 \cup \mathsf{C}[\![\, Y \leftarrow Y + 10 \,]\!]\, \mathcal{X}_5 \\ \mathcal{X}_4 = \mathsf{C}[\![\, X \geq 0 \,]\!]\, \mathcal{X}_3 \\ \mathcal{X}_5 = \mathsf{C}[\![\, X \leftarrow X - 1 \,]\!]\, \mathcal{X}_4 \\ \mathcal{X}_6 = \mathsf{C}[\![\, X < 0 \,]\!]\, \mathcal{X}_3 \end{cases}$$

CFG: $(\mathcal{L}, e, A)$
nodes: $\mathcal{L}$
entry node: $e \in \mathcal{L}$
arcs: $A \subseteq \mathcal{L} \times \texttt{com} \times \mathcal{L}$

$$\rightarrow \quad (\mathcal{X}_i)_{i \in \mathcal{L}} \quad \begin{cases} \mathcal{X}_e = \mathcal{E} \\ \mathcal{X}_i = \bigcup_{(j,c,i) \in A} \mathsf{C}[\![\, c \,]\!]\, \mathcal{X}_j \quad \text{if } i \neq e \end{cases}$$

Benefit: can also reason on unstructured programs.

# Transition semantics

Program execution as discrete transitions between states.

- $\Sigma$: set of states
- $\tau \subseteq \Sigma \times \Sigma$: a transition relation, written $\sigma \to_\tau \sigma'$, or $\sigma \to \sigma'$
  (sometimes, we use *labelled* transition systems instead: $\tau \subseteq \Sigma \times \mathcal{A} \times \Sigma$, $\sigma \xrightarrow{a} \sigma'$)

$\implies$ a form of small-step semantics.

Application: on our programming language

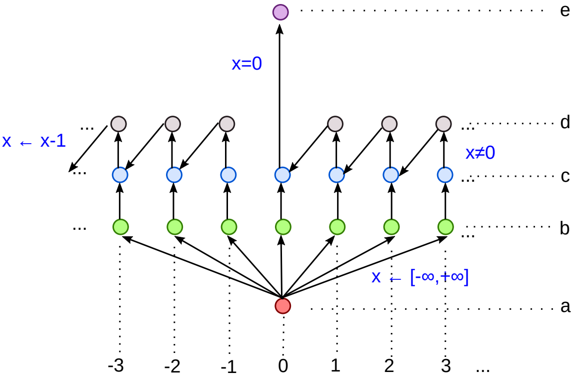
- $\Sigma \stackrel{\text{def}}{=} \mathcal{L} \times \mathcal{E}$: a control point and a memory state
- initial states $\mathcal{I} \stackrel{\text{def}}{=} \{\ell\} \times \mathcal{E}$ and
  final states $\mathcal{F} \stackrel{\text{def}}{=} \{\ell'\} \times \mathcal{E}$ for program ${}^{\ell}\mathtt{stat}^{\ell'}$
- $\tau$ defined by structural induction on ${}^{\ell}\mathtt{stat}^{\ell'}$ (next slides)

but transition systems can model many other languages: imperative languages, $\lambda-$calculus, abstract machines, concurrent programs, mobile systems, ...

# Transition semantics example

**Example**

$^a X \leftarrow [-\infty, \infty];$
$^b$**while** $^c X \neq 0$ **do** $^d X \leftarrow X - 1$ **done** $^e$



# From programs to transition relations

Transitions: $\tau[{}^{\ell}stat^{\ell'}] \subseteq \Sigma \times \Sigma$

$$\tau[{}^{\ell 1}X \leftarrow e^{\ell 2}] \stackrel{\text{def}}{=} \{ (\ell 1, \rho) \to (\ell 2, \rho[X \mapsto v]) \mid \rho \in \mathcal{E},\ v \in \mathsf{E}[\![\, e \,]\!]\, \rho \}$$

$$\begin{array}{l}\tau[{}^{\ell 1}\textbf{if } e \bowtie 0 \textbf{ then } {}^{\ell 2}s^{\ell 3}] \stackrel{\text{def}}{=} \\ \quad \{ (\ell 1, \rho) \to (\ell 2, \rho) \mid \rho \in \mathcal{E},\ \exists v \in \mathsf{E}[\![\, e \,]\!]\, \rho \colon v \bowtie 0 \} \cup \\ \quad \{ (\ell 1, \rho) \to (\ell 3, \rho) \mid \rho \in \mathcal{E},\ \exists v \in \mathsf{E}[\![\, e \,]\!]\, \rho \colon v \not\bowtie 0 \} \cup \tau[{}^{\ell 2}s^{\ell 3}]\end{array}$$

$$\begin{array}{l}\tau[{}^{\ell 1}\textbf{while } {}^{\ell 2}e \bowtie 0 \textbf{ do } {}^{\ell 3}s^{\ell 4} \textbf{ done}^{\ell 5}] \stackrel{\text{def}}{=} \\ \quad \{ (\ell 1, \rho) \to (\ell 2, \rho) \mid \rho \in \mathcal{E} \} \cup \\ \quad \{ (\ell 2, \rho) \to (\ell 3, \rho) \mid \rho \in \mathcal{E},\ \exists v \in \mathsf{E}[\![\, e \,]\!]\, \rho \colon v \bowtie 0 \} \cup \tau[{}^{\ell 3}s^{\ell 4}] \cup \\ \quad \{ (\ell 4, \rho) \to (\ell 2, \rho) \mid \rho \in \mathcal{E} \} \cup \\ \quad \{ (\ell 2, \rho) \to (\ell 5, \rho) \mid \rho \in \mathcal{E},\ \exists v \in \mathsf{E}[\![\, e \,]\!]\, \rho \colon v \not\bowtie 0 \}\end{array}$$

$$\tau[{}^{\ell 1}s_1; {}^{\ell 2}s_2{}^{\ell 3}] \stackrel{\text{def}}{=} \tau[{}^{\ell 1}s_1{}^{\ell 2}] \cup \tau[{}^{\ell 2}s_2{}^{\ell 3}]$$

# Reachability semantics and post-conditions

## Reachability semantics

- $\mathcal{R} \subseteq \Sigma$ states reachable from $\mathcal{I}$ by $\tau$ (transitively)
- $\mathcal{R} \cap \mathcal{F}$ final reachable states
  $\implies$ we can check program post-conditions and non-termination

## Link with the equational semantics

$$\mathcal{R} \cap (\{i\} \times \mathcal{E}) = \{i\} \times \mathcal{X}_i \simeq \mathcal{X}_i \qquad (\mathcal{X}_i \text{ are the reachable states at } i \in \mathcal{L})$$

### Alternate form for reachability

$\mathsf{C}[\![\, \texttt{stat} \,]\!]\, \mathcal{I} \subseteq \mathcal{E}$ defined by structural induction:

- $\mathsf{C}[\![\, X \leftarrow e \,]\!]$ and $\mathsf{C}[\![\, e \bowtie 0 \,]\!]$ as in the equational semantics
- $\mathsf{C}[\![\, s_1;\ s_2 \,]\!]\, \mathcal{X} \stackrel{\text{def}}{=} \mathsf{C}[\![\, s_2 \,]\!]\, (\mathsf{C}[\![\, s_1 \,]\!]\, \mathcal{X})$
- $\mathsf{C}[\![\, \textbf{if } e \bowtie 0 \textbf{ then } s \,]\!]\, \mathcal{X} \stackrel{\text{def}}{=} (\mathsf{C}[\![\, s \,]\!]\, (\mathsf{C}[\![\, e \bowtie 0 \,]\!]\, \mathcal{X})) \cup (\mathsf{C}[\![\, e \not\bowtie 0 \,]\!]\, \mathcal{X})$
- $\mathsf{C}[\![\, \textbf{while } e \bowtie 0 \textbf{ do } s \textbf{ done} \,]\!]\, \mathcal{X} \stackrel{\text{def}}{=} \mathsf{C}[\![\, e \not\bowtie 0 \,]\!]\, (\cup_{i \geq 0}\, (\mathsf{C}[\![\, s \,]\!] \circ \mathsf{C}[\![\, e \bowtie 0 \,]\!])^i \mathcal{X})$

# Trace semantics

## Semantics:

- trace: a sequence of states (finite or infinite)
- execution trace: a sequence of states linked by the transition relation $\tau$

The semantics of a program is now a set of traces.

## Trace properties:

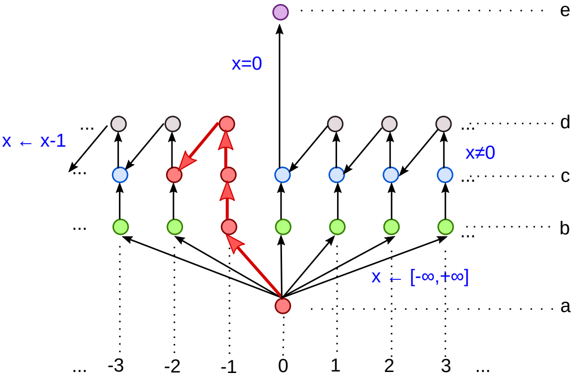
Traces carry more information than states
and can prove more expressive properties:

- temporal properties ($a$ occurs before $b$)
- computation length (possibly infinite)
- liveness (termination, inevitability)

# Trace semantics example

**Example**

$^{a}X \leftarrow [-\infty, \infty];$
$^{b}\textbf{while } {}^{c}X \neq 0 \textbf{ do } {}^{d}X \leftarrow X - 1 \textbf{ done } {}^{e}$



# Roadmap

Goal:

- express all these semantics as fixpoints
- relate these semantics by abstraction relations
- introduce variants (backward semantics, infinite trace semantics, ...)
- study which semantics to choose for which class of properties
- beyond trace properties

**Caveat:**

- start generally from transition systems (not high-level syntax)
  $\Longrightarrow$ uniform view of semantics independent from programming language
- remain at the level of concrete collecting semantics
  - express precisely all properties in a class of interest
  - uncomputable

the next course will return to numeric programs
and introduce computable abstractions to achieve computable static analysis

# State semantics and properties

# Forward semantics

# Forward reachability

Forward image: $\mathrm{post}_\tau : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma)$

$$\mathrm{post}_\tau(S) \stackrel{\text{def}}{=} \{\, \sigma' \mid \exists \sigma \in S : \sigma \to \sigma' \,\}$$

$\mathrm{post}_\tau$ is a strict, complete $\cup-$morphism in $(\mathcal{P}(\Sigma), \subseteq, \cup, \cap, \emptyset, \Sigma)$.

$\mathrm{post}_\tau(\cup_{i \in I} S_i) = \cup_{i \in I}\, \mathrm{post}_\tau(S_i)$, $\mathrm{post}_\tau(\emptyset) = \emptyset$

Blocking states: $\mathcal{B} \stackrel{\text{def}}{=} \{\, \sigma \mid \forall \sigma' \in \Sigma : \sigma \not\to \sigma' \,\}$

(states with no successor: valid final states but also errors)

$\mathcal{R}(\mathcal{I})$: states reachable from $\mathcal{I}$ in the transition system

$$\begin{aligned} \mathcal{R}(\mathcal{I}) &\stackrel{\text{def}}{=} \{\, \sigma \mid \exists n \geq 0, \sigma_0, \ldots, \sigma_n : \sigma_0 \in \mathcal{I}, \sigma = \sigma_n, \forall i : \sigma_i \to \sigma_{i+1} \,\} \\ &= \bigcup_{n \geq 0} \mathrm{post}_\tau^n(\mathcal{I}) \end{aligned}$$

(reachable $\iff$ reachable from $\mathcal{I}$ in $n$ steps of $\tau$ for some $n \geq 0$)

# Fixpoint formulation of forward reachability

$\mathcal{R}(\mathcal{I})$ can be expressed in fixpoint form:

$$\mathcal{R}(\mathcal{I}) = \mathsf{lfp}\ F_{\mathcal{R}} \text{ where } F_{\mathcal{R}}(S) \stackrel{\text{def}}{=} \mathcal{I} \cup \mathsf{post}_\tau(S)$$

$F_{\mathcal{R}}$ shifts $S$ and adds back $\mathcal{I}$

Alternate characterization: $\mathcal{R} = \mathsf{lfp}_{\mathcal{I}}\ G_{\mathcal{R}}$ where $G_{\mathcal{R}}(S) \stackrel{\text{def}}{=} S \cup \mathsf{post}_\tau(S)$.

$G_{\mathcal{R}}$ shifts $S$ by $\tau$ and accumulates the result with $S$

(proofs on next slide)

# Fixpoint formulation proof

proof: of $\mathcal{R}(\mathcal{I}) = \text{lfp}\, F_\mathcal{R}$ where $F_\mathcal{R}(S) \stackrel{\text{def}}{=} \mathcal{I} \cup \text{post}_\tau(S)$

$(\mathcal{P}(\Sigma), \subseteq)$ is a CPO and $\text{post}_\tau$ is continuous, hence $F_\mathcal{R}$ is continuous:
$F_\mathcal{R}(\cup_{i \in I} A_i) = \cup_{i \in I} F_\mathcal{R}(A_i)$.

By Kleene's theorem, $\text{lfp}\, F_\mathcal{R} = \cup_{n \in \mathbb{N}} F_\mathcal{R}^n(\emptyset)$.

We prove by recurrence on $n$ that: $\forall n$: $F_\mathcal{R}^n(\emptyset) = \cup_{i<n} \text{post}_\tau^i(\mathcal{I})$.
(states reachable in less than $n$ steps)

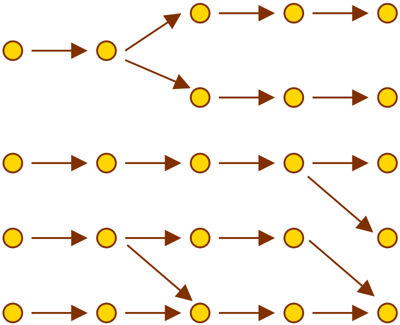
- $F_\mathcal{R}^0(\emptyset) = \emptyset$
- assuming the property at $n$,

$$\begin{array}{rcl}
F_\mathcal{R}^{n+1}(\emptyset) & = & F_\mathcal{R}(\bigcup_{i<n} \text{post}_\tau^i(\mathcal{I})) \\
& = & \mathcal{I} \cup \text{post}_\tau(\bigcup_{i<n} \text{post}_\tau^i(\mathcal{I})) \\
& = & \mathcal{I} \cup \bigcup_{i<n} \text{post}_\tau(\text{post}_\tau^i(\mathcal{I})) \\
& = & \mathcal{I} \cup \bigcup_{1 \leq i < n+1} \text{post}_\tau^i(\mathcal{I}) \\
& = & \bigcup_{i<n+1} \text{post}_\tau^i(\mathcal{I})
\end{array}$$

Hence: $\text{lfp}\, F_\mathcal{R} = \cup_{n \in \mathbb{N}} F_\mathcal{R}^n(\emptyset) = \cup_{i \in \mathbb{N}} \text{post}_\tau^i(\mathcal{I}) = \mathcal{R}(\mathcal{I})$.

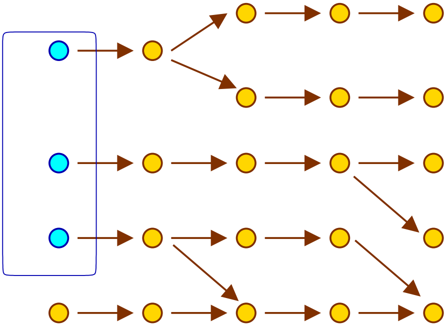
The proof is similar for the alternate form, given that $\text{lfp}_\mathcal{I}\, G_\mathcal{R} = \cup_{n \in \mathbb{N}} G_\mathcal{R}^n(\mathcal{I})$ and $G_\mathcal{R}^n(\mathcal{I}) = F_\mathcal{R}^{n+1}(\emptyset) = \cup_{i \leq n} \text{post}_\tau^i(\mathcal{I})$.

# Graphical illustration



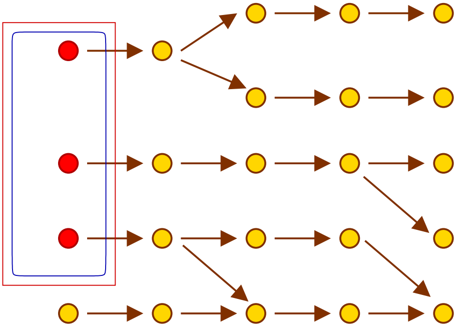
Transition system.

# Graphical illustration



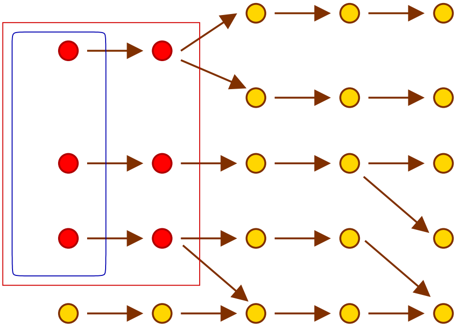
Initial states $\mathcal{I}$.

# Graphical illustration



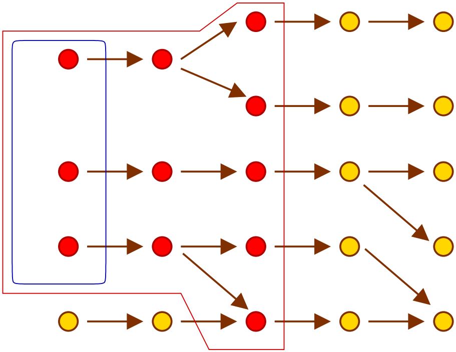
Iterate $F_{\mathcal{R}}^1(\mathcal{I})$.

# Graphical illustration



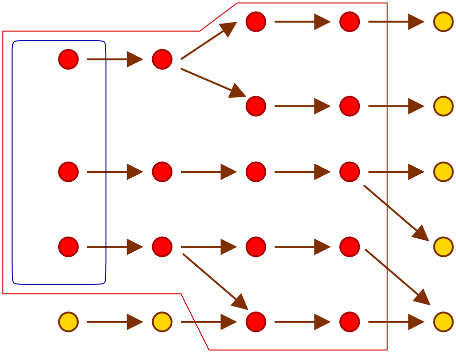
Iterate $F^2_{\mathcal{R}}(\mathcal{I})$.

# Graphical illustration



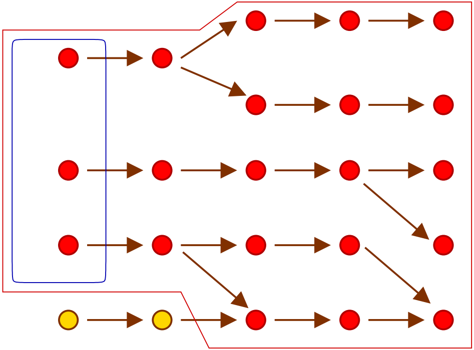
Iterate $F_{\mathcal{R}}^3(\mathcal{I})$.

# Graphical illustration



Iterate $F_{\mathcal{R}}^4(\mathcal{I})$.

# Graphical illustration



Iterate $F^5_{\mathcal{R}}(\mathcal{I})$.

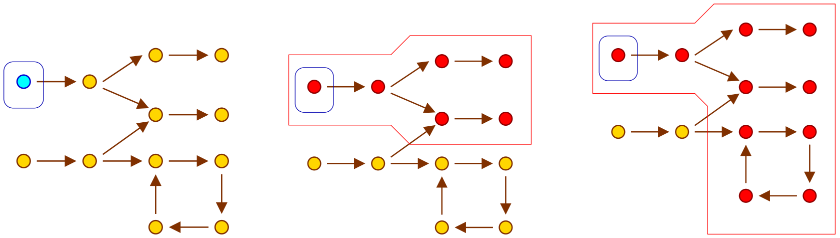
$F^6_{\mathcal{R}}(\mathcal{I}) = F^5_{\mathcal{R}}(\mathcal{I}) \Rightarrow$ we reached a fixpoint $\mathcal{R}(\mathcal{I}) = F^5_{\mathcal{R}}(\mathcal{I})$.

# Multiple forward fixpoints

Recall: $\mathcal{R}(\mathcal{I}) = \text{lfp } F_\mathcal{R}$ where $F_\mathcal{R}(S) \stackrel{\text{def}}{=} \mathcal{I} \cup \text{post}_\tau(S)$.

Note that $F_\mathcal{R}$ may have several fixpoints.

Example:



Initial state $\mathcal{I}$ $\qquad$ $\mathcal{R}(\mathcal{I}) = \text{lfp } F_\mathcal{R}$ $\qquad$ $\text{gfp } F_\mathcal{R}$

Exercise:

Compute all the fixpoints of $G_\mathcal{R}(S) \stackrel{\text{def}}{=} S \cup \text{post}_\tau(S)$ on this example.

# Example application of forward reachability

- Infer the set of possible states at program end: $\mathcal{R}(\mathcal{I}) \cap \mathcal{F}$.

$$
\begin{array}{ll}
\bullet & i \leftarrow 0; \\
& \textbf{while } i < 100 \textbf{ do} \\
& \quad i \leftarrow i + 1; \\
& \quad j \leftarrow j + [0, 1] \\
& \textbf{done } \bullet
\end{array}
$$

  - initial states $\mathcal{I}$: $j \in [0, 10]$ at control point $\bullet$,
  - final states $\mathcal{F}$: any memory state at control point $\bullet$,
  - $\implies \mathcal{R}(\mathcal{I}) \cap \mathcal{F}$: control at $\bullet$, $i = 100$, and $j \in [0, 110]$.

- Prove the absence of run-time error: $\mathcal{R}(\mathcal{I}) \cap \mathcal{B} \subseteq \mathcal{F}$.
  (never block except when reaching the end of the program)

  To ensure soundness, over-approximations are sufficient.
  (if $\mathcal{R}^\sharp(\mathcal{I}) \supseteq \mathcal{R}(\mathcal{I})$, then $\mathcal{R}^\sharp(\mathcal{I}) \cap \mathcal{B} \subseteq \mathcal{F} \implies \mathcal{R}(\mathcal{I}) \cap \mathcal{B} \subseteq \mathcal{F}$)

# Link with invariance proof methods

Invariance proof method: find an inductive invariant $I \subseteq \Sigma$

- $\mathcal{I} \subseteq I$ (contains initial states)
- $\forall \sigma \in I: \sigma \to \sigma' \implies \sigma' \in I$ (invariant by program transition)
- that implies the desired property: $I \subseteq P$.

Link with the state semantics $\mathcal{R}(\mathcal{I})$:

- if $I$ is an inductive invariant, then $F_{\mathcal{R}}(I) \subseteq I$
  $F_{\mathcal{R}}(I) = \mathcal{I} \cup \text{post}_\tau(I) \subseteq I \cup I = I$
  $\implies$ an inductive invariant is a post-fixpoint of $F_{\mathcal{R}}$
- $\mathcal{R}(\mathcal{I}) = \text{lfp}\, F_{\mathcal{R}}$
  $\implies \mathcal{R}(\mathcal{I})$ is the tightest inductive invariant

# Link with the equational semantics

By partitioning forward reachability wrt. control points, we retrieve the equation system form of program semantics.

## Control point partitioning

As $\Sigma \stackrel{\text{def}}{=} \mathcal{L} \times \mathcal{E}$, $\mathcal{P}(\Sigma) \simeq \mathcal{L} \to \mathcal{P}(\mathcal{E})$.

We have a Galois isomorphism:

$$(\mathcal{P}(\Sigma), \subseteq) \xleftarrow[\alpha_{\mathcal{L}}]{\gamma_{\mathcal{L}}} (\mathcal{L} \to \mathcal{P}(\mathcal{E}), \dot{\subseteq})$$

- $X \dot{\subseteq} Y \iff^{\text{def}} \forall \ell \in \mathcal{L}: X(\ell) \subseteq Y(\ell)$
- $\alpha_{\mathcal{L}}(S) \stackrel{\text{def}}{=} \lambda\ell.\{\, \rho \mid (\ell, \rho) \in S \,\}$
- $\gamma_{\mathcal{L}}(X) \stackrel{\text{def}}{=} \{\, (\ell, \rho) \mid \ell \in \mathcal{L}, \rho \in X(\ell) \,\}$
- given $F_{eq} \stackrel{\text{def}}{=} \alpha_{\mathcal{L}} \circ F_{\mathcal{R}} \circ \gamma_{\mathcal{L}}$
  we get back an equation system $\bigwedge_{\ell \in \mathcal{L}} \mathcal{X}_\ell = F_{eq,\ell}(\mathcal{X}_1, \ldots, \mathcal{X}_n)$
- $\alpha_{\mathcal{L}} \circ \gamma_{\mathcal{L}} = \gamma_{\mathcal{L}} \circ \alpha_{\mathcal{L}} = id$ (no abstraction)
  simply reorganize the states by control point
  after actual abstraction, partitioning makes a difference (flow-sensitivity)

# Link with Hoare logic

**Hoare logic:** proof method where we

- annotate program points with local sate invariants in $\mathcal{P}(\Sigma)$
- use logic rules to prove their correctness

$$\frac{}{\{P[e/X]\}\ X \leftarrow e\ \{P\}} \qquad \frac{\{P\}\ stat_1\ \{R\} \quad \{R\}\ stat_2\ \{Q\}}{\{P\}\ stat_1; stat_2\ \{Q\}}$$

$$\frac{\{P \wedge b\}\ stat\ \{Q\} \quad P \wedge \neg b \Rightarrow Q}{\{P\}\ \textbf{if}\ b\ \textbf{then}\ stat\ \{Q\}} \qquad \frac{\{P \wedge b\}\ stat\ \{P\}}{\{P\}\ \textbf{while}\ b\ \textbf{do}\ stat\ \{P \wedge \neg b\}}$$

$$\frac{\{P\}\ stat\ \{Q\} \quad P' \Rightarrow P \quad Q \Rightarrow Q'}{\{P'\}\ stat\ \{Q'\}}$$

Link with the state semantics $\mathcal{R}(\mathcal{I})$:

$F_{eq} \stackrel{\text{def}}{=} \alpha_{\mathcal{L}} \circ F_{\mathcal{R}} \circ \gamma_{\mathcal{L}}$ partitions $F_{\mathcal{R}}$ by control point and $\text{lfp}\, F_{\mathcal{R}}$ gives the tightest inductive invariant

- any post-fixpoint of $F_{eq}$ gives valid Hoare triples
- $\text{lfp}\, F_{eq}$ gives the tightest Hoare triples

# Solving the equational semantics

Solve $\bigwedge_{i\in[1,n]} \mathcal{X}_i = F_i(\mathcal{X}_1,\ldots,\mathcal{X}_n)$

Each $F_i$ is continuous in $\mathcal{P}(\mathcal{E})^n \to \mathcal{P}(\mathcal{E})$ (complete $\cup-$morphism)

aka $\vec{F} \stackrel{\text{def}}{=} (F_1,\ldots,F_n)$ is continuous in $\mathcal{P}(\mathcal{E})^n \to \mathcal{P}(\mathcal{E})^n$

By Taski's fixpoint theorem, lfp $\vec{F}$ exists.

**Tarksi's theorem: Jacobi iterations**

$$\left\{\begin{array}{l} \mathcal{X}_1^0 \stackrel{\text{def}}{=} \emptyset \\ \ldots \\ \mathcal{X}_i^0 \stackrel{\text{def}}{=} \emptyset \\ \ldots \\ \mathcal{X}_n^0 \stackrel{\text{def}}{=} \emptyset \end{array}\right. \qquad \left\{\begin{array}{l} \mathcal{X}_1^{k+1} \stackrel{\text{def}}{=} F_1(\mathcal{X}_1^k,\ldots,\mathcal{X}_n^k) \\ \ldots \\ \mathcal{X}_i^{k+1} \stackrel{\text{def}}{=} F_i(\mathcal{X}_1^k,\ldots,\mathcal{X}_n^k) \\ \ldots \\ \mathcal{X}_n^{k+1} \stackrel{\text{def}}{=} F_n(\mathcal{X}_1^k,\ldots,\mathcal{X}_n^k) \end{array}\right.$$

The limit of $(\mathcal{X}_1^k,\ldots,X_n^k)$ is lfp $\vec{F}$.

Naïve application of Tarski's theorem
called Jacobi iterations by analogy with linear algebra

# Solving the equational semantics (cont.)

Other iteration techniques exist [Cous92].

**Gauss-Seidl iterations**

$$\begin{cases} \mathcal{X}_1^{k+1} \stackrel{\text{def}}{=} F_1(\mathcal{X}_1^k, \ldots, \mathcal{X}_n^k) \\ \ldots \\ \mathcal{X}_i^{k+1} \stackrel{\text{def}}{=} F_i(\mathcal{X}_1^{k+1}, \ldots, \mathcal{X}_{i-1}^{k+1}, \mathcal{X}_i^k, \ldots, \mathcal{X}_n^k) \\ \ldots \\ \mathcal{X}_n^{k+1} \stackrel{\text{def}}{=} F_n(\mathcal{X}_1^{k+1}, \ldots, \mathcal{X}_{n-1}^{k+1}, \mathcal{X}_n^k) \end{cases}$$

use new results as soon available

**Chaotic iterations**

$$\mathcal{X}_i^{k+1} \stackrel{\text{def}}{=} \begin{cases} F_i(\mathcal{X}_1^k, \ldots, \mathcal{X}_n^k) & \text{if } i = \phi(k+1) \\ \mathcal{X}_i^k & \text{otherwise} \end{cases}$$

wrt. a fair schedule $\phi : \mathbb{N} \to [1, n]$
$\forall i \in [1, n]: \forall N > 0: \exists k > N: \phi(k) = i$

- worklist algorithms
- asynchonous iterations (parallel versions of chaotic iterations)

all give the same limit! (this will not be the case for abstract static analyses. . . )

# Inductive abstract interpreter

Principle:

- follow the control-flow of the program
- replace the global fixpoint with local fixpoints (loops)

$$\mathsf{C}[\![\, X \leftarrow e \,]\!]\,\mathcal{X} \stackrel{\text{def}}{=} \{\, \rho[X \mapsto v] \mid \rho \in \mathcal{X},\ v \in \mathsf{E}[\![\, e \,]\!]\,\rho \,\}$$

$$\mathsf{C}[\![\, e \bowtie 0 \,]\!]\,\mathcal{X} \stackrel{\text{def}}{=} \{\, \rho \in \mathcal{X} \mid \exists v \in \mathsf{E}[\![\, \rho \,]\!]\,\rho \colon v \bowtie 0 \,\}$$

$$\mathsf{C}[\![\, s_1; s_2 \,]\!]\,\mathcal{X} \stackrel{\text{def}}{=} \mathsf{C}[\![\, s_2 \,]\!]\,(\mathsf{C}[\![\, s_1 \,]\!]\,\mathcal{X})$$

$$\mathsf{C}[\![\, \textbf{if } e \bowtie 0 \textbf{ then } s \,]\!]\,\mathcal{X} \stackrel{\text{def}}{=} (\mathsf{C}[\![\, s \,]\!]\,(\mathsf{C}[\![\, e \bowtie 0 \,]\!]\,\mathcal{X})) \cup (\mathsf{C}[\![\, e \not\bowtie 0 \,]\!]\,\mathcal{X})$$

$$\mathsf{C}[\![\, \textbf{while } e \bowtie 0 \textbf{ do } s \textbf{ done} \,]\!]\,\mathcal{X} \stackrel{\text{def}}{=} \mathsf{C}[\![\, e \not\bowtie 0 \,]\!]\,(\mathrm{lfp}\, F)$$

where $F(\mathcal{Y}) \stackrel{\text{def}}{=} \mathcal{X} \cup \mathsf{C}[\![\, s \,]\!]\,(\mathsf{C}[\![\, e \bowtie 0 \,]\!]\,\mathcal{Y})$

informal justification for the loop semantics:

All the $\mathsf{C}[\![\, s \,]\!]$ functions are continuous, hence the fixoints exist.
By induction on $k$, $F^k(\emptyset) = \cup_{i \le k}\ (\mathsf{C}[\![\, s \,]\!] \circ \mathsf{C}[\![\, e \bowtie 0 \,]\!]\,)^i\mathcal{X}$
hence, $\mathrm{lfp}\, F = \cup_i\ (\mathsf{C}[\![\, s \,]\!] \circ \mathsf{C}[\![\, e \bowtie 0 \,]\!]\,)^i\mathcal{X}$
We fall back to a special case of (transfinite) chaotic iteration that stabilizes loops depth-first.

# Backward semantics

# Backward co-reachability

$\mathcal{C}(\mathcal{F})$: states co-reachable from $\mathcal{F}$ in the transition system:

$$\mathcal{C}(\mathcal{F}) \stackrel{\text{def}}{=} \{\, \sigma \mid \exists n \geq 0, \sigma_0, \ldots, \sigma_n : \sigma = \sigma_0, \sigma_n \in \mathcal{F}, \forall i : \sigma_i \to \sigma_{i+1} \,\}$$
$$= \bigcup_{n \geq 0} \mathrm{pre}_\tau^n(\mathcal{F})$$

where $\mathrm{pre}_\tau(S) \stackrel{\text{def}}{=} \{\, \sigma \mid \exists \sigma' \in S : \sigma \to \sigma' \,\}$ $\quad(\mathrm{pre}_\tau = \mathrm{post}_{\tau^{-1}})$

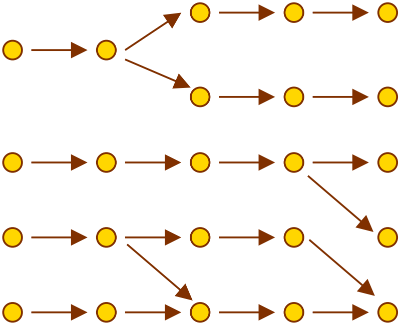
$\mathcal{C}(\mathcal{F})$ can also be expressed in fixpoint form:

$$\mathcal{C}(\mathcal{F}) = \mathrm{lfp}\, F_\mathcal{C} \text{ where } F_\mathcal{C}(S) \stackrel{\text{def}}{=} \mathcal{F} \cup \mathrm{pre}_\tau(S)$$

Justification: $\mathcal{C}(\mathcal{F})$ in $\tau$ is exactly $\mathcal{R}(\mathcal{F})$ in $\tau^{-1}$.

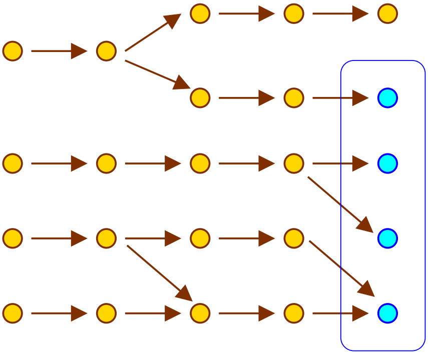
Alternate characterization: $\mathcal{C}(\mathcal{F}) = \mathrm{lfp}_\mathcal{F}\ G_\mathcal{C}$ where $G_\mathcal{C}(S) = S \cup \mathrm{pre}_\tau(S)$

# Graphical illustration



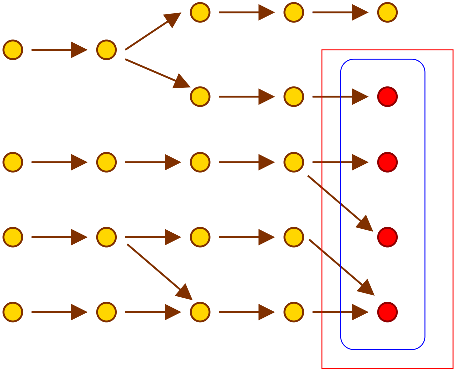
Transition system.

# Graphical illustration

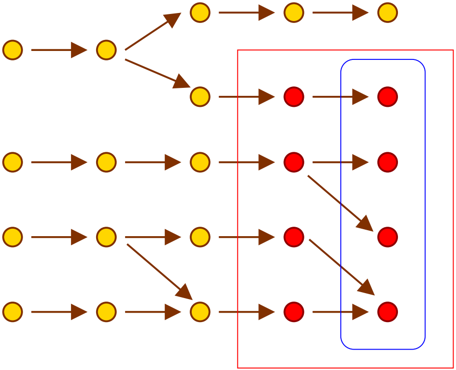


Final states $\mathcal{F}$.

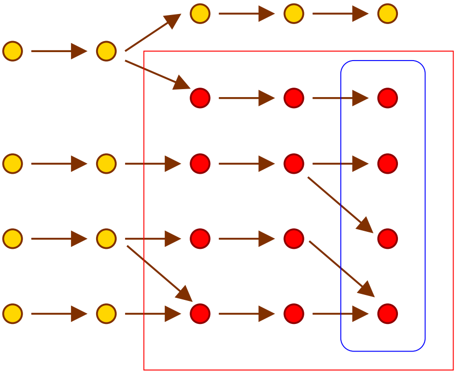
# Graphical illustration



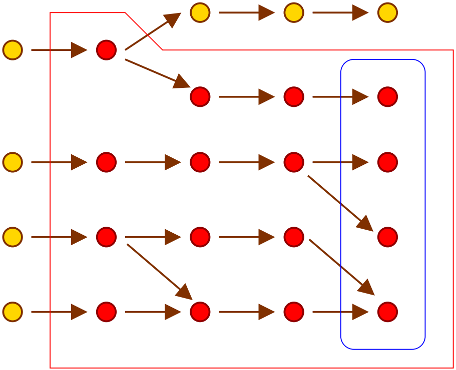
# Graphical illustration



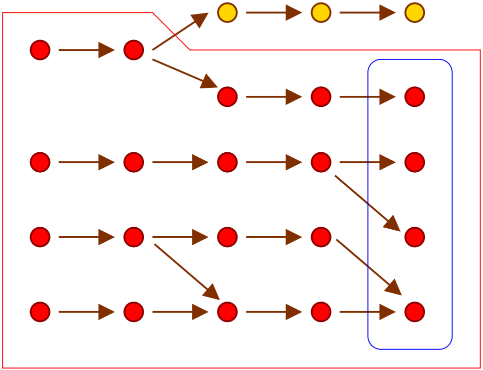
# Graphical illustration



# Graphical illustration



# Graphical illustration



States co-reachable from $\mathcal{F}$.

# Application of backward co-reachability

- $\mathcal{I} \cap \mathcal{C}(\mathcal{B} \setminus \mathcal{F})$
  Initial states that have at least one erroneous execution.

$$
\begin{array}{l}
\bullet \quad j \leftarrow 0; \\
\quad \textbf{while } i > 0 \textbf{ do} \\
\quad\quad i \leftarrow i - 1; \\
\quad\quad j \leftarrow j + [0, 10] \\
\quad\quad \textbf{assert } (j \leq 200) \\
\quad \textbf{done} \bullet
\end{array}
$$

- initial states $\mathcal{I}$: $i \in [0, 100]$ at •
- final states $\mathcal{F}$: any memory state at •
- blocking states $\mathcal{B}$: final,
  or $j > 200$ (assertion failure)
- $\mathcal{I} \cap \mathcal{C}(\mathcal{B} \setminus \mathcal{F})$: at •, $i > 20$

- Over-approximating $\mathcal{C}$ is useful to isolate possibly incorrect executions from those guaranteed to be correct.
- Iterate forward and backward analyses interactively
  $\Longrightarrow$ abstract debugging [Bour93].

# Backward co-reachability in equational form

**Principle:**

As before, reorganize transitions by label $\ell \in \mathcal{L}$,
to get an equation system on $(\mathcal{X}_\ell)_\ell$, with $\mathcal{X}_\ell \subseteq \mathcal{E}$

Example:

$\ell 1$ $j \leftarrow 0$;
$\ell 2$ **while** $\ell 3$ $i > 0$ **do**
  $\ell 4$ $i \leftarrow i - 1$;
  $\ell 5$ $j \leftarrow j + [0, 10]$
$\ell 6$

$$
\begin{aligned}
\mathcal{X}_1 &= \overleftarrow{C}[\![\, j \rightarrow 0 \,]\!]\, \mathcal{X}_2 \\
\mathcal{X}_2 &= \mathcal{X}_3 \\
\mathcal{X}_3 &= \overleftarrow{C}[\![\, i > 0 \,]\!]\, \mathcal{X}_4 \cup \overleftarrow{C}[\![\, i \leq 0 \,]\!]\, \mathcal{X}_6 \\
\mathcal{X}_4 &= \overleftarrow{C}[\![\, i \leftarrow i - 1 \,]\!]\, \mathcal{X}_5 \\
\mathcal{X}_5 &= \overleftarrow{C}[\![\, j \leftarrow j + [0, 10] \,]\!]\, \mathcal{X}_3 \\
\mathcal{X}_6 &= \mathcal{F}
\end{aligned}
$$

- final states $\{\ell 6\} \times \mathcal{F}$.
- $\overleftarrow{C}[\![\, X \leftarrow e \,]\!]\, \mathcal{X} \stackrel{\text{def}}{=} \{\, \rho \mid \exists v \in \mathrm{E}[\![\, e \,]\!]\, \rho \colon \rho[X \mapsto v] \in \mathcal{X} \,\}$.
- $\overleftarrow{C}[\![\, e \bowtie 0 \,]\!]\, \mathcal{X} \stackrel{\text{def}}{=} \{\, \rho \in \mathcal{X} \mid \exists v \in \mathrm{E}[\![\, \rho \,]\!]\, \rho \colon v \bowtie 0 \,\} = \mathrm{C}[\![\, e \bowtie 0 \,]\!]\, \mathcal{X}$

(also possible on control-flow graphs...)

# Sufficient precondition semantics

# Sufficient preconditions

$\mathcal{S}(\mathcal{Y})$: states with executions staying in $\mathcal{Y}$.

$$\mathcal{S}(\mathcal{Y}) \stackrel{\text{def}}{=} \{\, \sigma \mid \forall n \geq 0, \sigma_0, \ldots, \sigma_n \colon (\sigma = \sigma_0 \wedge \forall i \colon \sigma_i \to \sigma_{i+1}) \implies \sigma_n \in \mathcal{Y} \,\}$$
$$= \bigcap_{n \geq 0} \widetilde{\mathsf{pre}}_\tau^n(\mathcal{Y})$$

where $\widetilde{\mathsf{pre}}_\tau(S) \stackrel{\text{def}}{=} \{\, \sigma \mid \forall \sigma' \colon \sigma \to \sigma' \implies \sigma' \in S \,\}$

(states such that all successors satisfy $S$, $\widetilde{\mathsf{pre}}$ is a complete $\cap-$morphism)

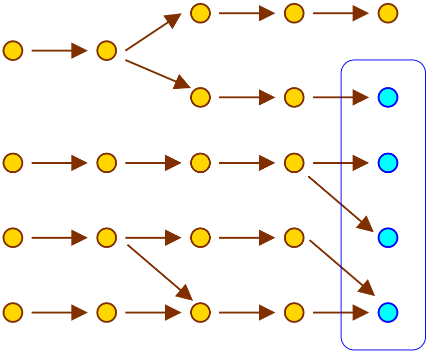
$\mathcal{S}(\mathcal{Y})$ can be expressed in fixpoint form:

$$\mathcal{S}(\mathcal{Y}) = \mathsf{gfp}\, F_\mathcal{S} \text{ where } F_\mathcal{S}(S) \stackrel{\text{def}}{=} \mathcal{Y} \cap \widetilde{\mathsf{pre}}_\tau(S)$$

proof sketch: similar to that of $\mathcal{R}(\mathcal{I})$, in the dual.

$F_\mathcal{S}$ is continuous in the dual CPO $(\mathcal{P}(\Sigma), \supseteq)$, because $\widetilde{\mathsf{pre}}_\tau$ is:
$F_\mathcal{S}(\cap_{i \in I} A_i) = \cap_{i \in I} F_\mathcal{S}(A_i)$.
By Kleene's theorem in the dual, $\mathsf{gfp}\, F_\mathcal{S} = \cap_{n \in \mathbb{N}} F_\mathcal{S}^n(\Sigma)$.
We would prove by recurrence that $F_\mathcal{S}^n(\Sigma) = \cap_{i<n} \widetilde{\mathsf{pre}}_\tau^i(\mathcal{Y})$.

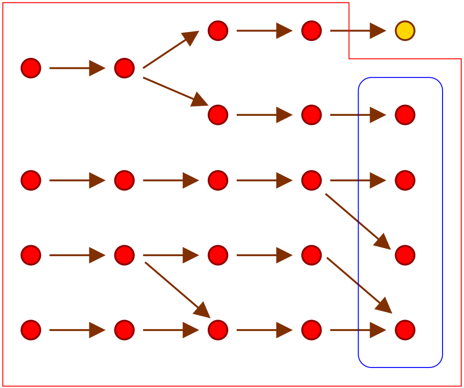
# Graphical illustration



Final states $\mathcal{F}$.

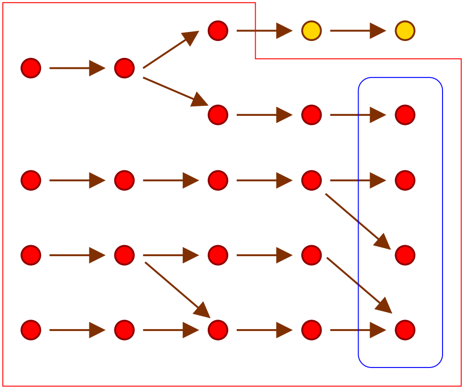
Goal: when stopping, stop in $\mathcal{F}$

# Graphical illustration



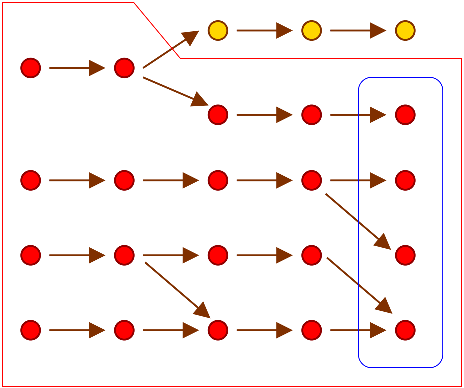
Final states $\mathcal{F}$.
Goal: stay in $\mathcal{Y} = \mathcal{F} \cup (\Sigma \setminus \mathcal{B})$
Iteration $F_{\mathcal{S}}^{0}(\mathcal{Y})$

# Graphical illustration



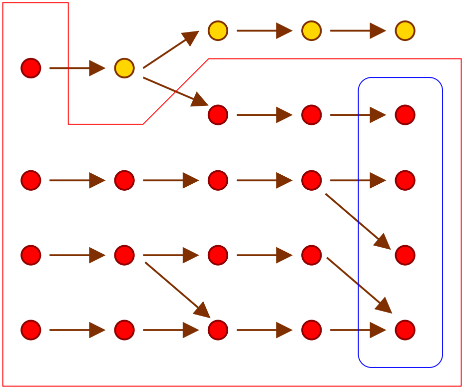
Final states $\mathcal{F}$.
Goal: stay in $\mathcal{Y} = \mathcal{F} \cup (\Sigma \setminus \mathcal{B})$
Iteration $F_{\mathcal{S}}^1(\mathcal{Y})$

# Graphical illustration



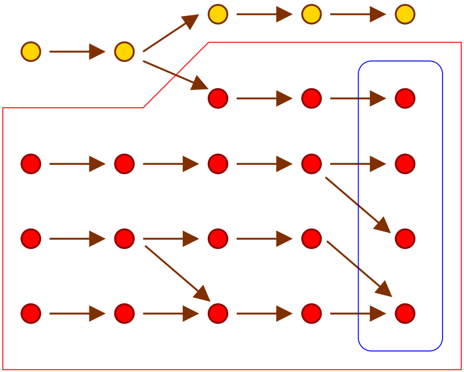
Final states $\mathcal{F}$.
Goal: stay in $\mathcal{Y} = \mathcal{F} \cup (\Sigma \setminus \mathcal{B})$
Iteration $F_{\mathcal{S}}^2(\mathcal{Y})$

# Graphical illustration



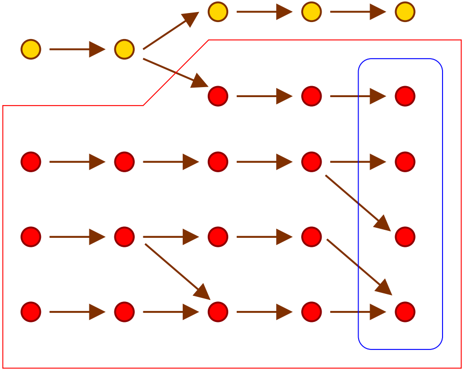
Final states $\mathcal{F}$.
Goal: stay in $\mathcal{Y} = \mathcal{F} \cup (\Sigma \setminus \mathcal{B})$
Iteration $F_{\mathcal{S}}^3(\mathcal{Y})$

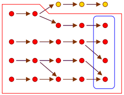
# Graphical illustration



Final states $\mathcal{F}$.
Goal: stay in $\mathcal{Y} = \mathcal{F} \cup (\Sigma \setminus \mathcal{B})$
Sufficient preconditions $\mathcal{S}(\mathcal{Y})$ to stop in $\mathcal{F}$

# Graphical illustration





Final states $\mathcal{F}$.
Goal: stay in $\mathcal{Y} = \mathcal{F} \cup (\Sigma \setminus \mathcal{B})$
Sufficient preconditions $\mathcal{S}(\mathcal{Y})$ to stop in $\mathcal{F}$ $\qquad \mathcal{C}(\mathcal{F})$

Note: $\mathcal{S}(\mathcal{Y}) \subsetneq \mathcal{C}(\mathcal{F})$

# Sufficient preconditions and reachability

## Correspondence with reachability:

We have a Galois connection:

$$(\mathcal{P}(\Sigma), \subseteq) \xleftrightarrow[\mathcal{R}]{\mathcal{S}} (\mathcal{P}(\Sigma), \subseteq)$$

- $\mathcal{R}(\mathcal{I}) \subseteq \mathcal{Y} \iff \mathcal{I} \subseteq \mathcal{S}(\mathcal{Y})$
  definition of a Galois connection
  all executions from $\mathcal{I}$ stay in $\mathcal{Y}$
  $\iff$ $\mathcal{I}$ includes only sufficient pre-conditions for $\mathcal{Y}$
- so $\mathcal{S}(\mathcal{Y}) = \bigcup \{ X \mid \mathcal{R}(X) \subseteq \mathcal{Y} \}$
  by Galois connection property
  $\mathcal{S}(\mathcal{Y})$ is the largest initial set whose reachability is in $\mathcal{Y}$

We retrieve Dijkstra's weakest liberal preconditions.

(proof sketch on next slide)

# Sufficient preconditions and reachability (proof)

proof sketch:

Recall that $\mathcal{R}(\mathcal{I}) = \mathrm{lfp}_{\mathcal{I}}\, G_{\mathcal{R}}$ where $G_{\mathcal{R}}(S) = S \cup \mathrm{post}_\tau(S)$.
Likewise, $\mathcal{S}(\mathcal{Y}) = \mathrm{gfp}_{\mathcal{Y}}\, G_{\mathcal{S}}$ where $G_{\mathcal{S}}(S) = S \cap \widetilde{\mathrm{pre}}_\tau(S)$.

We have a Galois connection: $(\mathcal{P}(\Sigma), \subseteq) \underset{\mathrm{post}_\tau}{\overset{\widetilde{\mathrm{pre}}_\tau}{\leftrightarrows}} (\mathcal{P}(\Sigma), \subseteq)$.

$$
\begin{array}{lcl}
\mathrm{post}_\tau(A) \subseteq B & \iff & \{\, \sigma' \mid \exists \sigma \in A : \sigma \to \sigma' \,\} \subseteq B \\
 & \iff & (\forall \sigma \in A : \sigma \to \sigma' \implies \sigma' \in B) \\
 & \iff & (A \subseteq \{\, \sigma \mid \forall \sigma' : \sigma \to \sigma' \implies \sigma' \in B \,\}) \\
 & \iff & A \subseteq \widetilde{\mathrm{pre}}_\tau(B)
\end{array}
$$

As a consequence $(\mathcal{P}(\Sigma), \subseteq) \underset{G_{\mathcal{R}}}{\overset{G_{\mathcal{S}}}{\leftrightarrows}} (\mathcal{P}(\Sigma), \subseteq)$.

The Galois connection can be lifted to fixpoint operators:

$(\mathcal{P}(\Sigma), \subseteq) \underset{x \mapsto \mathrm{lfp}_x\, G_{\mathcal{R}}}{\overset{x \mapsto \mathrm{gfp}_x\, G_{\mathcal{S}}}{\leftrightarrows}} (\mathcal{P}(\Sigma), \subseteq)$.

Exercise: complete the proof sketch.

# Application of sufficient preconditions

Initial states such that all executions are correct: $\mathcal{I} \cap \mathcal{S}(\mathcal{F} \cup (\Sigma \setminus \mathcal{B}))$.

(the only blocking states reachable from initial states are final states)

**program**

$\bullet$ $i \leftarrow 0;$
**while** $i < 100$ **do**
  $i \leftarrow i + 1;$
  $j \leftarrow j + [0, 1]$
  **assert** $(j \leq 105)$
**done** $\bullet$

- initial states $\mathcal{I}$: $j \in [0, 10]$ at $\bullet$
- final states $\mathcal{F}$: any memory state at $\bullet$
- blocking states $\mathcal{B}$: either final or $j > 105$ (assertion failure)
- $\mathcal{I} \cap \mathcal{S}(\mathcal{F} \cup (\Sigma \setminus \mathcal{B}))$: at $\bullet$, $j \in [0, 5]$
  (note that $\mathcal{I} \cap \mathcal{C}(\mathcal{F} \cup (\Sigma \setminus \mathcal{B}))$ gives $\mathcal{I}$)

- application to inferring function contracts
- application to inferring counter-examples
- requires under-approximations to build decidable abstractions
  but most analyses can only provide over-approximations!
  $\Longrightarrow$ **research topic**

# Finite trace semantics

# Traces

Trace: sequence of elements from $\Sigma$

- $\epsilon$: empty trace (unique)
- $\sigma$: trace of length 1 (assimilated to a state)
- $\sigma_0, \ldots, \sigma_{n-1}$: trace of length $n$

- $\Sigma^n$: the set of traces of length $n$
- $\Sigma^{\leq n} \stackrel{\text{def}}{=} \cup_{i \leq n} \Sigma^i$: the set of traces of length at most $n$
- $\Sigma^* \stackrel{\text{def}}{=} \cup_{i \in \mathbb{N}} \Sigma^i$: the set of finite traces

- state sets $\mathcal{I}, \mathcal{F} \subseteq \Sigma$ are also sets of traces, of length 1
- transition relation $\tau \subseteq \Sigma \times \Sigma$ is also a set of traces, of length 2

# Trace operations

Operations on traces:

- length: $|t| \in \mathbb{N}$ of a trace $t \in \Sigma^*$
- concatenation $\cdot$

  $(\sigma_0, \ldots, \sigma_n) \cdot (\sigma'_0, \ldots, \sigma'_m) \stackrel{\text{def}}{=} \sigma_0, \ldots, \sigma_n, \sigma'_0, \ldots, \sigma'_m$

  $\epsilon \cdot t \stackrel{\text{def}}{=} t \cdot \epsilon \stackrel{\text{def}}{=} t$
- junction $\frown$

  $(\sigma_0, \ldots, \sigma_n) \frown (\sigma'_0, \sigma'_1 \ldots, \sigma'_m) \stackrel{\text{def}}{=} \sigma_0, \ldots, \sigma_n, \sigma'_1, \ldots, \sigma'_m$
  when $\sigma_n = \sigma'_0$

  undefined if $\sigma_n \neq \sigma'_0$, and for $\epsilon$

  (join two consecutive traces, the common element $\sigma_n = \sigma'_0$ is not repeated)

# Trace operations (cont.)

Extension to sets of traces:

- $A \cdot B \stackrel{\text{def}}{=} \{\, a \cdot b \mid a \in A,\ b \in B \,\}$
  $\{\epsilon\}$ is the neutral element for $\cdot$
- $A ^\frown B \stackrel{\text{def}}{=} \{\, a ^\frown b \mid a \in A,\ b \in B,\ a ^\frown b \text{ defined} \,\}$
  $\Sigma$ is the neutral element for $^\frown$

$$
\begin{array}{lclclcl}
A^0 & \stackrel{\text{def}}{=} & \{\epsilon\} & \quad & A^{\frown 0} & \stackrel{\text{def}}{=} & \Sigma \\
A^{n+1} & \stackrel{\text{def}}{=} & A \cdot A^n & & A^{\frown n+1} & \stackrel{\text{def}}{=} & A ^\frown A^{\frown n} \\
A^* & \stackrel{\text{def}}{=} & \cup_{n<\omega} A^n & & A^{\frown *} & \stackrel{\text{def}}{=} & \cup_{n<\omega} A^{\frown n}
\end{array}
$$

Note: $A^n \neq \{\, a^n \mid a \in A \,\}$, $A^{\frown n} \neq \{\, a^{\frown n} \mid a \in A \,\}$ when $|A| > 1$

Note: $\cdot$ and $^\frown$ distribute $\cup$ and $\cap$

$(\cup_{i \in I} A_i) ^\frown (\cup_{j \in J} B_i) = \cup_{i \in I, j \in J} (A_i ^\frown B_j)$, etc.

# Finite prefix trace semantics

# Prefix trace semantics

$\mathcal{T}_p(\mathcal{I})$: finite partial execution traces starting in $\mathcal{I}$.

$$\begin{aligned}\mathcal{T}_p(\mathcal{I}) &\stackrel{\text{def}}{=} \{\, \sigma_0, \ldots, \sigma_n \mid n \geq 0, \sigma_0 \in \mathcal{I}, \forall i: \sigma_i \to \sigma_{i+1} \,\} \\ &= \bigcup_{n\geq 0} \mathcal{I}^\frown(\tau^{\frown n})\end{aligned}$$

(traces of length $n$, for any $n$, starting in $\mathcal{I}$ and following $\tau$)

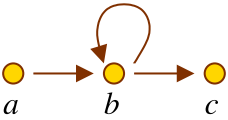
$\mathcal{T}_p(\mathcal{I})$ can be expressed in fixpoint form:

$$\mathcal{T}_p(\mathcal{I}) = \text{lfp}\ F_p \text{ where } F_p(T) \stackrel{\text{def}}{=} \mathcal{I} \cup T^\frown\tau$$

($F_p$ appends a transition to each trace, and adds back $\mathcal{I}$)

(proof on next slides)

# Prefix trace semantics: graphical illustration




$\mathcal{I} \stackrel{\text{def}}{=} \{a\}$
$\tau \stackrel{\text{def}}{=} \{(a,b),(b,b),(b,c)\}$

Iterates: $\mathcal{T}_p(\mathcal{I}) = \text{lfp}\, F_p$ where $F_p(T) \stackrel{\text{def}}{=} \mathcal{I} \cup T \frown \tau$.

- $F_p^0(\emptyset) = \emptyset$
- $F_p^1(\emptyset) = \mathcal{I} = \{a\}$
- $F_p^2(\emptyset) = \{a, ab\}$
- $F_p^3(\emptyset) = \{a, ab, abb, abc\}$
- $F_p^n(\emptyset) = \{\, a, ab^i, ab^j c \mid i \in [1, n-1], j \in [1, n-2] \,\}$
- $\mathcal{T}_p(\mathcal{I}) = \cup_{n \geq 0} F_p^n(\emptyset) = \{\, a, ab^i, ab^i c \mid i \geq 1 \,\}$

# Prefix trace semantics: proof

proof of: $\mathcal{T}_p(\mathcal{I}) = \text{lfp}\, F_p$ where $F_p(T) = \mathcal{I} \cup T \frown \tau$

Similar to the proof of $\mathcal{R}(\mathcal{I}) = \text{lfp}\, F_{\mathcal{R}}$ where $F_{\mathcal{R}}(S) \stackrel{\text{def}}{=} \mathcal{I} \cup \text{post}_\tau(S)$.

$F_p$ is continuous in a CPO $(\mathcal{P}(\Sigma^*), \subseteq)$:

$$\begin{array}{ll} & F_p(\cup_{i\in I}\, T_i) \\ = & \mathcal{I} \cup (\cup_{i\in I}\, T_i) \frown \tau \\ = & \mathcal{I} \cup (\cup_{i\in I}\, T_i \frown \tau) = \cup_{i\in I}\, (\mathcal{I} \cup T_i \frown \tau) \end{array}$$

hence (Kleene), $\text{lfp}\, F_p = \cup_{n\geq 0}\, F_p^i(\emptyset)$

We prove by recurrence on $n$ that $\forall n$: $F_p^n(\emptyset) = \cup_{i<n}\, \mathcal{I} \frown \tau^{\frown i}$:

- $F_p^0(\emptyset) = \emptyset$,
- $$\begin{array}{ll} & F_p^{n+1}(\emptyset) \\ = & \mathcal{I} \cup F_p^n(\emptyset) \frown \tau \\ = & \mathcal{I} \cup (\cup_{i<n}\, \mathcal{I} \frown \tau^{\frown i}) \frown \tau \\ = & \mathcal{I} \cup \cup_{i<n}\, (\mathcal{I} \frown \tau^{\frown i}) \frown \tau \\ = & \mathcal{I} \frown \tau^{\frown 0} \cup \cup_{i<n}\, (\mathcal{I} \frown \tau^{\frown i+1}) \\ = & \cup_{i<n+1}\, \mathcal{I} \frown \tau^{\frown i} \end{array}$$

Thus, $\text{lfp}\, F_p = \cup_{n\in\mathbb{N}}\, F_p^n(\emptyset) = \cup_{n\in\mathbb{N}}\, \cup_{i<n}\, \mathcal{I} \frown \tau^{\frown i} = \cup_{i\in\mathbb{N}}\, \mathcal{I} \frown \tau^{\frown i}$.

Note: we also have $\mathcal{T}_p(\mathcal{I}) = \text{lfp}_{\mathcal{I}}\, G_p$ where $G_p(T) = T \cup T \frown \tau$.

# Prefix trace semantics: expressive power

The prefix trace semantics is the collection of finite observations of program executions.

$\Longrightarrow$ this is the semantics of testing!

Limitations:

- no information on infinite executions,
  (we will add infinite traces later)
- can bound maximal execution time: $\mathcal{T}_p(\mathcal{I}) \subseteq \Sigma^{\leq n}$
  but cannot bound minimal execution time.
  (we will consider maximal traces later)
- cannot distinguish between finished and unfinished executions
  $\Longrightarrow$ no liveness property (see later)

# Abstracting traces into states

**Idea:** view state semantics as abstractions of traces semantics.

A state in the state semantics
corresponds to any partial execution trace terminating in this state.

We have a Galois embedding between finite traces and states:

$$(\mathcal{P}(\Sigma^*), \subseteq) \xleftarrow[\alpha_p]{\gamma_p} (\mathcal{P}(\Sigma), \subseteq)$$

- $\alpha_p(T) \stackrel{\text{def}}{=} \{\, \sigma \in \Sigma \mid \exists \sigma_0, \ldots, \sigma_n \in T \colon \sigma = \sigma_n \,\}$
  (last state in traces in $T$)
- $\gamma_p(S) \stackrel{\text{def}}{=} \{\, \sigma_0, \ldots, \sigma_n \in \Sigma^* \mid \sigma_n \in S \,\}$
  (traces ending in a state in $S$)

(proof on next slide)

# Abstracting traces into states (proof)

proof of: $(\alpha_p, \gamma_p)$ forms a Galois embedding.

Instead of the definition $\alpha(c) \subseteq a \iff c \subseteq \gamma(a)$, we use the alternate characterization of Galois connections: $\alpha$ and $\gamma$ are monotonic, $\gamma \circ \alpha$ is extensive, and $\alpha \circ \gamma$ is reductive.
Embedding means that, additionally, $\alpha \circ \gamma = id$.

- $\alpha_p$, $\gamma_p$ are $\cup-$morphisms, hence monotonic
- $(\gamma_p \circ \alpha_p)(T)$
  $= \{ \sigma_0, \ldots, \sigma_n \mid \sigma_n \in \alpha_p(T) \}$
  $= \{ \sigma_0, \ldots, \sigma_n \mid \exists \sigma'_0, \ldots, \sigma'_m \in T : \sigma_n = \sigma'_m \}$
  $\supseteq T$
- $(\alpha_p \circ \gamma_p)(S)$
  $= \{ \sigma \mid \exists \sigma_0, \ldots, \sigma_n \in \gamma_p(S) : \sigma = \sigma_n \}$
  $= \{ \sigma \mid \exists \sigma_0, \ldots, \sigma_n : \sigma_n \in S,\ \sigma = \sigma_n \}$
  $= S$

# Abstracting prefix trace semantics into reachability

We can abstract semantic operators and their least fixpoint.

Recall that:

- $\mathcal{T}_p(\mathcal{I}) = \mathrm{lfp}\, F_p$ where $F_p(T) \stackrel{\text{def}}{=} \mathcal{I} \cup T \frown \tau$,
- $\mathcal{R}(\mathcal{I}) = \mathrm{lfp}\, F_\mathcal{R}$ where $F_\mathcal{R}(S) \stackrel{\text{def}}{=} \mathcal{I} \cup \mathrm{post}_\tau(S)$,
- $(\mathcal{P}(\Sigma^*), \subseteq) \xleftarrow[\alpha_p]{\gamma_p} (\mathcal{P}(\Sigma), \subseteq)$.

We have: $\alpha_p \circ F_p = F_\mathcal{R} \circ \alpha_p$;

by fixpoint transfer, we get: $\alpha_p(\mathcal{T}_p(\mathcal{I})) = \mathcal{R}(\mathcal{I})$.

(proof on next slide)

# Abstracting prefix traces into reachability (proof)

proof: of $\alpha_p \circ F_p = F_{\mathcal{R}} \circ \alpha_p$

$$\begin{aligned}
&(\alpha_p \circ F_p)(T)\\
&= \alpha_p(\mathcal{I} \cup T^\frown\tau)\\
&= \{\, \sigma \mid \exists \sigma_0, \ldots, \sigma_n \in \mathcal{I} \cup T^\frown\tau \colon \sigma = \sigma_n \,\}\\
&= \mathcal{I} \cup \{\, \sigma \mid \exists \sigma_0, \ldots, \sigma_n \in T^\frown\tau \colon \sigma = \sigma_n \,\}\\
&= \mathcal{I} \cup \{\, \sigma \mid \exists \sigma_0, \ldots, \sigma_n \in T \colon \sigma_n \to \sigma \,\}\\
&= \mathcal{I} \cup \mathrm{post}_\tau(\{\, \sigma \mid \exists \sigma_0, \ldots, \sigma_n \in T \colon \sigma = \sigma_n \,\})\\
&= \mathcal{I} \cup \mathrm{post}_\tau(\alpha_p(T))\\
&= (F_{\mathcal{R}} \circ \alpha_p)(T)
\end{aligned}$$

# Abstracting traces into states (example)

**program**

$j \leftarrow 0;$
$i \leftarrow 0;$
**while** $i < 100$ **do**
    $i \leftarrow i + 1;$
    $j \leftarrow j + [0, 1]$
**done**

- prefix trace semantics:
  $i$ and $j$ are increasing and $0 \leq j \leq i \leq 100$
- forward reachable state semantics:
  $0 \leq j \leq i \leq 100$

$\Longrightarrow$ the abstraction forgets the ordering of states.

# Prefix closure

Prefix partial order: $\preceq$ on $\Sigma^*$

$$x \preceq y \stackrel{\text{def}}{\iff} \exists u \in \Sigma^*: x \cdot u = y$$

Note: $(\Sigma^*, \preceq)$ is not a CPO

Prefix closure: $\rho_p : \mathcal{P}(\Sigma^*) \to \mathcal{P}(\Sigma^*)$

$$\rho_p(T) \stackrel{\text{def}}{=} \{\, u \mid \exists t \in T: u \preceq t,\ u \neq \epsilon \,\}$$

$\rho_p$ is an upper closure operator on $\mathcal{P}(\Sigma^* \setminus \{\epsilon\})$.

(monotonic, extensive $T \subseteq \rho_p(T)$, idempotent $\rho_p \circ \rho_p = \rho_p$)

The prefix trace semantics is closed by prefix:

$$\rho_p(\mathcal{T}_p(\mathcal{I})) = \mathcal{T}_p(\mathcal{I}).$$

(note that $\epsilon \notin \mathcal{T}_p(\mathcal{I})$, which is why we disallowed $\epsilon$ in $\rho_p$)

# Another state/trace abstraction: Ordering abstraction

Another Galois embedding between finite traces and states:

$$(\mathcal{P}(\Sigma^*), \subseteq) \xleftarrow[\alpha_o]{\gamma_o} (\mathcal{P}(\Sigma), \subseteq)$$

- $\alpha_o(T) \stackrel{\text{def}}{=} \{\, \sigma \mid \exists \sigma_0, \ldots, \sigma_n \in T, i \leq n : \sigma = \sigma_i \,\}$

  (set of all states appearing in some trace in $T$)

- $\gamma_o(S) \stackrel{\text{def}}{=} \{\, \sigma_0, \ldots, \sigma_n \mid n \geq 0, \forall i \leq n : \sigma_i \in S \,\}$

  (traces composed of elements from $S$)

proof sketch:

$\alpha_o$ and $\gamma_o$ are monotonic, and $\alpha_o \circ \gamma_o = id$.

$(\gamma_o \circ \alpha_o)(T) = \{\, \sigma_0, \ldots, \sigma_n \mid \forall i \leq n : \exists \sigma'_0, \ldots, \sigma'_m \in T, j \leq m : \sigma_i = \sigma'_j \,\} \supseteq T$.

# Semantic correspondence by ordering abstraction

We have: $\alpha_o(\mathcal{T}_p(\mathcal{I})) = \mathcal{R}(\mathcal{I})$.

proof:

We have $\alpha_o = \alpha_p \circ \rho_p$ (i.e.: a state is in a trace if it is the last state of one of its prefix).
Recall the prefix trace abstraction into states: $\mathcal{R}(\mathcal{I}) = \alpha_p(\mathcal{T}_p(\mathcal{I}))$ and the fact that the prefix trace semantics is closed by prefix: $\rho_p(\mathcal{T}_p(\mathcal{I})) = \mathcal{T}_p(\mathcal{I})$.
We get $\alpha_o(\mathcal{T}_p(\mathcal{I})) = \alpha_p(\rho_p(\mathcal{T}_p(\mathcal{I}))) = \alpha_p(\mathcal{T}_p(\mathcal{I})) = \mathcal{R}(\mathcal{I})$.

This is a direct proof, not a fixpoint transfer proof (our theorems do not apply ...)

alternate proof: generalized fixpoint transfer

Recall that $\mathcal{T}_p(\mathcal{I}) = \mathrm{lfp}\, F_p$ where $F_p(T) \stackrel{\text{def}}{=} \mathcal{I} \cup T \frown \tau$ and $\mathcal{R}(\mathcal{I}) = \mathrm{lfp}\, F_\mathcal{R}$ where $F_\mathcal{R}(S) \stackrel{\text{def}}{=} \mathcal{I} \cup \mathrm{post}_\tau(S)$, but $\alpha_o \circ F_p = F_\mathcal{R} \circ \alpha_o$ does not hold in general, so, fixpoint transfer theorems do not apply directly.

However, $\alpha_o \circ F_p = F_\mathcal{R} \circ \alpha_o$ holds for sets of traces closed by prefix. By induction, the Kleene iterates $a_p^n$ and $a_\mathcal{R}^n$ involved in the computation of $\mathrm{lfp}\, F_p$ and $\mathrm{lfp}\, F_\mathcal{R}$ satisfy $\forall n\colon \alpha_o(a_p^n) = a_\mathcal{R}^n$, and so $\alpha_o(\mathrm{lfp}\, F_p) = \mathrm{lfp}\, F_\mathcal{R}$.

# Finite suffix trace semantics

# Suffix trace semantics

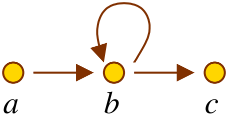
Similar results on the suffix trace semantics,
going backwards from $\mathcal{F}$:

- $\mathcal{T}_s(\mathcal{F}) \stackrel{\text{def}}{=} \{\, \sigma_0, \ldots, \sigma_n \mid n \geq 0, \sigma_n \in \mathcal{F}, \forall i: \sigma_i \to \sigma_{i+1} \,\}$
  (traces following $\tau$ and ending in a state in $\mathcal{F}$)
- $\mathcal{T}_s(\mathcal{F}) = \bigcup_{n \geq 0} (\tau^{\frown n})^\frown \mathcal{F}$
- $\mathcal{T}_s(\mathcal{F}) = \operatorname{lfp} F_s$ where $F_s(T) \stackrel{\text{def}}{=} \mathcal{F} \cup \tau^\frown T$
  ($F_s$ prepends a transition to each trace, and adds back $\mathcal{F}$)

Backward state co-rechability abstracts the suffix trace semantics:

- $\alpha_s(\mathcal{T}_s(\mathcal{F})) = \mathcal{C}(\mathcal{F})$ where $\alpha_s(T) \stackrel{\text{def}}{=} \{\, \sigma \mid \exists \sigma_0, \ldots, \sigma_n \in T: \sigma = \sigma_0 \,\}$
- $\rho_s(\mathcal{T}_s(\mathcal{F})) = \mathcal{T}_s(\mathcal{F})$ where $\rho_s(T) \stackrel{\text{def}}{=} \{\, u \mid \exists t \in \Sigma^*: t \cdot u \in T, u \neq \epsilon \,\}$
  (closed by suffix)

# Graphical illustration




$\mathcal{F} \stackrel{\text{def}}{=} \{c\}$
$\tau \stackrel{\text{def}}{=} \{(a,b),(b,b),(b,c)\}$

Iterates: $\mathcal{T}_s(\mathcal{F}) = \text{lfp}\ F_s$ where $F_s(T) \stackrel{\text{def}}{=} \mathcal{F} \cup \tau \frown T$.

- $F_s^0(\emptyset) = \emptyset$
- $F_s^1(\emptyset) = \mathcal{F} = \{c\}$
- $F_s^2(\emptyset) = \{c, bc\}$
- $F_s^3(\emptyset) = \{c, bc, bbc, abc\}$
- $F_s^n(\emptyset) = \{\, c, b^i c, ab^j c \mid i \in [1, n-1], j \in [1, n-2] \,\}$
- $\mathcal{T}_s(\mathcal{F}) = \cup_{n \geq 0} F_s^n(\emptyset) = \{\, c, b^i c, ab^i c \mid i \geq 1 \,\}$

# Finite partial trace semantics

# Symmetric finite partial trace semantics

$\mathcal{T}$: all the finite partial execution traces.

(not necessarily starting in $\mathcal{I}$ or ending in $\mathcal{F}$)

$$\begin{aligned}\mathcal{T} &\stackrel{\text{def}}{=} \{\, \sigma_0, \ldots, \sigma_n \mid n \geq 0, \forall i: \sigma_i \rightarrow \sigma_{i+1} \,\} \\ &= \bigcup_{n\geq 0} \Sigma^\frown \tau^{\frown n} \\ &= \bigcup_{n\geq 0} \tau^{\frown n \frown} \Sigma\end{aligned}$$

The semantics (and iterates) are forward/backward symmetric:

- $\mathcal{T} = \mathcal{T}_p(\Sigma)$, hence $\mathcal{T} = \text{lfp}\, F_{p*}$ where $F_{p*}(T) \stackrel{\text{def}}{=} \Sigma \cup T^\frown \tau$
  (prefix partial traces from any initial state)
- $\mathcal{T} = \mathcal{T}_s(\Sigma)$, hence $\mathcal{T} = \text{lfp}\, F_{s*}$ where $F_{s*}(T) \stackrel{\text{def}}{=} \Sigma \cup \tau^\frown T$
  (suffix partial traces to any final state)
- $F^n_{p*}(\emptyset) = F^n_{s*}(\emptyset) = \bigcup_{i<n} \Sigma^\frown \tau^{\frown i} = \bigcup_{i<n} \tau^{\frown i \frown} \Sigma = \mathcal{T} \cap \Sigma^{<n}$

# Abstracting partial traces into prefix traces

Prefix traces abstract partial traces
as we forget all about partial traces not starting in $\mathcal{I}$.

Galois connection:

$$(\mathcal{P}(\Sigma^*), \subseteq) \xleftrightarrow[\alpha_{\mathcal{I}}]{\gamma_{\mathcal{I}}} (\mathcal{P}(\Sigma^*), \subseteq)$$

- $\alpha_{\mathcal{I}}(T) \stackrel{\text{def}}{=} T \cap (\mathcal{I} \cdot \Sigma^*)$ (keep only traces starting in $\mathcal{I}$)
- $\gamma_{\mathcal{I}}(T) \stackrel{\text{def}}{=} T \cup ((\Sigma \setminus \mathcal{I}) \cdot \Sigma^*)$ (add all traces not starting in $\mathcal{I}$)

We then have: $\mathcal{T}_p(\mathcal{I}) = \alpha_{\mathcal{I}}(\mathcal{T})$.

similarly for the suffix traces: $\mathcal{T}_s(\mathcal{F}) = \alpha_{\mathcal{F}}(\mathcal{T})$ where $\alpha_{\mathcal{F}}(T) \stackrel{\text{def}}{=} \mathcal{T} \cap (\Sigma^* \cdot \mathcal{F})$

(proof on next slide)

# Abstracting partial traces into prefix traces (proof)

proof

$\alpha_\mathcal{I}$ and $\gamma_\mathcal{I}$ are monotonic.
$(\alpha_\mathcal{I} \circ \gamma_\mathcal{I})(T) = (T \cup (\Sigma \setminus \mathcal{I}) \cdot \Sigma^*) \cap \mathcal{I} \cdot \Sigma^*) = T \cap \mathcal{I} \cdot \Sigma^* \subseteq T$.
$(\gamma_\mathcal{I} \circ \alpha_\mathcal{I})(T) = (T \cap \mathcal{I} \cdot \Sigma^*) \cup (\Sigma \setminus \mathcal{I}) \cdot \Sigma^* = T \cup (\Sigma \setminus \mathcal{I}) \cdot \Sigma^* \supseteq T$.
So, we have a Galois connection.

A direct proof of $\mathcal{T}_p(\mathcal{I}) = \alpha_\mathcal{I}(\mathcal{T})$ is straightforward,
by definition of $\mathcal{T}_p$, $\alpha_\mathcal{I}$, and $\mathcal{T}$.

We can also retrieve the result by fixpoint transfer.

$\mathcal{T} = \text{lfp}\, F_{p*}$ where $F_{p*}(T) \stackrel{\text{def}}{=} \Sigma \cup T \frown \tau$.

$\mathcal{T}_p = \text{lfp}\, F_p$ where $F_p(T) \stackrel{\text{def}}{=} \mathcal{I} \cup T \frown \tau$.

We have: $(\alpha_\mathcal{I} \circ F_{p*})(T) = (\Sigma \cup T \frown \tau) \cap (\mathcal{I} \cdot \Sigma^*) = \mathcal{I} \cup ((T \frown \tau) \cap (\mathcal{I} \cdot \Sigma^*) =$
$\mathcal{I} \cup ((T \cap (\mathcal{I} \cdot \Sigma^*)) \frown \tau) = (F_p \circ \alpha_\mathcal{I})(T)$.

# A first hierarchy of semantics

$$\begin{tikzcd}
\mathcal{R}(\mathcal{I}) & & \mathcal{C}(\mathcal{F}) \\
\mathcal{T}_p(\mathcal{I}) \arrow[u, "\alpha_p"] & & \mathcal{T}_s(\mathcal{F}) \arrow[u, "\alpha_p"'] \\
& \mathcal{T} \arrow[ul, "\alpha_{\mathcal{I}}"] \arrow[ur, "\alpha_{\mathcal{F}}"'] &
\end{tikzcd}$$

forward/backward states

prefix/suffix traces

partial finite traces

# Maximal trace semantics

# The need for maximal traces

The partial trace semantics cannot distinguish between:

**while** $^a$ $0 = 0$ **do done**

**while** $^a$ $[0, 1] = 0$ **do done**

(we get $a^*$ for both programs)

Principle: restrict the semantics to complete executions only

- keep only executions finishing in a blocking state $\mathcal{B}$
- add back infinite executions

  the partial semantics took into account infinite execution by including all their finite parts, but we no longer keep them as they are not maximal!

Benefit:

- avoid confusing prefix of infinite executions with finite executions
- allow reasoning on trace length
- allow reasoning on infinite traces (non-termination, inevitability, liveness)

# Infinite traces

Notations:

- $\sigma_0, \ldots, \sigma_n, \ldots$: an infinite trace (length $\omega$)
- $\Sigma^\omega$: the set of all infinite traces
- $\Sigma^\infty \stackrel{\text{def}}{=} \Sigma^* \cup \Sigma^\omega$: the set of all traces

Extending the operators:

- $(\sigma_0, \ldots, \sigma_n) \cdot (\sigma'_0, \ldots) \stackrel{\text{def}}{=} \sigma_0, \ldots, \sigma_n, \sigma'_0, \ldots$
  (append to a finite trace)
- $t \cdot t' \stackrel{\text{def}}{=} t$ if $t \in \Sigma^\omega$ (append to an infinite trace does nothing)
- $(\sigma_0, \ldots, \sigma_n)^\frown(\sigma'_0, \sigma'_1 \ldots) \stackrel{\text{def}}{=} \sigma_0, \ldots, \sigma_n, \sigma'_1, \ldots$ when $\sigma_n = \sigma'_0$
- $t^\frown t' \stackrel{\text{def}}{=} t$, if $t \in \Sigma^\omega$
- prefix: $x \preceq y \stackrel{\text{def}}{\iff} \exists u \in \Sigma^\omega : x \cdot u = y$ $(\Sigma^\omega, \preceq)$ is a CPO

$\cdot$ distributes infinite $\cup$ and $\cap$

$^\frown$ distributes infinite $\cup$, but not infinite $\cap$

$\{a^\omega\}^\frown(\cap_{n\in\mathbb{N}} \{ a^m \mid n \geq m \}) = \{a^\omega\}^\frown\emptyset = \emptyset$ but

$\cap_{n\in\mathbb{N}} (\{a^\omega\}^\frown \{ a^m \mid n \geq m \}) = \cap_{n\in\mathbb{N}} \{a^\omega\} = \{a^\omega\}$

However $A^\frown(\cap_{i\in I} B_i) = \cup_{i\in I} (A^\frown B_i)$ if $A \subseteq \Sigma^*$.

# Maximal traces

Maximal traces: $\mathcal{M}_\infty \in \mathcal{P}(\Sigma^\infty)$

- sequences of states linked by the transition relation $\tau$,
- start in any state ($\mathcal{I} = \Sigma$),
- either finite and stop in a blocking state ($\mathcal{F} = \mathcal{B}$),
- or infinite.

$$\mathcal{M}_\infty \stackrel{\text{def}}{=} \{\, \sigma_0, \ldots, \sigma_n \in \Sigma^* \,|\, \sigma_n \in \mathcal{B}, \forall i < n \colon \sigma_i \to \sigma_{i+1} \,\} \cup \{\, \sigma_0, \ldots, \sigma_n, \ldots \in \Sigma^\omega \,|\, \forall i < \omega \colon \sigma_i \to \sigma_{i+1} \,\}$$

(can be anchored at $\mathcal{I}$ and $\mathcal{F}$ as: $\mathcal{M}_\infty \cap (\mathcal{I} \cdot \Sigma^\infty) \cap ((\Sigma^* \cdot \mathcal{F}) \cup \Sigma^\omega)$)

# Partitioned fixpoint formulation of maximal traces

**Goal:** we look for a fixpoint characterization of $\mathcal{M}_\infty$.

We consider separately finite and infinite maximal traces.

- Finite traces: already done!

  From the suffix partial trace semantics, recall:

  $\mathcal{M}_\infty \cap \Sigma^* = \mathcal{T}_s(\mathcal{B}) = \mathrm{lfp}\ F_s$

  recall that $F_s(T) \stackrel{\text{def}}{=} \mathcal{B} \cup \tau \frown T$ in $(\mathcal{P}(\Sigma^*), \subseteq)\ldots$

- Infinite traces:

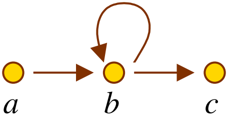
  Additionally, we will prove: $\mathcal{M}_\infty \cap \Sigma^\omega = \mathrm{gfp}\ G_s$

  where $G_s(T) \stackrel{\text{def}}{=} \tau \frown T$ in $(\mathcal{P}(\Sigma^\omega), \subseteq)$.

Note: only backward fixpoint formulation of maximal traces exist!

(proof in following slides)

# Infinite trace semantics: graphical illustration




$$\mathcal{B} \stackrel{\text{def}}{=} \{c\}$$
$$\tau \stackrel{\text{def}}{=} \{(a,b),(b,b),(b,c)\}$$

Iterates: $\mathcal{M}_\infty \cap \Sigma^\omega = \text{gfp } G_s$ where $G_s(T) \stackrel{\text{def}}{=} \tau \frown T$.

- $G_s^0(\Sigma^\omega) = \Sigma^\omega$
- $G_s^1(\Sigma^\omega) = ab\Sigma^\omega \cup bb\Sigma^\omega \cup bc\Sigma^\omega$
- $G_s^2(\Sigma^\omega) = abb\Sigma^\omega \cup bbb\Sigma^\omega \cup abc\Sigma^\omega \cup bbc\Sigma^\omega$
- $G_s^3(\Sigma^\omega) = abbb\Sigma^\omega \cup bbbb\Sigma^\omega \cup abbc\Sigma^\omega \cup bbbc\Sigma^\omega$
- $G_s^n(\Sigma^\omega) = \{\, ab^n t,\ b^{n+1} t,\ ab^{n-1} ct,\ b^n ct \mid t \in \Sigma^\omega \,\}$
- $\mathcal{M}_\infty \cap \Sigma^\omega = \cap_{n \geq 0}\, G_s^n(\Sigma^\omega) = \{ab^\omega,\ b^\omega\}$

# Infinite trace semantics: proof

$\mathcal{M}_\infty \cap \Sigma^\omega = \text{gfp } G_s$
where $G_s(T) \stackrel{\text{def}}{=} \tau^\frown T$ in $(\mathcal{P}(\Sigma^\omega), \subseteq)$

proof:

$G_s$ is continuous in $(\mathcal{P}(\Sigma^\omega), \supseteq)$: $G_s(\cap_{i \in I} T_i) = \cap_{i \in I} G_s(T_i)$.
By Kleene's theorem in the dual: $\text{gfp } G_s = \cap_{n \in \mathbb{N}} G_s^n(\Sigma^\omega)$.
We prove by recurrence on $n$ that $\forall n$: $G_s^n(\Sigma^\omega) = (\tau^{\frown n})^\frown \Sigma^\omega$:

- $G_s^0(\Sigma^\omega) = \Sigma^\omega = (\tau^{\frown 0})^\frown \Sigma^\omega$,
- $G_s^{n+1}(\Sigma^\omega) = \tau^\frown G_s^n(\Sigma^\omega) = \tau^\frown((\tau^{\frown n})^\frown \Sigma^\omega) = (\tau^{\frown n+1})^\frown \Sigma^\omega$.

$$\begin{array}{rcl}
\text{gfp } G_s & = & \cap_{n \in \mathbb{N}} (\tau^{\frown n})^\frown \Sigma^\omega \\
& = & \{\, \sigma_0, \ldots \in \Sigma^\omega \mid \forall n \geq 0 \colon \sigma_0, \ldots, \sigma_{n-1} \in \tau^{\frown n} \,\} \\
& = & \{\, \sigma_0, \ldots \in \Sigma^\omega \mid \forall n \geq 0 \colon \forall i < n \colon \sigma_i \to \sigma_{i+1} \,\} \\
& = & \mathcal{M}_\infty \cap \Sigma^\omega
\end{array}$$

# Least fixpoint formulation of maximal traces

**Idea:** To get a least fixpoint formulation for whole $\mathcal{M}_\infty$,
merge finite and infinite maximal trace least fixpoint forms.

## Fixpoint fusion

$\mathcal{M}_\infty \cap \Sigma^*$ is best defined on $(\mathcal{P}(\Sigma^*), \subseteq, \cup, \cap, \emptyset, \Sigma^*)$.
$\mathcal{M}_\infty \cap \Sigma^\omega$ is best defined on $(\mathcal{P}(\Sigma^\omega), \supseteq, \cap, \cup, \Sigma^\omega, \emptyset)$, the dual lattice
(we transform the greatest fixpoint into a least fixpoint!)

We mix them into a new complete lattice $(\mathcal{P}(\Sigma^\infty), \sqsubseteq, \sqcup, \sqcap, \bot, \top)$:

- $A \sqsubseteq B \overset{\text{def}}{\iff} (A \cap \Sigma^*) \subseteq (B \cap \Sigma^*) \wedge (A \cap \Sigma^\omega) \supseteq (B \cap \Sigma^\omega)$
- $A \sqcup B \overset{\text{def}}{=} ((A \cap \Sigma^*) \cup (B \cap \Sigma^*)) \cup ((A \cap \Sigma^\omega) \cap (B \cap \Sigma^\omega))$
- $A \sqcap B \overset{\text{def}}{=} ((A \cap \Sigma^*) \cap (B \cap \Sigma^*)) \cup ((A \cap \Sigma^\omega) \cup (B \cap \Sigma^\omega))$
- $\bot \overset{\text{def}}{=} \Sigma^\omega$
- $\top \overset{\text{def}}{=} \Sigma^*$

In this lattice, $\mathcal{M}_\infty = \text{lfp } F_s$ where $F_s(T) \overset{\text{def}}{=} \mathcal{B} \cup \tau \frown T$.

(proof on next slides)

# Fixpoint fusion theorem

**Theorem:** fixpoint fusion

If $X_1 = \text{lfp}\, F_1$ in $(\mathcal{P}(\mathcal{D}_1), \sqsubseteq_1)$ and $X_2 = \text{lfp}\, F_2$ in $(\mathcal{P}(\mathcal{D}_2), \sqsubseteq_2)$
and $\mathcal{D}_1 \cap \mathcal{D}_2 = \emptyset$,
then $X_1 \cup X_2 = \text{lfp}\, F$ in $(\mathcal{P}(\mathcal{D}_1 \cup \mathcal{D}_2), \sqsubseteq)$ where:

- $F(X) \stackrel{\text{def}}{=} F_1(X \cap \mathcal{D}_1) \cup F_2(X \cap \mathcal{D}_2)$,
- $A \sqsubseteq B \stackrel{\text{def}}{\iff} (A \cap \mathcal{D}_1) \sqsubseteq_1 (B \cap \mathcal{D}_1) \wedge (A \cap \mathcal{D}_2) \sqsubseteq_2 (B \cap \mathcal{D}_2)$.

proof:
We have:
$F(X_1 \cup X_2) = F_1((X_1 \cup X_2) \cap \mathcal{D}_1) \cup F_2((X_1 \cup X_2) \cap \mathcal{D}_2) = F_1(X_1) \cup F_2(X_2) = X_1 \cup X_2$,
hence $X_1 \cup X_2$ is a fixpoint of $F$.

Let $Y$ be a fixpoint. Then $Y = F(Y) = F_1(Y \cap \mathcal{D}_1) \cup F_2(Y \cap \mathcal{D}_2)$, hence, $Y \cap \mathcal{D}_1 = F_1(Y \cap \mathcal{D}_1)$ and $Y \cap \mathcal{D}_1$ is a fixpoint of $F_1$. Thus, $X_1 \sqsubseteq_1 Y \cap \mathcal{D}_1$. Likewise, $X_2 \sqsubseteq_2 Y \cap \mathcal{D}_2$. We deduce that $X = X_1 \cup X_2 \sqsubseteq (Y \cap \mathcal{D}_1) \cup (Y \cap \mathcal{D}_2) = Y$, and so, $X$ is $F$'s least fixpoint.

note: we also have $\text{gfp}\, F = \text{gfp}\, F_1 \cup \text{gfp}\, F_2$.

# Least fixpoint formulation of maximal traces (proof)

We are now ready to finish the proof that $\mathcal{M}_\infty = \text{lfp } F_s$ in $(\mathcal{P}(\Sigma^\infty), \sqsubseteq)$ with $F_s(T) \stackrel{\text{def}}{=} \mathcal{B} \cup \tau \frown T$

proof:

We have:

- $\mathcal{M}_\infty \cap \Sigma^* = \text{lfp } F_s$ in $(\mathcal{P}(\Sigma^*), \subseteq)$,
- $\mathcal{M}_\infty \cap \Sigma^\omega = \text{lfp } G_s$ in $(\mathcal{P}(\Sigma^\omega), \supseteq)$ where $G_s(T) \stackrel{\text{def}}{=} \tau \frown T$,
- in $\mathcal{P}(\Sigma^\infty)$, we have
  $F_s(A) = (F_s(A) \cap \Sigma^*) \cup (F_s(A) \cap \Sigma^\omega) = F_s(A \cap \Sigma^*) \cup G_s(A \cap \Sigma^\omega)$.

So, by fixpoint fusion in $(\mathcal{P}(\Sigma^\infty), \sqsubseteq)$, we have:

$\mathcal{M}_\infty = (\mathcal{M}_\infty \cap \Sigma^*) \cup (\mathcal{M}_\infty \cap \Sigma^\omega) = \text{lfp } F_s$.

Note: a greatest fixpoint formulation in $(\Sigma^\infty, \subseteq)$ also exists!

# Abstracting maximal traces into partial traces

# Finite and infinite partial trace semantics

Two steps to go from maximal to finite partial traces:

- add all partial traces
- remove infinite traces (in this order!)

## Partial trace semantics $\mathcal{T}_\infty$

all finite and infinite sequences of states
linked by the transition relation $\tau$:

$$\begin{array}{rl}\mathcal{T}_\infty \stackrel{\text{def}}{=} & \{\, \sigma_0, \ldots, \sigma_n \in \Sigma^* \,|\, \forall i < n : \sigma_i \to \sigma_{i+1} \,\} \cup \\ & \{\, \sigma_0, \ldots, \sigma_n, \ldots \in \Sigma^\omega \,|\, \forall i < \omega : \sigma_i \to \sigma_{i+1} \,\}\end{array}$$

(partial finite traces do not necessarily end in a blocking state)

Fixpoint form similar to $\mathcal{M}_\infty$:
$\mathcal{T}_\infty = \operatorname{lfp} F_{s*}$ in $(\mathcal{P}(\Sigma^\infty), \sqsubseteq)$ where $F_{s*}(T) \stackrel{\text{def}}{=} \Sigma \cup \tau \frown T$,

proof: similar to the proof of $\mathcal{M}_\infty = \operatorname{lfp} F_s$.

# Finite trace abstraction

Finite partial traces $\mathcal{T}$ are an abstraction of all partial traces $\mathcal{T}_\infty$
(forget about infinite executions)

We have a Galois embedding:

$$(\mathcal{P}(\Sigma^\infty), \sqsubseteq) \xleftarrow[\alpha_*]{\gamma_*} (\mathcal{P}(\Sigma^*), \subseteq)$$

- $\sqsubseteq$ is the fused ordering on $\Sigma^* \cup \Sigma^\omega$:
  $A \sqsubseteq B \iff^{\text{def}} (A \cap \Sigma^*) \subseteq (B \cap \Sigma^*) \wedge (A \cap \Sigma^\omega) \supseteq (B \cap \Sigma^\omega)$
- $\alpha_*(T) \stackrel{\text{def}}{=} T \cap \Sigma^*$
  (remove infinite traces)
- $\gamma_*(T) \stackrel{\text{def}}{=} T$
  (embedding)
- $\mathcal{T} = \alpha_*(\mathcal{T}_\infty)$

(proof on next slide)

# Finite trace abstraction (proof)

proof:

We have Galois embedding because:

- $\alpha_*$ and $\gamma_*$ are monotonic,
- given $T \subseteq \Sigma^*$, we have $(\alpha_* \circ \gamma_*)(T) = T \cap \Sigma^* = T$,
- $(\gamma_* \circ \alpha_*)(T) = T \cap \Sigma^* \sqsupseteq T$, as we only remove infinite traces.

Recall that $\mathcal{T}_\infty = \mathrm{lfp}\, F_{s*}$ in $(\mathcal{P}(\Sigma^\infty), \sqsubseteq)$ and $\mathcal{T} = \mathrm{lfp}\, F_{s*}$ in $(\mathcal{P}(\Sigma^*), \subseteq)$, where $F_{s*}(T) \stackrel{\text{def}}{=} \Sigma \cup T \frown \tau$.

As $\alpha_* \circ F_{s*} = F_{s*} \circ \alpha_*$ and $\alpha_*(\emptyset) = \emptyset$, we can apply the fixpoint transfer theorem to get $\alpha_*(\mathcal{T}_\infty) = \mathcal{T}$.

# Prefix abstraction

**Idea:** complete maximal traces by adding (non-empty) prefixes.

We have a Galois connection:

$$(\mathcal{P}(\Sigma^\infty \setminus \{\epsilon\}), \subseteq) \xleftarrow[\alpha_\preceq]{\gamma_\preceq} (\mathcal{P}(\Sigma^\infty \setminus \{\epsilon\}), \subseteq)$$

- $\alpha_\preceq(T) \stackrel{\text{def}}{=} \{\, t \in \Sigma^\infty \setminus \{\epsilon\} \,|\, \exists u \in T: t \preceq u \,\}$
  (set of all non-empty prefixes of traces in $T$)
- $\gamma_\preceq(T) \stackrel{\text{def}}{=} \{\, t \in \Sigma^\infty \setminus \{\epsilon\} \,|\, \forall u \in \Sigma^\infty \setminus \{\epsilon\}: u \preceq t \implies u \in T \,\}$
  (traces with non-empty prefixes in $T$)

proof:

$\alpha_\preceq$ and $\gamma_\preceq$ are monotonic.

$(\alpha_\preceq \circ \gamma_\preceq)(T) = \{\, t \in T \,|\, \rho_p(t) \subseteq T \,\} \subseteq T$ (prefix-closed trace sets).

$(\gamma_\preceq \circ \alpha_\preceq)(T) = \rho_p(T) \supseteq T$.

# Abstraction from maximal traces to partial traces

Finite and infinite partial traces $\mathcal{T}_\infty$ are an abstraction of maximal traces $\mathcal{M}_\infty$: $\mathcal{T}_\infty = \alpha_\preceq(\mathcal{M}_\infty)$.

proof:

Firstly, $\mathcal{T}_\infty$ and $\alpha_\preceq(\mathcal{M}_\infty)$ coincide on infinite traces. Indeed, $\mathcal{T}_\infty \cap \Sigma^\omega = \mathcal{M}_\infty \cap \Sigma^\omega$ and $\alpha_\preceq$ does not add infinite traces, so: $\mathcal{T}_\infty \cap \Sigma^\omega = \alpha_\preceq(\mathcal{M}_\infty) \cap \Sigma^\omega$.

We now prove that they also coincide on finite traces. Assume $\sigma_0, \ldots, \sigma_n \in \alpha_\preceq(\mathcal{M}_\infty)$, then $\forall i < n$: $\sigma_i \to \sigma_{i+1}$, so, $\sigma_0, \ldots, \sigma_n \in \mathcal{T}_\infty$.
Assume $\sigma_0, \ldots, \sigma_n \in \mathcal{T}_\infty$, then it can be completed into a maximal trace, either finite or infinite, and so, $\sigma_0, \ldots, \sigma_n \in \alpha_\preceq(\mathcal{M}_\infty)$.

Note: no fixpoint transfer applies here.

# Enriched hierarchy of semantics

$$
\begin{tikzcd}
\mathcal{R}(\mathcal{I}) & & \mathcal{C}(\mathcal{F}) & \text{forward/backward states} \\
\mathcal{T}_p(\mathcal{I}) \arrow[u, "\alpha_p"] & & \mathcal{T}_s(\mathcal{F}) \arrow[u, "\alpha_p"'] & \text{prefix/suffix finite traces} \\
& \mathcal{T} \arrow[ul, "\alpha_{\mathcal{I}}"] \arrow[ur, "\alpha_{\mathcal{F}}"'] & & \text{partial finite traces} \\
& \mathcal{T}_\infty \arrow[u, "\alpha_*"] & & \text{partial traces} \\
& \mathcal{M}_\infty \arrow[u, "\alpha_\preceq"] & & \text{maximal traces}
\end{tikzcd}
$$

See [Cous02] for more semantics in this diagram.

# Trace properties

# Trace properties

Trace property: $P \in \mathcal{P}(\Sigma^\infty)$

Verification problem: $\mathcal{M}_\infty \cap (\mathcal{I} \cdot \Sigma^\infty) \subseteq P$

or, equivalently, as $\mathcal{M}_\infty \subseteq P'$ where $P' \stackrel{\text{def}}{=} P \cup ((\Sigma \setminus \mathcal{I}) \cdot \Sigma^\infty)$

Examples:

- termination: $P \stackrel{\text{def}}{=} \Sigma^*$,
- non-termination: $P \stackrel{\text{def}}{=} \Sigma^\omega$,
- any state property $S \subseteq \Sigma$: $P \stackrel{\text{def}}{=} S^\infty$,
- maximal execution time: $P \stackrel{\text{def}}{=} \Sigma^{\leq k}$,
- minimal execution time: $P \stackrel{\text{def}}{=} \Sigma^{\geq k}$,
- ordering, e.g.: $P \stackrel{\text{def}}{=} (\Sigma \setminus \{b\})^* \cdot a \cdot \Sigma^* \cdot b \cdot \Sigma^\infty$.

  ($a$ and $b$ occur, and $a$ occurs before $b$)

# Safety properties for traces

**Idea:** a safety property $P$ models that "nothing bad ever occurs"

- $P$ is provable by exhaustive testing;
  (observe the prefix trace semantics: $\mathcal{T}_p(\mathcal{I}) \subseteq P$)
- $P$ is disprovable by finding a single finite execution not in $P$.

Examples:

- any state property: $P \stackrel{\text{def}}{=} S^\infty$ for $S \subseteq \Sigma$,
- ordering: $P \stackrel{\text{def}}{=} \Sigma^\infty \setminus ((\Sigma \setminus \{a\})^* \cdot b \cdot \Sigma^\infty)$,
  no $b$ can appear without an $a$ before,
  but we can have only $a$, or neither $a$ nor $b$
  (not a state property)
- but termination $P \stackrel{\text{def}}{=} \Sigma^*$ is not a safety property.
  disproving requires exhibiting an *infinite* execution

# Definition of safety properties

**Reminder:** finite prefix abstraction (simplified to allow $\epsilon$)

$$(\mathcal{P}(\Sigma^\infty), \subseteq) \underset{\alpha_{*\preceq}}{\overset{\gamma_{*\preceq}}{\leftrightarrows}} (\mathcal{P}(\Sigma^*), \subseteq)$$

- $\alpha_{*\preceq}(T) \stackrel{\text{def}}{=} \{\, t \in \Sigma^* \mid \exists u \in T : t \preceq u \,\}$
- $\gamma_{*\preceq}(T) \stackrel{\text{def}}{=} \{\, t \in \Sigma^\infty \mid \forall u \in \Sigma^* : u \preceq t \implies u \in T \,\}$

The associated upper closure $\rho_{*\preceq} \stackrel{\text{def}}{=} \gamma_{\preceq} \circ \alpha_{\preceq}$ is:

$\rho_{*\preceq} = \lim \circ \rho_p$ where:

- $\rho_p(T) \stackrel{\text{def}}{=} \{\, u \in \Sigma^\infty \mid \exists t \in T : u \preceq t \,\}$,
- $\lim(T) \stackrel{\text{def}}{=} T \cup \{\, t \in \Sigma^\omega \mid \forall u \in \Sigma^* : u \preceq t \implies u \in T \,\}$.

**Definition:** $P \in \mathcal{P}(\Sigma^\infty)$ is a safety property if $P = \rho_{*\preceq}(P)$.

# Definition of safety properties (examples)

**Definition:** $P \subseteq \mathcal{P}(\Sigma^\infty)$ is a safety property if $P = \rho_{*\preceq}(P)$.

Examples and counter-examples:

- state property $P \stackrel{\text{def}}{=} S^\infty$ for $S \subseteq \Sigma$:

  $\rho_p(S^\infty) = \lim(S^\infty) = S^\infty \implies$ safety;

- termination $P \stackrel{\text{def}}{=} \Sigma^*$:

  $\rho_p(\Sigma^*) = \Sigma^*$, but $\lim(\Sigma^*) = \Sigma^\infty \neq \Sigma^* \implies$ not safety;

- even number of steps $P \stackrel{\text{def}}{=} (\Sigma^2)^\infty$:

  $\rho_p((\Sigma^2)^\infty) = \Sigma^\infty \neq (\Sigma^2)^\infty \implies$ not safety.

# Proving safety properties

**Invariance proof method:** find an inductive invariant $I$

- set of finite traces $I \subseteq \Sigma^*$
- $\mathcal{I} \subseteq I$

  (contains traces reduced to an initial state)
- $\forall \sigma_0, \ldots, \sigma_n \in I: \sigma_n \to \sigma_{n+1} \implies \sigma_0, \ldots, \sigma_n, \sigma_{n+1} \in I$

  (invariant by program transition)

and implies the desired property: $I \subseteq P$.

Link with the finite prefix trace semantics $\mathcal{T}_p(\mathcal{I})$:

An inductive invariant is a post-fixpoint of $F_p$: $F_p(I) \subseteq I$

where $F_p(T) \stackrel{\text{def}}{=} \mathcal{I} \cup T \frown \tau$.

$\mathcal{T}_p(\mathcal{I}) = \text{lfp}\, F_p$ is the tightest inductive invariant.

# Correctness of the invariant method for safety

**Soundness:**

if $P$ is a safety property and an inductive invariant $I$ exists
then: $\mathcal{M}_\infty \cap (\mathcal{I} \cdot \Sigma^\infty) \subseteq P$

proof:

Using the Galois connection between $\mathcal{M}_\infty$ and $\mathcal{T}$, we get:
$\mathcal{M}_\infty \cap (\mathcal{I} \cdot \Sigma^\infty) \subseteq \rho_{*\preceq}(\mathcal{M}_\infty \cap (\mathcal{I} \cdot \Sigma^\infty)) = \gamma_{*\preceq}(\alpha_{*\preceq}(\mathcal{M}_\infty \cap (\mathcal{I} \cdot \Sigma^\infty))) =$
$\gamma_{*\preceq}(\alpha_{*\preceq}(\mathcal{M}_\infty) \cap (\mathcal{I} \cdot \Sigma^*)) = \gamma_{*\preceq}(\mathcal{T} \cap (\mathcal{I} \cdot \Sigma^*)) = \gamma_{*\preceq}(\mathcal{T}_p(\mathcal{I}))$.
Using the link between invariants and the finite prefix trace semantics, we have:
$\mathcal{T}_p(\mathcal{I}) \subseteq I \subseteq P$.

As $P$ is a safety property, $P = \gamma_{*\preceq}(P)$, so, $\gamma_{*\preceq}(\mathcal{T}_p(\mathcal{I})) \subseteq \gamma_{*\preceq}(P) = P$, and so,
$\mathcal{M}_\infty \cap (\mathcal{I} \cdot \Sigma^\infty) \subseteq P$.

**Completeness:** an inductive invariant always exists

proof: $\mathcal{T}_p(\mathcal{I})$ provides an inductive invariant.

# Disproving safety properties

**Proof method:**

A safety property $P$ can be disproved by constructing a finite prefix of execution that does not satisfy the property:

$$\mathcal{M}_\infty \cap (\mathcal{I} \cdot \Sigma^\infty) \not\subseteq P \implies \exists t \in \mathcal{T}_p(\mathcal{I}) \colon t \notin P$$

proof:

By contradiction, assume that no such trace exists, i.e., $\mathcal{T}_p(\mathcal{I}) \subseteq P$.

We proved in the previous slide that this implies $\mathcal{M}_\infty \cap (\mathcal{I} \cdot \Sigma^\infty) \subseteq P$.

Examples:

- disproving a state property $P \stackrel{\text{def}}{=} S^\infty$:
  $\Rightarrow$ find a partial execution containing a state in $\Sigma \setminus S$;
- disproving an order property $P \stackrel{\text{def}}{=} \Sigma^\infty \setminus ((\Sigma \setminus \{a\})^* \cdot b \cdot \Sigma^\infty)$
  $\Rightarrow$ find a partial execution where $b$ appears and not $a$.

# Liveness properties

**Idea:** liveness property $P \in \mathcal{P}(\Sigma^\infty)$

Liveness properties model that "something good eventually occurs"

- $P$ cannot be proved by testing
  (if nothing good happens in a prefix execution,
  it can still happen in the rest of the execution)
- disproving $P$ requires exhibiting an infinite execution not in $P$

Examples:

- termination: $P \stackrel{\text{def}}{=} \Sigma^*$,
- inevitability: $P \stackrel{\text{def}}{=} \Sigma^* \cdot a \cdot \Sigma^\infty$,

  ($a$ eventually occurs in all executions)
- state properties are not liveness properties.

# Definition of liveness properties

**Definition:** $P \in \mathcal{P}(\Sigma^\infty)$ is a liveness property if $\rho_{*\preceq}(P) = \Sigma^\infty$.

Examples and counter-examples:

- termination $P \stackrel{\text{def}}{=} \Sigma^*$:

  $\rho_p(\Sigma^*) = \Sigma^*$ and $\lim(\Sigma^*) = \Sigma^\infty \implies$ liveness;

- inevitability: $P \stackrel{\text{def}}{=} \Sigma^* \cdot a \cdot \Sigma^\infty$

  $\rho_p(P) = P \cup \Sigma^*$ and $\lim(P \cup \Sigma^*) = \Sigma^\infty \implies$ liveness;

- state property $P \stackrel{\text{def}}{=} S^\infty$ for $S \subseteq \Sigma$:

  $\rho_p(S^\infty) = \lim(S^\infty) = S^\infty \neq \Sigma^\infty$ if $S \neq \Sigma \implies$ not liveness;

- maximal execution time $P \stackrel{\text{def}}{=} \Sigma^{\leq k}$:

  $\rho_p(\Sigma^{\leq k}) = \lim(\Sigma^{\leq k}) = \Sigma^{\leq k} \neq \Sigma^\infty \implies$ not liveness;

- the only property which is both safety and liveness is $\Sigma^\infty$.

# Proving liveness properties

**Variance proof method:** (informal definition)

Find a decreasing quantity until something good happens.

Example: termination proof

- find $f : \Sigma \to \mathcal{S}$ where $(\mathcal{S}, \sqsubseteq)$ is well-ordered;

  ($f$ is called a "ranking function")

- $\sigma \in \mathcal{B} \implies f = \min \mathcal{S}$;
- $\sigma \to \sigma' \implies f(\sigma') \sqsubset f(\sigma)$.

($f$ counts the number of steps remaining before termination)

# Disproving liveness properties

**Property:**

If $P$ is a liveness property, then $\forall t \in \Sigma^*: \exists u \in P: t \preceq u$.

proof:

By definition of liveness, $\rho_{*\preceq}(P) = \Sigma^\infty$, so $t \in \rho_{*\preceq}(P) = \lim(\alpha_p(P))$.
As $t \in \Sigma^*$ and lim only adds infinite traces, $t \in \alpha_p(P)$.

By definition of $\alpha_p$, $\exists u \in P: t \preceq u$.

Consequence:

- liveness cannot be disproved by testing.

# Trace topology

A topology on a set can be defined as:
– either a family of open sets (closed under union)
– or family of closed sets (closed under intersection)

**Trace topology:** on sets of traces in $\Sigma^\infty$

- the closed sets are: $\mathcal{C} \stackrel{\text{def}}{=} \{\, P \in \mathcal{P}(\Sigma^\infty) \mid P \text{ is a safety property} \,\}$
- the open sets can be derived as $\mathcal{O} \stackrel{\text{def}}{=} \{\, \Sigma^\infty \setminus c \mid c \in \mathcal{C} \,\}$

Topological closure: $\rho : \mathcal{P}(X) \to \mathcal{P}(X)$

- $\rho(x) \stackrel{\text{def}}{=} \cap \{\, c \in \mathcal{C} \mid x \subseteq c \,\}$ (upper closure operator in $(\mathcal{P}(X), \subseteq)$)
- on our trace topology, $\rho = \rho_{*\preceq}$.

Dense sets:

- $x \subseteq X$ is dense if $\rho(x) = X$;
- on our trace topology, dense sets are liveness properties.

# Decomposition theorem

**Theorem:** decomposition on a topological space

Any set $x \subseteq X$ is the intersection of a closed set and a dense set.

proof:

We have $x = \rho(x) \cap (x \cup (X \setminus \rho(x)))$. Indeed:
$\rho(x) \cap (x \cup (X \setminus \rho(x))) = (\rho(x) \cap x) \cup (\rho(x) \cap (X \setminus \rho(x))) = \rho(x) \cap x = x$ as $x \subseteq \rho(x)$.

- $\rho(x)$ is closed
- $x \cup (X \setminus \rho(x))$ is dense because: $\begin{aligned}\rho(x \cup (X \setminus \rho(x))) &\supseteq \rho(x) \cup \rho(X \setminus \rho(x)) \\ &\supseteq \rho(x) \cup (X \setminus \rho(x)) \\ &= X\end{aligned}$

**Consequence:** on trace properties

Every trace property is the conjunction of
a safety property and a liveness property.

proving a trace property can be decomposed into
a soundness proof and a liveness proof

# Beyond trace properties

# Properties

We generalize the notion of properties and program verification.

**General setting:**

- programs: $prog \in Prog$
- semantics: $[\![\, \cdot \,]\!] : Prog \to \mathcal{D}$ in some semantic domain $\mathcal{D}$
- property: the set of allowed program semantics $P \in \mathcal{P}(\mathcal{D})$

  $\subseteq$ gives an information order on properties

  $P \subseteq P'$ means that $P'$ is weaker than $P$ (allows more semantics)
- verification problem: $[\![\, prog \,]\!] \in P$

# Collecting semantics

**Collecting semantics:** $\mathit{Col} : \mathit{Prog} \to \mathcal{P}(\mathcal{D})$

- $\mathit{Col}(\mathit{prog}) \stackrel{\text{def}}{=} \{\llbracket\, \mathit{prog}\, \rrbracket\}$
- $\mathit{Col}(\mathit{prog})$ is the strongest property of a program in $\mathcal{P}(\mathcal{D})$
  (relative to the choice of the semantic domain $\mathcal{D}$ and function $\llbracket \cdot \rrbracket$)
- we can interpret program verification as property inclusion:
  $\mathit{Col}(\mathit{prog}) \subseteq P$
  $P$ is weaker than $\mathit{Col}(\mathit{prog})$ in the information order of properties
- generally, the collecting semantics cannot be computed;
  we settle for a weaker property $S^\sharp$ that
  - is sound: $\mathit{Col}(\mathit{prog}) \subseteq S^\sharp$
  - implies the desired property: $S^\sharp \subseteq P$

# Retrieving state and trace properties

Reachability state semantics:

- $\mathcal{D} \stackrel{\text{def}}{=} \mathcal{P}(\Sigma)$
- $[\![ \cdot ]\!] \stackrel{\text{def}}{=} \mathcal{R}(\mathcal{I})$

Trace semantics:

- $\mathcal{D} \stackrel{\text{def}}{=} \mathcal{P}(\Sigma^\infty)$
- $[\![ \cdot ]\!] \stackrel{\text{def}}{=} \mathcal{M}_\infty \cap (\mathcal{I} \cdot \Sigma^\infty)$

State and trace properties: interpreted in $\mathcal{P}(\mathcal{D})$

$\rho_\downarrow(x)$ for some $x \in \mathcal{D}$
where $\rho_\downarrow(x) \stackrel{\text{def}}{=} \{\, y \in \mathcal{D} \,|\, y \subseteq x \,\} \in \mathcal{P}(\mathcal{D})$

(proof: $A \subseteq B \iff A \in \rho_\downarrow(B)$)

# Non-trace properties

Note: expressing properties in $\mathcal{P}(\mathcal{D})$
is more general than expressing properties in $\mathcal{D}$

Example: non-interference for variable $X$

$$P \stackrel{\text{def}}{=} \{\, T \in \mathcal{P}(\Sigma^*) \mid \forall \sigma_0, \ldots, \sigma_n \in T: \forall \sigma_0': \sigma_0 \equiv \sigma_0' \implies \exists \sigma_0', \ldots, \sigma_m' \in T: \sigma_m' \equiv \sigma_m \,\}$$

where $(\ell, \rho) \equiv (\ell', \rho') \iff \ell = \ell' \wedge \forall V \neq X: \rho(V) = \rho'(V)$

(changing the initial value of $X$ does not affect the set of final environments up to the value of $X$)

There is no $Q \subseteq \Sigma^\infty$ such that $P = \rho_\downarrow(Q)$.
$\implies$ non-interference is not a trace property in $\mathcal{P}(\Sigma^\infty)$.

Reading assignment: hyperproperties.

# Bibliography

# Bibliography

[Bour93] **F. Bourdoncle**. *Abstract debugging of higher-order imperative languages.* In PLDI, 46-55, ACM Press, 1993.

[Cous92] **P. Cousot & R. Cousot**. *Abstract interpretation and application to logic programs.* In Journal of Logic Programming, 13(2–3):103–179, 1992..

[Cous02] **P. Cousot**. *Constructive design of a hierarchy of semantics of a transition system by abstract interpretation.* In Theoretical Comp. Sc., 277(1–2):47–103.