Relational Numerical Abstract Domains

MPRI 2–6: Abstract Interpretation,
application to verification and static analysis

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Outline

- The need for relational domains
- Presentation of a few relational numerical abstract domains
  - linear equality domain
  - polyhedra domain
  - weakly relational domains: zones, octagons
- Bibliography

Shortcomings of non-relational domains

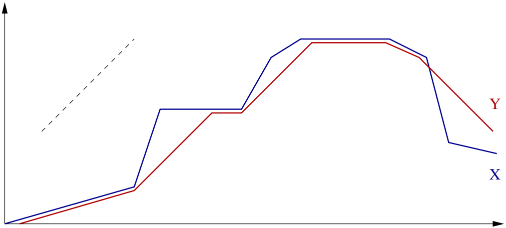
Accumulated loss of precision

Non-relation domains cannot represent variable relationships

Rate limiter

```
Y ← 0; while ● 1=1 do
  X ← [-128,128]; D ← [0,16];
  S ← Y; Y ← X; R ← X - S;
  if R ≤ -D then Y ← S - D fi;
  if R ≥ D then Y ← S + D fi
done
```

X: input signal
Y: output signal
S: last output
R: delta $Y - S$
D: max. allowed for $|R|$



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Iterations in the interval domain (without widening):

$\mathcal{X}^{\sharp 0}_{\bullet}$	$\mathcal{X}^{\sharp 1}_{\bullet}$	$\mathcal{X}^{\sharp 2}_{\bullet}$	...	$\mathcal{X}^{\sharp n}_{\bullet}$
$Y = 0$	$\|Y\| \leq 144$	$\|Y\| \leq 160$	...	$\|Y\| \leq 128 + 16n$

In fact, $Y \in [-128, 128]$ always holds.

To prove that, e.g. $Y \geq -128$, we must be able to:

- represent the properties $R = X - S$ and $R \leq -D$
- combine them to deduce $S - X \geq D$, and then $Y = S - D \geq X$

The need for relational loop invariants

To prove some invariant after the end of a loop,
we often need to find a loop invariant of a more complex form

relational loop invariant

```
X ← 0; I ← 1;
while ● I < 5000 do
  if [0,1] = 1 then X ← X + 1 else X ← X - 1 fi;
  I ← I + 1
done ♦
```

A non-relational analysis finds at ♦ that $I = 5000$ and $X \in \mathbb{Z}$

The best invariant is: $(I = 5000) \wedge (X \in [-4999, 4999]) \wedge (X \equiv 0\ [2])$

To find this non-relational invariant, we must find a relational loop invariant at ●: $(-I < X < I) \wedge (X + I \equiv 1\ [2]) \wedge (I \in [1, 5000])$,
and apply the loop exit condition $C^\sharp[\![\, I \geq 5000 \,]\!]$

Modular analysis

store the maximum of X,Y,0 into Z

```
max(X,Y,Z)

  Z ← X ;
  if Y  > Z  then Z ← Y ;
  if Z  < 0 then Z ← 0;
```

Modular analysis:

- analyze a procedure once (procedure summary)
- reuse the summary at each call site (instantiation)

  $\Longrightarrow$ improved efficiency

Modular analysis

store the maximum of X,Y,0 into Z'

```
max(X,Y,Z)
  X' ← X; Y' ← Y; Z' ← Z;
  Z' ← X';
  if Y' > Z' then Z' ← Y';
  if Z' < 0 then Z' ← 0;
```

$(Z' \geq X \wedge Z' \geq Y \wedge Z' \geq 0 \wedge X' = X \wedge Y' = Y)$

Modular analysis:

- analyze a procedure once (procedure summary)
- reuse the summary at each call site (instantiation)
  $\Longrightarrow$ improved efficiency
- infer a relation between input X, Y, Z and output X', Y', Z' values, in $\mathcal{P}((\mathbb{V} \to \mathbb{R}) \times (\mathbb{V} \to \mathbb{R})) \equiv \mathcal{P}((\mathbb{V} \times \mathbb{V}) \to \mathbb{R})$
- requires inferring relational information

[Anco10], [Jean09]

Linear equality domain

The affine equality domain

Here $\mathbb{I} \in \{\mathbb{Q}, \mathbb{R}\}$.

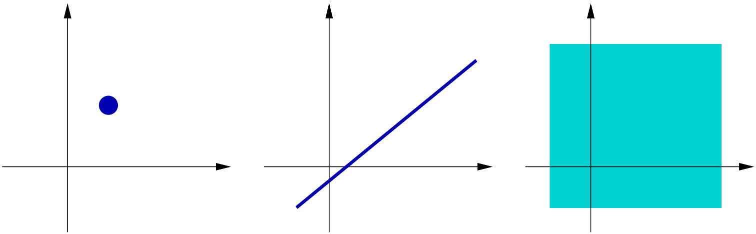
We look for invariants of the form:

$$\bigwedge_j \left(\sum_{i=1}^{n} \alpha_{ij} V_i = \beta_j\right), \ \alpha_{ij}, \beta_j \in \mathbb{I}$$

where all the α_{ij} and β_j are inferred automatically.

We use a domain of affine spaces proposed by [Karr76]:

$$\mathcal{D}^\sharp \stackrel{\text{def}}{=} \{\text{ affine subspaces of } \mathbb{V} \to \mathbb{I}\ \}$$



Affine equality representation

Machine representation: an affine subspace is represented as

- either the constant $\bot^\sharp$,
- or a pair $\langle \mathbf{M}, \vec{C} \rangle$ where
  - $\mathbf{M} \in \mathbb{I}^{m \times n}$ is a $m \times n$ matrix, $n = |\mathbb{V}|$ and $m \leq n$,
  - $\vec{C} \in \mathbb{I}^m$ is a row-vector with m rows.

$\langle \mathbf{M}, \vec{C} \rangle$ represents an equation system, with solutions:

$$\gamma(\langle \mathbf{M}, \vec{C} \rangle) \stackrel{\text{def}}{=} \{\, \vec{V} \in \mathbb{I}^n \mid \mathbf{M} \times \vec{V} = \vec{C} \,\}$$

M should be in row echelon form:

- $\forall i \leq m: \exists k_i: M_{ik_i} = 1$ and $\forall c < k_i: M_{ic} = 0$, $\forall l \neq i: M_{lk_i} = 0$,
- if $i < i'$ then $k_i < k_{i'}$ *(leading index)*

example:

$$\begin{bmatrix} 1 & 0 & 0 & 5 & 0 \\ 0 & 1 & 0 & 6 & 0 \\ 0 & 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Remarks:

- the representation is unique
- as $m \leq n = |\mathbb{V}|$, the memory cost is in $\mathcal{O}(n^2)$ at worst
- $\top$ is represented as the empty equation system: $m = 0$

Galois connection

Galois connection: (actually, a Galois insertion)

between arbitrary subsets and affine subsets

$$(\mathcal{P}(\mathbb{I}^n), \subseteq) \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} (Aff(\mathbb{I}^n), \subseteq)$$

- $\gamma(X) \stackrel{\text{def}}{=} X$ (identity)
- $\alpha(X) \stackrel{\text{def}}{=}$ smallest affine subset containing X

  $Aff(\mathbb{I}^n)$ is closed under arbitrary intersections, so we have:

  $\alpha(X) = \cap \{ Y \in Aff(\mathbb{I}^n) \mid X \subseteq Y \}$

  $Aff(\mathbb{I}^n)$ contains every point in $\mathbb{I}^n$

  we can also construct $\alpha(X)$ by abstract union:
  $\alpha(X) = \cup^\sharp \{ \{x\} \mid x \in X \}$

Notes:

- we have assimilated $\mathbb{V} \to \mathbb{I}$ to $\mathbb{I}^n$
- we have used $Aff(\mathbb{I}^n)$ instead of the matrix representation $\mathcal{D}^\sharp$ for simplicity; a Galois connection also exists between $\mathcal{P}(\mathbb{I}^n)$ and $\mathcal{D}^\sharp$

Normalisation and emptiness testing

Let $\mathbf{M} \times \vec{V} = \vec{C}$ be a system, not necessarily in normal form.

The Gaussian reduction $Gauss(\langle \mathbf{M}, \vec{C} \rangle)$ tells in $\mathcal{O}(n^3)$ time:

- whether the system is satisfiable, and in that case
- gives an equivalent system $\langle \mathbf{M}', \vec{C}' \rangle$ in normal form

i.e. returns an element in $\mathcal{D}^\sharp$.

Principle: reorder lines, and combine lines linearly to eliminate variables

Example:

$$\left\{ \begin{array}{rcrcrcr} 2X & + & Y & + & Z & = & 19 \\ 2X & + & Y & - & Z & = & 9 \\ & & & & 3Z & = & 15 \end{array} \right.$$

$$\Downarrow$$

$$\left\{ \begin{array}{rcrcrcr} X & + & 0.5Y & & & = & 7 \\ & & & & Z & = & 5 \end{array} \right.$$

Affine equality operators

Applications

If $\mathcal{X}^\sharp, \mathcal{Y}^\sharp \neq \bot^\sharp$, we define:

$$\mathcal{X}^\sharp \cap^\sharp \mathcal{Y}^\sharp \stackrel{\text{def}}{=} \textit{Gauss}\left(\left\langle \begin{bmatrix} \mathbf{M}_{\mathcal{X}^\sharp} \\ \mathbf{M}_{\mathcal{Y}^\sharp} \end{bmatrix}, \begin{bmatrix} \vec{C}_{\mathcal{X}^\sharp} \\ \vec{C}_{\mathcal{Y}^\sharp} \end{bmatrix} \right\rangle\right)$$

$$\mathcal{X}^\sharp =^\sharp \mathcal{Y}^\sharp \stackrel{\text{def}}{\iff} \mathbf{M}_{\mathcal{X}^\sharp} = \mathbf{M}_{\mathcal{Y}^\sharp} \quad \text{and} \quad \vec{C}_{\mathcal{X}^\sharp} = \vec{C}_{\mathcal{Y}^\sharp}$$

$$\mathcal{X}^\sharp \subseteq^\sharp \mathcal{Y}^\sharp \stackrel{\text{def}}{\iff} \mathcal{X}^\sharp \cap^\sharp \mathcal{Y}^\sharp =^\sharp \mathcal{X}^\sharp$$

$$\mathsf{C}^\sharp[\![\textstyle\sum_j \alpha_j V_j - \beta = 0]\!]\, \mathcal{X}^\sharp \stackrel{\text{def}}{=} \textit{Gauss}\left(\left\langle \begin{bmatrix} \mathbf{M}_{\mathcal{X}^\sharp} \\ \alpha_1 \cdots \alpha_n \end{bmatrix}, \begin{bmatrix} \vec{C}_{\mathcal{X}^\sharp} \\ \beta \end{bmatrix} \right\rangle\right)$$

$$\mathsf{C}^\sharp[\![e \bowtie 0]\!]\, \mathcal{X}^\sharp \stackrel{\text{def}}{=} \mathcal{X}^\sharp \qquad \text{for other tests}$$

Remark:

$\subseteq^\sharp$, $=^\sharp$, $\cap^\sharp$, $=^\sharp$ and $\mathsf{C}^\sharp[\![\sum_j \alpha_j V_j - \beta = 0]\!]$ are exact:

$$\mathcal{X}^\sharp \subseteq^\sharp \mathcal{Y}^\sharp \iff \gamma(\mathcal{X}^\sharp) \subseteq \gamma(\mathcal{Y}^\sharp), \quad \gamma(\mathcal{X}^\sharp \cap^\sharp \mathcal{Y}^\sharp) = \gamma(\mathcal{X}^\sharp) \cap \gamma(\mathcal{Y}^\sharp), \ldots$$

Generator representation

Generator representation

An affine subspace can also be represented as a set of vector generators $\vec{G}_1, \ldots, \vec{G}_m$ and an origin point $\vec{O}$, denoted as $[\mathbf{G}, \vec{O}]$.

$$\gamma([\mathbf{G}, \vec{O}]) \stackrel{\text{def}}{=} \{\, \mathbf{G} \times \vec{\lambda} + \vec{O} \mid \vec{\lambda} \in \mathbb{I}^m \,\} \qquad (\mathbf{G} \in \mathbb{I}^{n \times m},\ \vec{O} \in \mathbb{I}^n)$$

We can switch between a generator and a constraint representation:

- From generators to constraints: $\langle \mathbf{M}, \vec{C} \rangle = \mathit{Cons}([\mathbf{G}, \vec{O}])$

  Write the system $\vec{V} = \mathbf{G} \times \vec{\lambda} + \vec{O}$ with variables $\vec{V}, \vec{\lambda}$.
  Solve it in $\vec{\lambda}$ (by row operations).
  Keep the constraints involving only $\vec{V}$.

  e.g. $\left\{ \begin{array}{lcl} X & = & \lambda + 2 \\ Y & = & 2\lambda + \mu + 3 \\ Z & = & \mu \end{array} \right. \Longrightarrow \left\{ \begin{array}{rcl} X - 2 & = & \lambda \\ -2X + Y + 1 & = & \mu \\ 2X - Y + Z - 1 & = & 0 \end{array} \right.$

  The result is: $2X - Y + Z = 1$.

Generator representation (cont.)

- From constraints to generators: $[\mathbf{G}, \vec{O}] \stackrel{\text{def}}{=} \mathit{Gen}(\langle \mathbf{M}, \vec{C} \rangle)$

  Assume $\langle \mathbf{M}, \vec{C} \rangle$ is normalized.
  For each non-leading variable V, assign a distinct λ_V,
  solve leading variables in terms of non-leading ones.

  e.g. $\left\{ \begin{array}{rcl} X + 0.5Y & = & 7 \\ Z & = & 5 \end{array} \right. \implies \begin{bmatrix} -0.5 \\ 1 \\ 0 \end{bmatrix} \lambda_Y + \begin{bmatrix} 7 \\ 0 \\ 5 \end{bmatrix}$

Affine equality operators (cont.)

Applications

Given $\mathcal{X}^\sharp, \mathcal{Y}^\sharp \neq \bot^\sharp$, we define:

$$\mathcal{X}^\sharp \cup^\sharp \mathcal{Y}^\sharp \stackrel{\text{def}}{=} \mathit{Cons}\left(\left[\mathbf{G}_{\mathcal{X}^\sharp}\ \mathbf{G}_{\mathcal{Y}^\sharp}\ (\vec{O}_{\mathcal{Y}^\sharp} - \vec{O}_{\mathcal{X}^\sharp}),\ \vec{O}_{\mathcal{X}^\sharp}\right]\right)$$

$$\mathsf{C}^\sharp[\![\, V_j \leftarrow [-\infty, +\infty] \,]\!]\, \mathcal{X}^\sharp \stackrel{\text{def}}{=} \mathit{Cons}\left(\left[\mathbf{G}_{\mathcal{X}^\sharp}\ \vec{x}_j,\ \vec{O}_{\mathcal{X}^\sharp}\right]\right)$$

$$\mathsf{C}^\sharp[\![\, V_j \leftarrow \textstyle\sum_i \alpha_i V_i + \beta \,]\!]\, \mathcal{X}^\sharp \stackrel{\text{def}}{=}$$

if $\alpha_j = 0, (\mathsf{C}^\sharp[\![\, \sum_i \alpha_i V_i - V_j + \beta = 0 \,]\!] \circ \mathsf{C}^\sharp[\![\, V_j \leftarrow [-\infty, +\infty] \,]\!])\mathcal{X}^\sharp$

if $\alpha_j \neq 0, \mathcal{X}^\sharp$ where V_j is replaced with $(V_j - \sum_{i \neq j} \alpha_i V_i - \beta)/\alpha_j$

(proofs on next slide)

$$\mathsf{C}^\sharp[\![\, V_j \leftarrow e \,]\!]\, \mathcal{X}^\sharp \stackrel{\text{def}}{=} \mathsf{C}^\sharp[\![\, V_j \leftarrow [-\infty, +\infty] \,]\!]\, \mathcal{X}^\sharp \text{ for other assignments}$$

Remarks:

- $\cup^\sharp$ is optimal, but not exact.
- $\mathsf{C}^\sharp[\![\, V_j \leftarrow \sum_i \alpha_i V_i + \beta \,]\!]$ and $\mathsf{C}^\sharp[\![\, V_j \leftarrow [-\infty, +\infty] \,]\!]$ are exact.

Affine assignments: proofs

$C^\sharp[\![\, V_j \leftarrow \sum_i \alpha_i V_i + \beta \,]\!]\, \mathcal{X}^\sharp \stackrel{\text{def}}{=}$

if $\alpha_j = 0, (C^\sharp[\![\, \sum_i \alpha_i V_i - V_j + \beta = 0 \,]\!] \circ C^\sharp[\![\, V_j \leftarrow [-\infty,+\infty] \,]\!])\mathcal{X}^\sharp$

if $\alpha_j \neq 0, \mathcal{X}^\sharp$ where V_j is replaced with $(V_j - \sum_{i\neq j} \alpha_i V_i - \beta)/\alpha_j$

Proof sketch:

we use the following identities in the concrete

non-invertible assignment: $\alpha_j = 0$

$C[\![\, V_j \leftarrow e \,]\!] = C[\![\, V_j \leftarrow e \,]\!] \circ C[\![\, V_j \leftarrow [-\infty,+\infty] \,]\!]$ as the value of V_j is not used in e

so: $C[\![\, V_j \leftarrow e \,]\!] = C[\![\, V_j - e = 0 \,]\!] \circ C[\![\, V_j \leftarrow [-\infty,+\infty] \,]\!]$

$\Longrightarrow$ reduces the assignment to a test

invertible assignment: $\alpha_j \neq 0$

$C[\![\, V_j \leftarrow e \,]\!] \subsetneq C[\![\, V_j \leftarrow e \,]\!] \circ C[\![\, V_j \leftarrow [-\infty,+\infty] \,]\!]$ as e depends on V

(e.g., $C[\![\, V \leftarrow V+1 \,]\!] \neq C[\![\, V \leftarrow V+1 \,]\!] \circ C[\![\, V \leftarrow [-\infty,+\infty] \,]\!]$)

$$\begin{aligned}
\rho \in C[\![\, V_j \leftarrow e \,]\!]\, R &\iff \exists \rho' \in R\colon \rho = \rho'[V_j \mapsto \textstyle\sum_i \alpha_i \rho'(V_i) + \beta] \\
&\iff \exists \rho' \in R\colon \rho[V_j \mapsto (\rho(V_j) - \textstyle\sum_{i\neq j} \alpha_i \rho'(V_i) - \beta)/\alpha_j] = \rho' \\
&\iff \rho[V_j \mapsto (\rho(V_j) - \textstyle\sum_{i\neq j} \alpha_i \rho(V_i) - \beta)/\alpha_j] \in R
\end{aligned}$$

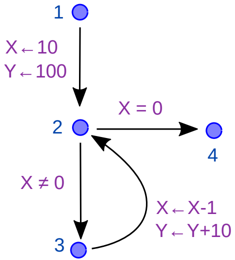
$\Longrightarrow$ reduces the assignment to a substitution by the inverse expression

Analysis example

No infinite increasing chain: we can iterate without widening.

Forward analysis example:

```
¹X ← 10; Y ← 100;
while ²X ≠ 0 do³
  X ← X-1;
  Y ← Y+10
done⁴
```



ℓ	$\mathcal{X}_\ell^{\sharp 0}$	$\mathcal{X}_\ell^{\sharp 1}$	$\mathcal{X}_\ell^{\sharp 2}$	$\mathcal{X}_\ell^{\sharp 3}$	$\mathcal{X}_\ell^{\sharp 4}$
1	$\top^\sharp$	$\top^\sharp$	$\top^\sharp$	$\top^\sharp$	$\top^\sharp$
2	$\bot^\sharp$	$(10, 100)$	$(10, 100)$	$10X + Y = 200$	$10X + Y = 200$
3	$\bot^\sharp$	$\bot^\sharp$	$(10, 100)$	$(10, 100)$	$10X + Y = 200$
4	$\bot^\sharp$	$\bot^\sharp$	$\bot^\sharp$	$\bot^\sharp$	$(0, 200)$

Note in particular:

$$\mathcal{X}_2^{\sharp 3} = \{(10, 100)\} \cup^\sharp \{(9, 110)\} = \{\ (X, Y) \mid 10X + Y = 200\ \}$$

Backward affine equality operators

Backward assignments:

$\overleftarrow{C}^\sharp[\![V_j \leftarrow [-\infty, +\infty]]\!](\mathcal{X}^\sharp, \mathcal{R}^\sharp) \stackrel{\text{def}}{=} \mathcal{X}^\sharp \cap^\sharp (\mathrm{C}^\sharp[\![V_j \leftarrow [-\infty, +\infty]]\!] \mathcal{R}^\sharp)$

$\overleftarrow{C}^\sharp[\![V_j \leftarrow \sum_i \alpha_i V_i + \beta]\!](\mathcal{X}^\sharp, \mathcal{R}^\sharp) \stackrel{\text{def}}{=}$
$\mathcal{X}^\sharp \cap^\sharp (\mathcal{R}^\sharp$ where V_j is replaced with $(\sum_i \alpha_i V_i + \beta))$
(reduces to a substitution by the (non-inverted) expression)

$\overleftarrow{C}^\sharp[\![V_j \leftarrow e]\!](\mathcal{X}^\sharp, \mathcal{R}^\sharp) \stackrel{\text{def}}{=} \overleftarrow{C}^\sharp[\![V_j \leftarrow [-\infty, +\infty]]\!](\mathcal{X}^\sharp, \mathcal{R}^\sharp)$
for other assignments

Remarks:

- $\overleftarrow{C}^\sharp[\![V_j \leftarrow \sum_i \alpha_i V_i + \beta]\!]$ and $\overleftarrow{C}^\sharp[\![V_j \leftarrow [-\infty, +\infty]]\!]$ are exact

A note on integers

Suppose now that $\mathbb{I} = \mathbb{Z}$.

- $\mathbb{Z}$ is not closed under affine operations: $(x/y) \times y \neq x$,
- Gaussian reduction implemented in $\mathbb{Z}$ is unsound.

  (e.g. unsound normalization $2X + Y = 19 \not\Longrightarrow X = 9$, by truncation)

One possible solution:

- keep a representation using matrices with coefficients in $\mathbb{Q}$,
- keep all abstract operators as in $\mathbb{Q}$,
- change the concretization into: $\gamma_{\mathbb{Z}}(\mathcal{X}^\sharp) \stackrel{\text{def}}{=} \gamma(\mathcal{X}^\sharp) \cap \mathbb{Z}^n$.

With respect to $\gamma_{\mathbb{Z}}$, the operators are **no longer best / exact**.

Example: where $\mathcal{X}^\sharp$ is the equation $Y = 2X$

- $\gamma_{\mathbb{Z}}(\mathcal{X}^\sharp) = \{ (X, Y) \mid X \in \mathbb{Z},\ Y = 2X \}$
- $(\mathsf{C}[\![X \leftarrow 0]\!] \circ \gamma_{\mathbb{Z}})\mathcal{X}^\sharp = \{ (X, Y) \mid X = 0,\ Y \text{ is even} \}$
- $(\gamma_{\mathbb{Z}} \circ \mathsf{C}^\sharp[\![X \leftarrow 0]\!])\mathcal{X}^\sharp = \{ (X, Y) \mid X = 0,\ Y \in \mathbb{Z} \}$

$\Longrightarrow$ The analysis forgets the "intergerness" of variables.

The congruence equality domain

Another possible solution: use a more expressive domain.

We look for invariants of the form: $\bigwedge_j \left(\sum_{i=1}^{n} m_{ij} V_i \equiv c_j \; [k_j] \right).$

Algorithms:

- there exists minimal forms (but not unique), computed using an extension of Euclide's algorithm,
- there is a dual representation: $\{ \mathbf{G} \times \vec{\lambda} + \vec{O} \mid \vec{\lambda} \in \mathbb{Z}^m \}$, and passage algorithms,
- see [Gran91].

Polyhedron domain

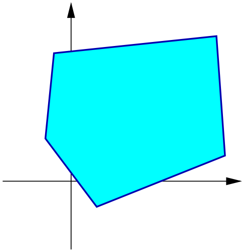
The polyhedron domain

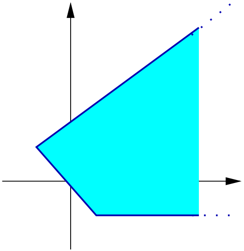
Here again, $\mathbb{I} \in \{\mathbb{Q}, \mathbb{R}\}$.

We look for invariants of the form: $\bigwedge_j \left(\sum_{i=1}^{n} \alpha_{ij} V_i \geq \beta_j \right)$.

We use the polyhedron domain proposed by [Cous78]:

$\mathcal{D}^\sharp \stackrel{\text{def}}{=} \{$closed convex polyhedra of $\mathbb{V} \to \mathbb{I}\}$





Note: polyhedra need not be bounded ($\neq$ polytopes).

Double description of polyhedra

Polyhedra have dual representations (Weyl–Minkowski Theorem). (see [Schr86])

Constraint representation

$\langle \mathbf{M}, \vec{C} \rangle$ with $\mathbf{M} \in \mathbb{I}^{m \times n}$ and $\vec{C} \in \mathbb{I}^m$

represents: $\gamma(\langle \mathbf{M}, \vec{C} \rangle) \stackrel{\text{def}}{=} \{ \vec{V} \mid \mathbf{M} \times \vec{V} \geq \vec{C} \}$

We will also often use a constraint set notation $\{ \sum_i \alpha_{ij} V_i \geq \beta_j \}$.

Generator representation

$[\mathbf{P}, \mathbf{R}]$ where

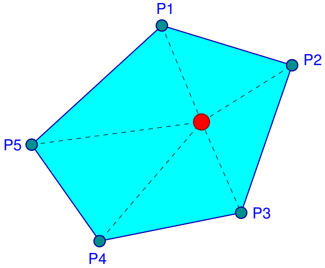
- $\mathbf{P} \in \mathbb{I}^{n \times p}$ is a set of p points: $\vec{P}_1, \ldots, \vec{P}_p$
- $\mathbf{R} \in \mathbb{I}^{n \times r}$ is a set of r rays: $\vec{R}_1, \ldots, \vec{R}_r$

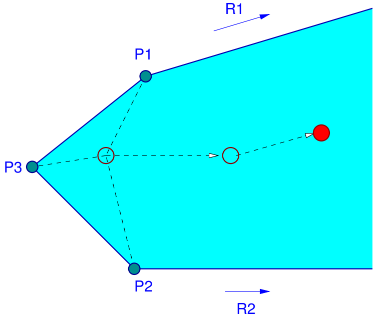
$$\gamma([\mathbf{P}, \mathbf{R}]) \stackrel{\text{def}}{=} \left\{ \left(\sum_{j=1}^{p} \alpha_j \vec{P}_j \right) + \left(\sum_{j=1}^{r} \beta_j \vec{R}_j \right) \mid \forall j, \alpha_j, \beta_j \geq 0, \ \sum_{j=1}^{p} \alpha_j = 1 \right\}$$

Double description of polyhedra (cont.)

Generator representation examples:

$$\gamma([\mathbf{P},\mathbf{R}]) \stackrel{\text{def}}{=} \{\, (\sum_{j=1}^{p} \alpha_j \vec{P}_j) + (\sum_{j=1}^{r} \beta_j \vec{R}_j) \mid \forall j, \alpha_j, \beta_j \geq 0 \colon \sum_{j=1}^{p} \alpha_j = 1 \,\}$$





- the points define a bounded convex hull
- the rays allow unbounded polyhedra

Origin of duality

Dual $\quad A^* \stackrel{\text{def}}{=} \{ \vec{x} \in \mathbb{R}^n \mid \forall \vec{a} \in A,\ \vec{a} \cdot \vec{x} \leq 0 \}$

- $\{\vec{a}\}^*$ and $\{\lambda \vec{r} \mid \lambda \geq 0\}^*$ are half-spaces,
- $(A \cup B)^* = A^* \cap B^*$,
- if A is convex, closed, and $\vec{0} \in A$, then $A^{**} = A$.

Duality on polyhedral cones:

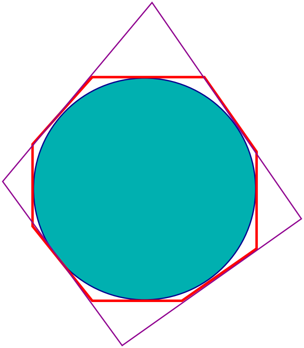
Cone: $C = \{\vec{V} \mid \mathbf{M} \times \vec{V} \geq \vec{0}\}$ or $C = \{ \sum_{j=1}^{r} \beta_j \vec{R}_j \mid \forall j,\ \beta_j \geq 0\}$
(polyhedron with no vertex, except $\vec{0}$)

- C^* is also a polyhedral cone,
- $C^{**} = C$,
- a ray of C corresponds to a constraint of C^*,
- a constraint of C corresponds to a ray of C^*.

Extension to polyhedra: by homogenisation to polyhedral cones:

$C(P) \stackrel{\text{def}}{=} \{ \lambda \vec{V} \mid \lambda \geq 0,\ (V_1, \ldots, V_n) \in \gamma(P),\ V_{n+1} = 1 \} \subseteq \mathbb{R}^{n+1}$
(polyhedron in $\mathbb{R}^n \simeq$ polyhedral cone in $\mathbb{R}^{n+1}$)

Polyhedra representations



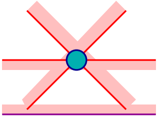
- No best abstraction α
  (e.g., a disc has infinitely many polyhedral over-approximations, but no best one)
- No memory bound on the representations

Polyhedra representations

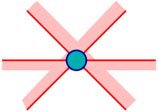
Minimal representations

- A constraint / generator system is minimal if no constraint / generator can be omitted without changing the concretization
- Minimal representations are not unique
- No memory bound even on minimal representations

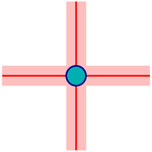
Example: three different constraint representations for a point



(a)



(b)



(c)

- (a) $y + x \geq 0, y - x \geq 0, y \leq 0, y \geq -5$ (non mimimal)
- (b) $y + x \geq 0, y - x \geq 0, y \leq 0$ (minimal)
- (c) $x \leq 0, x \geq 0, y \leq 0, y \geq 0$ (minimal)

Chernikova's algorithm

Algorithm by [Cher68], improved by [LeVe92] to switch from a constraint system to an equivalent generator system

Why? most operators are easier on one representation

Notes:

- By duality, we can use the same algorithm to switch from generators to constraints
- The minimal generator system can be exponential in the original constraint system
  (e.g., hypercube: $2n$ constraints, 2^n vertices)
- Equality constraints and lines (pairs of opposed rays) may be handled separately and more efficiently

Chernikova's algorithm (cont.)

Algorithm: incrementally add constraints one by one

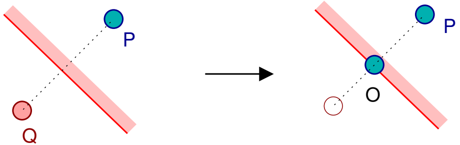
Start with: $\begin{cases} \mathbf{P}_0 = \{\, (0,\ldots,0) \,\} & \text{(origin)} \\ \mathbf{R}_0 = \{\, \vec{x}_i,\ -\vec{x}_i \mid 1 \leq i \leq n \,\} & \text{(axes)} \end{cases}$

For each constraint $\vec{M}_k \cdot \vec{V} \geq C_k \in \langle \mathbf{M}, \vec{C} \rangle$, update $[\mathbf{P}_{k-1}, \mathbf{R}_{k-1}]$ to $[\mathbf{P}_k, \mathbf{R}_k]$.

Start with $\mathbf{P}_k = \mathbf{R}_k = \emptyset$,

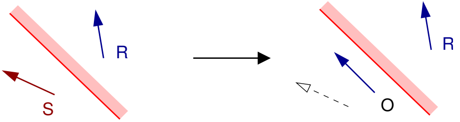
- for any $\vec{P} \in \mathbf{P}_{k-1}$ s.t. $\vec{M}_k \cdot \vec{P} \geq C_k$, add $\vec{P}$ to $\mathbf{P}_k$
- for any $\vec{R} \in \mathbf{R}_{k-1}$ s.t. $\vec{M}_k \cdot \vec{R} \geq 0$, add $\vec{R}$ to $\mathbf{R}_k$
- for any $\vec{P}, \vec{Q} \in \mathbf{P}_{k-1}$ s.t. $\vec{M}_k \cdot \vec{P} > C_k$ and $\vec{M}_k \cdot \vec{Q} < C_k$, add to $\mathbf{P}_k$:
  $\vec{O} \stackrel{\text{def}}{=} \frac{C_k - \vec{M}_k \cdot \vec{Q}}{\vec{M}_k \cdot \vec{P} - \vec{M}_k \cdot \vec{Q}} \vec{P} - \frac{C_k - \vec{M}_k \cdot \vec{P}}{\vec{M}_k \cdot \vec{P} - \vec{M}_k \cdot \vec{Q}} \vec{Q}$

  i.e., move Q towards P along $[Q, P]$ until it saturates the constraint

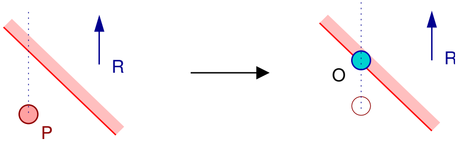


Chernikova's algorithm (cont.)

- for any $\vec{R}, \vec{S} \in \mathbf{R}_{k-1}$ s.t. $\vec{M}_k \cdot \vec{R} > 0$ and $\vec{M}_k \cdot \vec{S} < 0$, add to $\mathbf{R}_k$:
  $\vec{O} \stackrel{\text{def}}{=} (\vec{M}_k \cdot \vec{S})\vec{R} - (\vec{M}_k \cdot \vec{R})\vec{S}$
  i.e., rotate S towards R until it is parallel to the constraint

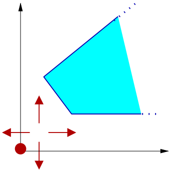


- for any $\vec{P} \in \mathbf{P}_{k-1}$, $\vec{R} \in \mathbf{R}_{k-1}$ s.t.
  either $\vec{M}_k \cdot \vec{P} > C_k$ and $\vec{M}_k \cdot \vec{R} < 0$, or $\vec{M}_k \cdot \vec{P} < C_k$ and $\vec{M}_k \cdot \vec{R} > 0$
  add to $\mathbf{P}_k$: $\vec{O} \stackrel{\text{def}}{=} \vec{P} + \frac{C_k - \vec{M}_k \cdot \vec{P}}{\vec{M}_k \cdot \vec{R}} \vec{R}$



Chernikova's algorithm example

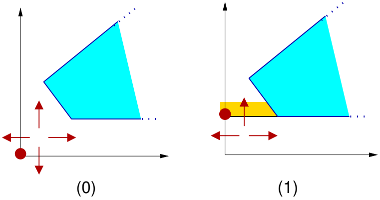
Example:



$\mathbf{P}_0 = \{(0,0)\}$ $\mathbf{R}_0 = \{(1,0),\ (-1,0),\ (0,1),\ (0,-1)\}$

Chernikova's algorithm example

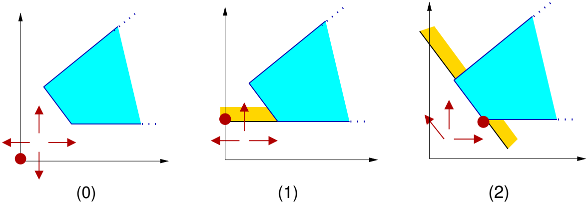
Example:



	$\mathbf{P}_0 = \{(0,0)\}$	$\mathbf{R}_0 = \{(1,0),\ (-1,0),\ (0,1),\ (0,-1)\}$
$Y \geq 1$	$\mathbf{P}_1 = \{(0,1)\}$	$\mathbf{R}_1 = \{(1,0),\ (-1,0),\ (0,1)\}$

Chernikova's algorithm example

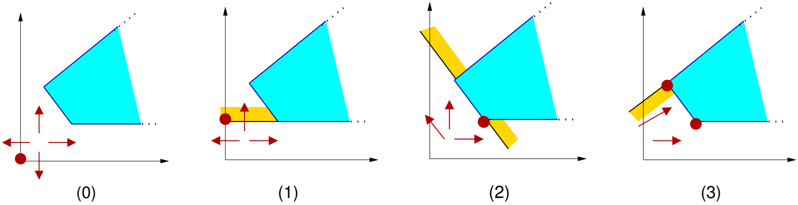
Example:



$$
\begin{array}{lll}
 & \mathbf{P}_0 = \{(0,0)\} & \mathbf{R}_0 = \{(1,0),\ (-1,0),\ (0,1),\ (0,-1)\} \\
Y \geq 1 & \mathbf{P}_1 = \{(0,1)\} & \mathbf{R}_1 = \{(1,0),\ (-1,0),\ (0,1)\} \\
X + Y \geq 3 & \mathbf{P}_2 = \{(2,1)\} & \mathbf{R}_2 = \{(1,0),\ (-1,1),\ (0,1)\}
\end{array}
$$

Chernikova's algorithm example

Example:



	$\mathbf{P}_0 = \{(0,0)\}$	$\mathbf{R}_0 = \{(1,0),\ (-1,0),\ (0,1),\ (0,-1)\}$
$Y \geq 1$	$\mathbf{P}_1 = \{(0,1)\}$	$\mathbf{R}_1 = \{(1,0),\ (-1,0),\ (0,1)\}$
$X + Y \geq 3$	$\mathbf{P}_2 = \{(2,1)\}$	$\mathbf{R}_2 = \{(1,0),\ (-1,1),\ (0,1)\}$
$X - Y \leq 1$	$\mathbf{P}_3 = \{(2,1),\ (1,2)\}$	$\mathbf{R}_3 = \{(0,1),\ (1,1)\}$

Redundancy removal

Goal: only introduce non-redundant points and rays during Chernikova's algorithm

Definitions (for rays in polyhedral cones)

Given $C = \{ \vec{V} \mid \mathbf{M} \times \vec{V} \geq \vec{0} \} = \{ \mathbf{R} \times \vec{\beta} \mid \vec{\beta} \geq \vec{0} \}$.

- $\vec{R}$ saturates $\vec{M}_k \cdot \vec{V} \geq 0 \stackrel{\text{def}}{\iff} \vec{M}_k \cdot \vec{R} = 0$
- $S(\vec{R}, C) \stackrel{\text{def}}{=} \{ k \mid \vec{M}_k \cdot \vec{R} = 0 \}$.

Theorem:

assume C has no line ($\nexists \vec{L} \neq \vec{0}$ s.t. $\forall \alpha, \alpha \vec{L} \in C$)
$\vec{R}$ is non-redundant w.r.t. $\mathbf{R} \iff \nexists \vec{R}_i \in \mathbf{R}, S(\vec{R}, C) \subsetneq S(\vec{R}_i, C)$

- $S(\vec{R}_i, C)$, $\vec{R}_i \in \mathbf{R}$ is maintained during Chernikova's algorithm in a saturation matrix
- extension to (non-conic) polyhedra and to lines
- various improvements exist [LeVe92]

Operators on polyhedra

Given $\mathcal{X}^\sharp, \mathcal{Y}^\sharp \neq \bot^\sharp$, we define:

$$\mathcal{X}^\sharp \subseteq^\sharp \mathcal{Y}^\sharp \quad \stackrel{\text{def}}{\iff} \quad \begin{cases} \forall \vec{P} \in \mathbf{P}_{\mathcal{X}^\sharp},\ \mathbf{M}_{\mathcal{Y}^\sharp} \times \vec{P} \ \geq\ \vec{C}_{\mathcal{Y}^\sharp} \\ \forall \vec{R} \in \mathbf{R}_{\mathcal{X}^\sharp},\ \mathbf{M}_{\mathcal{Y}^\sharp} \times \vec{R} \ \geq\ \vec{0} \end{cases}$$

(every generator of $\mathcal{X}^\sharp$ must satisfy every constraint in $\mathcal{Y}^\sharp$)

$$\mathcal{X}^\sharp =^\sharp \mathcal{Y}^\sharp \quad \stackrel{\text{def}}{\iff} \quad \mathcal{X}^\sharp \subseteq^\sharp \mathcal{Y}^\sharp \ \text{ and } \ \mathcal{Y}^\sharp \subseteq^\sharp \mathcal{X}^\sharp$$

$$\mathcal{X}^\sharp \cap^\sharp \mathcal{Y}^\sharp \quad \stackrel{\text{def}}{=} \quad \left\langle \begin{bmatrix} \mathbf{M}_{\mathcal{X}^\sharp} \\ \mathbf{M}_{\mathcal{Y}^\sharp} \end{bmatrix}, \begin{bmatrix} \vec{C}_{\mathcal{X}^\sharp} \\ \vec{C}_{\mathcal{Y}^\sharp} \end{bmatrix} \right\rangle$$

(set union of sets of constraints)

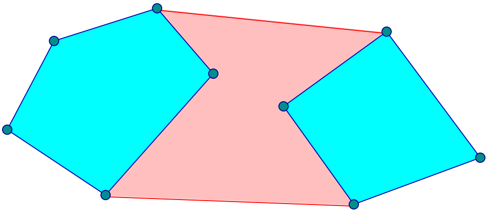
Remarks:

- $\subseteq^\sharp$, $=^\sharp$ and $\cap^\sharp$ are exact.

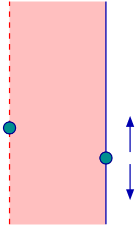
Operators on polyhedra: join

Join: $\mathcal{X}^\sharp \cup^\sharp \mathcal{Y}^\sharp \stackrel{\text{def}}{=} [\,[\mathbf{P}_{\mathcal{X}^\sharp}\ \mathbf{P}_{\mathcal{Y}^\sharp}],\ [\mathbf{R}_{\mathcal{X}^\sharp}\ \mathbf{R}_{\mathcal{Y}^\sharp}]\,]$ (join generator sets)

Examples:



two polytopes



a point and a line

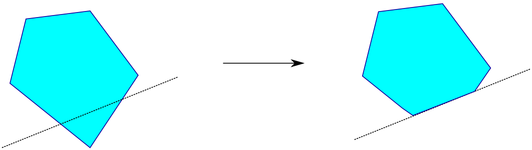
$\cup^\sharp$ is optimal:

we get the topological closure of the convex hull of $\gamma(\mathcal{X}^\sharp) \cup \gamma(\mathcal{Y}^\sharp)$

Operators on polyhedra (cont.)

Forward operators: affine tests

$$C^\sharp[\![\, \sum_i \alpha_i V_i + \beta \geq 0 \,]\!]\, \mathcal{X}^\sharp \stackrel{\text{def}}{=} \left\langle \begin{bmatrix} \mathbf{M}_{\mathcal{X}^\sharp} \\ \alpha_1 \cdots \alpha_n \end{bmatrix}, \begin{bmatrix} \vec{C}_{\mathcal{X}^\sharp} \\ -\beta \end{bmatrix} \right\rangle$$

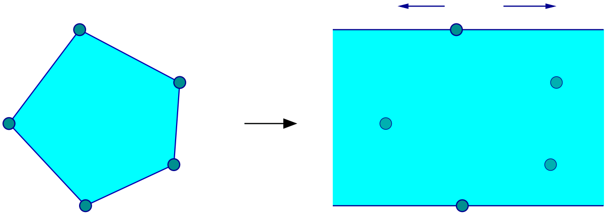


These test operators are exact.

Operators on polyhedra (cont.)

Forward operators: forget

$$C^\sharp[\![\, V_j \leftarrow [-\infty,+\infty] \,]\!]\, \mathcal{X}^\sharp \stackrel{\text{def}}{=} [\, \mathbf{P}_{\mathcal{X}^\sharp},\, [\, \mathbf{R}_{\mathcal{X}^\sharp} \;\; \vec{x}_j \;\; (-\vec{x}_j) \,]\,]$$



This operator is exact.

It is also a sound abstraction for any assignment.

Operators on polyhedra (cont.)

Forward operators: affine assignments

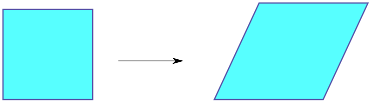
$$\mathsf{C}^\sharp[\![\, V_j \leftarrow \textstyle\sum_i \alpha_i V_i + \beta \,]\!]\, \mathcal{X}^\sharp \stackrel{\text{def}}{=}$$

$$\text{if } \alpha_j = 0, (\mathsf{C}^\sharp[\![\, \textstyle\sum_i \alpha_i V_i - V_j + \beta = 0 \,]\!] \circ \mathsf{C}^\sharp[\![\, V_j \leftarrow [-\infty, +\infty] \,]\!])\mathcal{X}^\sharp$$

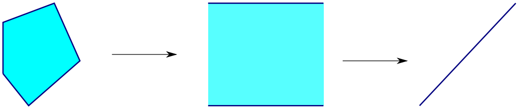
$$\text{if } \alpha_j \neq 0, \langle \mathbf{M}, \vec{C} \rangle \text{ where } V_j \text{ is replaced with } \tfrac{1}{\alpha_j}(V_j - \textstyle\sum_{i \neq j} \alpha_i V_i - \beta)$$

Examples :

$X \leftarrow X + Y$



$X \leftarrow Y$



Affine assignments are exact.

They could also be defined on generator systems.

Operators on polyhedra (cont.)

Backward assignments:

$\overleftarrow{C}^\sharp[\![\, V_j \leftarrow [-\infty,+\infty] \,]\!](\mathcal{X}^\sharp,\mathcal{R}^\sharp) \stackrel{\text{def}}{=} \mathcal{X}^\sharp \cap^\sharp (\mathsf{C}^\sharp[\![\, V_j \leftarrow [-\infty,+\infty] \,]\!]\,\mathcal{R}^\sharp)$

$\overleftarrow{C}^\sharp[\![\, V_j \leftarrow \sum_i \alpha_i V_i + \beta \,]\!](\mathcal{X}^\sharp,\mathcal{R}^\sharp) \stackrel{\text{def}}{=}$
$\mathcal{X}^\sharp \cap^\sharp (\mathcal{R}^\sharp$ where V_j is replaced with $(\sum_i \alpha_i V_i + \beta))$

$\overleftarrow{C}^\sharp[\![\, V_j \leftarrow e \,]\!](\mathcal{X}^\sharp,\mathcal{R}^\sharp) \stackrel{\text{def}}{=} \overleftarrow{C}^\sharp[\![\, V_j \leftarrow [-\infty,+\infty] \,]\!](\mathcal{X}^\sharp,\mathcal{R}^\sharp)$
for other assignments

Note: identical to the case of linear equalities.

Polyhedra widening

$\mathcal{D}^\sharp$ has strictly increasing infinite chains $\implies$ we need a widening

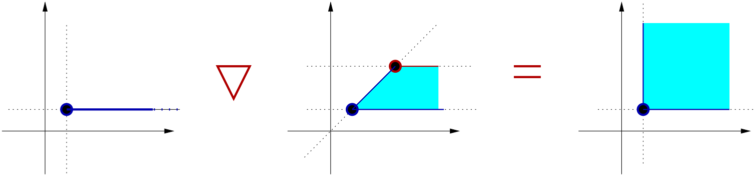
Definition:

Take $\mathcal{X}^\sharp$ and $\mathcal{Y}^\sharp$ in minimal constraint-set form

$$\mathcal{X}^\sharp \mathbin{\triangledown} \mathcal{Y}^\sharp \stackrel{\text{def}}{=} \{\, c \in \mathcal{X}^\sharp \mid \mathcal{Y}^\sharp \subseteq^\sharp \{c\} \,\}$$

We suppress any unstable constraint $c \in \mathcal{X}^\sharp$, i.e., $\mathcal{Y}^\sharp \not\subseteq^\sharp \{c\}$

Example:



Polyhedra widening

$\mathcal{D}^\sharp$ has strictly increasing infinite chains $\Longrightarrow$ we need a widening

Definition:

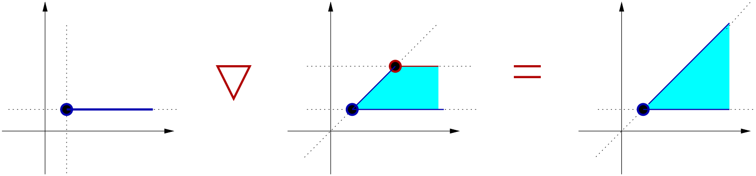
Take $\mathcal{X}^\sharp$ and $\mathcal{Y}^\sharp$ in minimal constraint-set form

$$\begin{array}{rcl} \mathcal{X}^\sharp \mathbin{\triangledown} \mathcal{Y}^\sharp & \stackrel{\text{def}}{=} & \{\, c \in \mathcal{X}^\sharp \,|\, \mathcal{Y}^\sharp \subseteq^\sharp \{c\} \,\} \\ & \cup & \{\, c \in \mathcal{Y}^\sharp \,|\, \exists c' \in \mathcal{X}^\sharp \colon \mathcal{X}^\sharp =^\sharp (\mathcal{X}^\sharp \setminus c') \cup \{c\} \,\} \end{array}$$

We suppress any unstable constraint $c \in \mathcal{X}^\sharp$, i.e., $\mathcal{Y}^\sharp \not\subseteq^\sharp \{c\}$

We also keep constraints $c \in \mathcal{Y}^\sharp$ equivalent to those in $\mathcal{X}^\sharp$, i.e., when $\exists c' \in \mathcal{X}^\sharp \colon \mathcal{X}^\sharp =^\sharp (\mathcal{X}^\sharp \setminus c') \cup \{c\}$

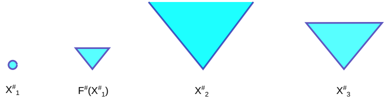
Example:



Example analysis

```
X ← 2; I ← 0;
while ● I < 10 do
  if [0,1] = 0 then X ← X + 2 else X ← X - 3 fi;
  I ← I + 1
done ♦
```

Loop invariant:



Increasing iterations with widening at ● give:

$$\begin{array}{lcl} \mathcal{X}_1^\sharp & = & \{X = 2, I = 0\} \\ \mathcal{X}_2^\sharp & = & \{X = 2, I = 0\} \triangledown (\{X = 2, I = 0\} \cup^\sharp \{X \in [-1, 4],\ I = 1\}) \\ & = & \{X = 2, I = 0\} \triangledown \{ I \in [0, 1],\ 2 - 3I \leq X \leq 2I + 2 \} \\ & = & \{I \geq 0,\ 2 - 3I \leq X \leq 2I + 2\} \end{array}$$

Decreasing iterations *(to find $I \leq 10$)*:

$$\begin{array}{lcl} \mathcal{X}_3^\sharp & = & \{X = 2, I = 0\} \cup^\sharp \{ I \in [1, 10],\ 2 - 3I \leq X \leq 2I + 2 \} \\ & = & \{I \in [0, 10],\ 2 - 3I \leq X \leq 2I + 2\} \end{array}$$

We find, at the end of the loop ♦: $I = 10 \wedge X \in [-28, 22]$.

Other polyhedra widenings

Widening with thresholds:

Given a finite set T of constraints, we add to $\mathcal{X}^\sharp \mathbin{\triangledown} \mathcal{Y}^\sharp$ all the constraints from T satisfied by both $\mathcal{X}^\sharp$ and $\mathcal{Y}^\sharp$.

Delayed widening:

We replace $\mathcal{X}^\sharp \mathbin{\triangledown} \mathcal{Y}^\sharp$ with $\mathcal{X}^\sharp \cup^\sharp \mathcal{Y}^\sharp$ a finite number of times
(this works for any widening and abstract domain).

See also [Bagn03].

Integer polyhedra

How can we deal with $\mathbb{I} = \mathbb{Z}$?

Issue: integer linear programming is difficult.

Example: satsfiability of conjunctions of linear constraints:

- polynomial cost in $\mathbb{Q}$,
- NP-complete cost in $\mathbb{Z}$.

Possible solutions:

- Use some complete integer algorithms.
  (e.g. Presburger arithmetics)
  Costly, and we do not have any abstract domain structure.
- Keep $\mathbb{Q}-$polyhedra as representation, and change the concretization into: $\gamma_{\mathbb{Z}}(\mathcal{X}^\sharp) \stackrel{\text{def}}{=} \gamma(\mathcal{X}^\sharp) \cap \mathbb{Z}^n$.
  However, operators are no longer exact / optimal.

Weakly relational domains

Zone domain

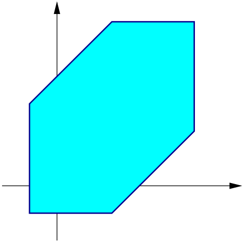
The zone domain

Here, $\mathbb{I} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$.

We look for invariants of the form:

$$\bigwedge \; V_i - V_j \leq c \text{ or } \pm V_i \leq c, \quad c \in \mathbb{I}$$

A subset of $\mathbb{I}^n$ bounded by such constraints is called a **zone**.



[Mine01a]

Machine representation

A potential constraint has the form: $V_j - V_i \leq c$.

Potential graph: directed, weighted graph $\mathcal{G}$

- nodes are labelled with variables in $\mathbb{V}$,
- we add an arc with weight c from V_i to V_j for each constraint $V_j - V_i \leq c$.

Difference Bound Matrix (DBM)

Adjacency matrix $\mathbf{m}$ of $\mathcal{G}$:

- $\mathbf{m}$ is square, with size $n \times n$, and elements in $\mathbb{I} \cup \{+\infty\}$,
- $m_{ij} = c < +\infty$ denotes the constraint $V_j - V_i \leq c$,
- $m_{ij} = +\infty$ if there is no upper bound on $V_j - V_i$.

Concretization:

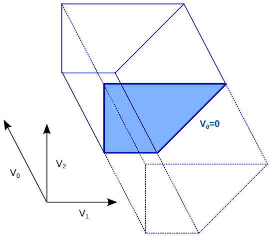
$$\gamma(\mathbf{m}) \stackrel{\text{def}}{=} \{ (v_1, \ldots, v_n) \in \mathbb{I}^n \mid \forall i, j,\ v_j - v_i \leq m_{ij} \}.$$

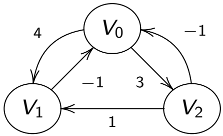
Machine representation (cont.)

Unary constraints add a constant null variable V_0.

- $\mathbf{m}$ has size $(n+1) \times (n+1)$;
- $V_i \leq c$ is denoted as $V_i - V_0 \leq c$, i.e., $m_{i0} = c$;
- $V_i \geq c$ is denoted as $V_0 - V_i \leq -c$, i.e., $m_{0i} = -c$;
- γ is now: $\gamma_0(\mathbf{m}) \stackrel{\text{def}}{=} \{ (v_1, \ldots, v_n) \mid (0, v_1, \ldots, v_n) \in \gamma(\mathbf{m}) \}$.

Example:





	V_0	V_1	V_2
V_0	$+\infty$	4	3
V_1	-1	$+\infty$	$+\infty$
V_2	-1	1	$+\infty$

The DBM lattice

$\mathcal{D}^\sharp$ contains all DBMs, plus $\bot^\sharp$.

$\leq$ on $\mathbb{I} \cup \{+\infty\}$ is extended point-wisely.

If $\mathbf{m}, \mathbf{n} \neq \bot^\sharp$:

$$\begin{array}{ccc}
\mathbf{m} \subseteq^\sharp \mathbf{n} & \stackrel{\text{def}}{\iff} & \forall i,j,\ m_{ij} \leq n_{ij} \\
\mathbf{m} =^\sharp \mathbf{n} & \stackrel{\text{def}}{\iff} & \forall i,j,\ m_{ij} = n_{ij} \\
\left[\mathbf{m} \cap^\sharp \mathbf{n}\right]_{ij} & \stackrel{\text{def}}{=} & \min(m_{ij}, n_{ij}) \\
\left[\mathbf{m} \cup^\sharp \mathbf{n}\right]_{ij} & \stackrel{\text{def}}{=} & \max(m_{ij}, n_{ij}) \\
\left[\top^\sharp\right]_{ij} & \stackrel{\text{def}}{=} & +\infty
\end{array}$$

$(\mathcal{D}^\sharp, \subseteq^\sharp, \cup^\sharp, \cap^\sharp, \bot^\sharp, \top^\sharp)$ is a lattice.

Remarks:

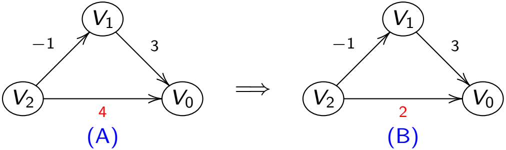
- $\mathcal{D}^\sharp$ is complete if $\leq$ is ($\mathbb{I} = \mathbb{R}$ or $\mathbb{Z}$, but not $\mathbb{Q}$),
- $\mathbf{m} \subseteq^\sharp \mathbf{n} \implies \gamma_0(\mathbf{m}) \subseteq \gamma_0(\mathbf{n})$, but not the converse,
- $\mathbf{m} =^\sharp \mathbf{n} \implies \gamma_0(\mathbf{m}) = \gamma_0(\mathbf{n})$, but not the converse.

Normal form, equality and inclusion testing

Issue: how can we compare $\gamma_0(\mathbf{m})$ and $\gamma_0(\mathbf{n})$?

Idea: find a normal form by propagating/tightening constraints.

$$\begin{cases} V_0 - V_1 \leq 3 \\ V_1 - V_2 \leq -1 \\ V_0 - V_2 \leq 4 \end{cases} \qquad \begin{cases} V_0 - V_1 \leq 3 \\ V_1 - V_2 \leq -1 \\ V_0 - V_2 \leq 2 \end{cases}$$



Definition: shortest-path closure $\mathbf{m}^*$

$$m^*_{ij} \stackrel{\text{def}}{=} \min_{\substack{N \\ \langle i = i_1, \ldots, i_N = j \rangle}} \sum_{k=1}^{N-1} m_{i_k\, i_{k+1}}$$

Exists only when $\mathbf{m}$ has no cycle with strictly negative weight.

Floyd–Warshall algorithm

Properties:

- $\gamma_0(\mathbf{m}) = \emptyset \iff \mathcal{G}$ has a cycle with strictly negative weight.
- if $\gamma_0(\mathbf{m}) \neq \emptyset$, the shortest-path graph $\mathbf{m}^*$ is a normal form:

  $\mathbf{m}^* = \min_{\subseteq^\sharp} \{ \mathbf{n} \mid \gamma_0(\mathbf{m}) = \gamma_0(\mathbf{n}) \}$
- If $\gamma_0(\mathbf{m}), \gamma_0(\mathbf{n}) \neq \emptyset$, then
  - $\gamma_0(\mathbf{m}) = \gamma_0(\mathbf{n}) \iff \mathbf{m}^* =^\sharp \mathbf{n}^*$,
  - $\gamma_0(\mathbf{m}) \subseteq \gamma_0(\mathbf{n}) \iff \mathbf{m}^* \subseteq^\sharp \mathbf{n}$.

Floyd–Warshall algorithm

$$\begin{cases} m_{ij}^0 & \stackrel{\text{def}}{=} & m_{ij} \\ m_{ij}^{k+1} & \stackrel{\text{def}}{=} & \min(m_{ij}^k, m_{ik}^k + m_{kj}^k) \end{cases}$$

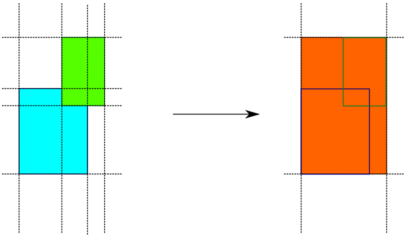
- If $\gamma_0(\mathbf{m}) \neq \emptyset$, then $\mathbf{m}^* = \mathbf{m}^{n+1}$, (normal form)
- $\gamma_0(\mathbf{m}) = \emptyset \iff \exists i,\ m_{ii}^{n+1} < 0$, (emptiness testing)
- $\mathbf{m}^{n+1}$ can be computed in $\mathcal{O}(n^3)$ time.

Abstract operators

Abstract join: naïve version $\cup^\sharp$ *(element-wise* max*)*

- $\cup^\sharp$ is a sound abstraction of $\cup$

  but $\gamma_0(\mathbf{m} \cup^\sharp \mathbf{n})$ is not necessarily the smallest zone containing $\gamma_0(\mathbf{m})$ and $\gamma_0(\mathbf{n})$!



The union of two zones with $\cup^\sharp$ is no more precise in the zone domain than in the interval domain!

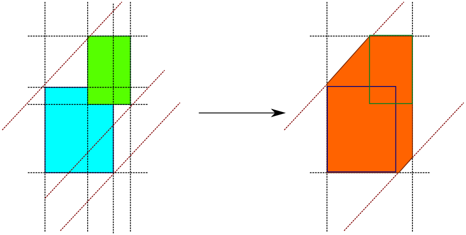
Abstract operators (cont.)

Abstract join: precise version: $\cup^\sharp$ after closure

- $(\mathbf{m}^*) \cup^\sharp (\mathbf{n}^*)$ is however optimal

  we have: $(\mathbf{m}^*) \cup^\sharp (\mathbf{n}^*) = \min_{\subseteq^\sharp} \{ \mathbf{o} \mid \gamma_0(\mathbf{o}) \supseteq \gamma_0(\mathbf{m}) \cup \gamma_0(\mathbf{n}) \}$

  which implies:
  $\gamma_0((\mathbf{m}^*) \cup^\sharp (\mathbf{n}^*)) = \min_{\subseteq} \{ \gamma_0(\mathbf{o}) \mid \gamma_0(\mathbf{o}) \supseteq \gamma_0(\mathbf{m}) \cup \gamma_0(\mathbf{n}) \}$



after closure, new constraints $c \leq X - Y \leq d$ give an increase in precision

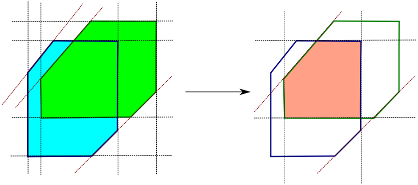
- $(\mathbf{m}^*) \cup^\sharp (\mathbf{n}^*)$ is always closed.

Abstract operators (cont.)

Abstract intersection $\cap^\sharp$: element-wise min

- $\cap^\sharp$ is an exact abstraction of $\cap$ *(zones are closed under intersection)*:

$$\gamma_0(\mathbf{m} \cap^\sharp \mathbf{n}) = \gamma_0(\mathbf{m}) \cap \gamma_0(\mathbf{n})$$



- $(\mathbf{m}^*) \cap^\sharp (\mathbf{n}^*)$ is not necessarily closed...

Remark

The set of closed matrices, with $\perp^\sharp$, and the operations $\subseteq^\sharp$, $\cup^\sharp$, $\lambda\mathbf{m},\mathbf{n}.(\mathbf{m} \cap^\sharp \mathbf{n})^*$ is a sub-lattice, where γ_0 is injective.

Abstract operators (cont.)

We can define:

$$\left[C^\sharp[\![\, V_{j_0} - V_{i_0} \leq c \,]\!]\, \mathbf{m}\right]_{ij} \stackrel{\text{def}}{=} \begin{cases} \min(m_{ij}, c) & \text{if } (i,j) = (i_0, j_0), \\ m_{ij} & \text{otherwise.} \end{cases}$$

$$\left[C^\sharp[\![\, V_{j_0} \leftarrow [-\infty, +\infty] \,]\!]\, \mathbf{m}\right]_{ij} \stackrel{\text{def}}{=} \begin{cases} +\infty & \text{if } i = j_0 \text{ or } j = j_0, \\ m^*_{ij} & \text{otherwise.} \end{cases}$$

(not optimal on non-closed arguments)

$$C^\sharp[\![\, V_{j_0} \leftarrow V_{i_0} + a \,]\!]\, \mathbf{m} \stackrel{\text{def}}{=} (C^\sharp[\![\, V_{j_0} - V_{i_0} = a \,]\!] \circ C^\sharp[\![\, V_{j_0} \leftarrow [-\infty, +\infty] \,]\!]\,)\mathbf{m} \quad \text{if } i_0 \neq j_0$$

$$\left[C^\sharp[\![\, V_{j_0} \leftarrow V_{j_0} + a \,]\!]\, \mathbf{m}\right]_{ij} \stackrel{\text{def}}{=} \begin{cases} m_{ij} - a & \text{if } i = j_0 \text{ and } j \neq j_0 \\ m_{ij} + a & \text{if } i \neq j_0 \text{ and } j = j_0 \\ m_{ij} & \text{otherwise.} \end{cases}$$

These transfer functions are exact.

Abstract operators (cont.)

Backward assignment:

$$\overleftarrow{C}^\sharp[\![\, V_{j_0} \leftarrow [-\infty, +\infty] \,]\!](\mathbf{m}, \mathbf{r}) \stackrel{\text{def}}{=} \mathbf{m} \cap^\sharp (\mathrm{C}^\sharp[\![\, V_{j_0} \leftarrow [-\infty, +\infty] \,]\!]\, \mathbf{r})$$

$$\overleftarrow{C}^\sharp[\![\, V_{j_0} \leftarrow V_{j_0} + a \,]\!](\mathbf{m}, \mathbf{r}) \stackrel{\text{def}}{=} \mathbf{m} \cap^\sharp (\mathrm{C}^\sharp[\![\, V_{j_0} \leftarrow V_{j_0} - a \,]\!]\, \mathbf{r})$$

$$\left[\overleftarrow{C}^\sharp[\![\, V_{j_0} \leftarrow V_{i_0} + a \,]\!](\mathbf{m}, \mathbf{r})\right]_{ij} \stackrel{\text{def}}{=}$$

$$\mathbf{m} \cap^\sharp \begin{cases} \min(\mathbf{r}^*_{ij}, \mathbf{r}^*_{j_0 j} + a) & \text{if } i = i_0 \text{ and } j \neq i_0, j_0 \\ \min(\mathbf{r}^*_{ij}, \mathbf{r}^*_{i j_0} - a) & \text{if } j = i_0 \text{ and } i \neq i_0, j_0 \\ +\infty & \text{if } i = j_0 \text{ or } j = j_0 \\ \mathbf{r}^*_{ij} & \text{otherwise.} \end{cases}$$

Abstract operators (cont.)

Issue: given an arbitrary linear assignment $V_{j_0} \leftarrow a_0 + \sum_k a_k \times V_k$

- there is no exact abstraction, in general;
- the best abstraction $\alpha \circ \mathsf{C}[\![c]\!] \circ \gamma$ is costly to compute.
  (e.g. convert to a polyhedron and back, with exponential cost)

Possible solution:

Given a (more general) assignment $e = [a_0, b_0] + \sum_k [a_k, b_k] \times V_k$

we define an approximate operator as follows:

$$\left[\mathsf{C}^\sharp[\![V_{j_0} \leftarrow e]\!]\,\mathbf{m}\right]_{ij} \stackrel{\text{def}}{=} \begin{cases} \max(\mathsf{E}^\sharp[\![e]\!]\,\mathbf{m}) & \text{if } i = 0 \text{ and } j = j_0 \\ -\min(\mathsf{E}^\sharp[\![e]\!]\,\mathbf{m}) & \text{if } i = j_0 \text{ and } j = 0 \\ \max(\mathsf{E}^\sharp[\![e - V_i]\!]\,\mathbf{m}) & \text{if } i \neq 0, j_0 \text{ and } j = j_0 \\ -\min(\mathsf{E}^\sharp[\![e + V_j]\!]\,\mathbf{m}) & \text{if } i = j_0 \text{ and } j \neq 0, j_0 \\ m_{ij} & \text{otherwise} \end{cases}$$

where $\mathsf{E}^\sharp[\![e]\!]\,\mathbf{m}$ evaluates e using interval arithmetics with $V_k \in [-m^*_{k0}, m^*_{0k}]$.

Quadratic total cost (plus the cost of closure).

Abstract operators (cont.)

Example:

Argument

$$\begin{cases} 0 \leq Y \leq 10 \\ 0 \leq Z \leq 10 \\ 0 \leq Y - Z \leq 10 \end{cases}$$

$$\Downarrow \; X \leftarrow Y - Z$$

$$\begin{cases} -10 \leq X \leq 10 \\ -20 \leq X - Y \leq 10 \\ -20 \leq X - Z \leq 10 \end{cases}$$

Intervals

$$\begin{cases} -10 \leq X \leq 10 \\ -10 \leq X - Y \leq 0 \\ -10 \leq X - Z \leq 10 \end{cases}$$

Approximate solution

$$\begin{cases} 0 \leq X \leq 10 \\ -10 \leq X - Y \leq 0 \\ -10 \leq X - Z \leq 10 \end{cases}$$

Best (polyhedra)

We have a good trade-off between cost and precision.

The same idea can be used for tests and backward assignments.

Widening and narrowing

The zone domain has both strictly increasing and decreasing infinite chains.

Widening $\triangledown$

$$[\mathbf{m} \triangledown \mathbf{n}]_{ij} \stackrel{\text{def}}{=} \begin{cases} m_{ij} & \text{if } n_{ij} \leq m_{ij} \\ +\infty & \text{otherwise} \end{cases}$$

Unstable constraints are deleted.

Narrowing $\vartriangle$

$$[\mathbf{m} \vartriangle \mathbf{n}]_{ij} \stackrel{\text{def}}{=} \begin{cases} n_{ij} & \text{if } m_{ij} = +\infty \\ m_{ij} & \text{otherwise} \end{cases}$$

Only $+\infty$ bounds are refined.

Remarks:

- We can construct widenings with thresholds.
- $\triangledown$ (resp. $\vartriangle$) can be seen as a point-wise extension of an interval widening (resp. narrowing).

Interaction between closure and widening

Widening $\triangledown$ and closure $*$ cannot always be mixed safely:

- $\mathbf{m}_{i+1} \stackrel{\text{def}}{=} \mathbf{m}_i \triangledown (\mathbf{n}_i^*)$ OK
- $\mathbf{m}_{i+1} \stackrel{\text{def}}{=} (\mathbf{m}_i^*) \triangledown \mathbf{n}_i$ wrong!
- $\mathbf{m}_{i+1} \stackrel{\text{def}}{=} (\mathbf{m}_i \triangledown \mathbf{n}_i)^*$ wrong

otherwise the sequence $(\mathbf{m}_i)$ may be infinite!

Example:

```
X ← 0; Y ← [-1,1];
while • 1 = 1 do
  R ← [-1,1];
  if X = Y then Y ← X + R
  else X ← Y + R fi
done
```

iter.	X	Y	$X - Y$
0	0	$[-1, 1]$	$[-1, 1]$
1	$[-2, 2]$	$[-1, 1]$	$[-1, 1]$
2	$[-2, 2]$	$[-3, 3]$	$[-1, 1]$
...	...	...	...
$2j$	$[-2j, 2j]$	$[-2j-1, 2j+1]$	$[-1, 1]$
$2j+1$	$[-2j-2, 2j+2]$	$[-2j-1, 2j+1]$	$[-1, 1]$

Applying the closure after the widening at • prevents convergence.
Without the closure, we would find in finite time $X - Y \in [-1, 1]$.

Note: this situation also occurs in reduced products.

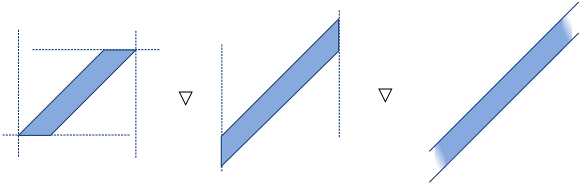
(here, $\mathcal{D}^\sharp$ $\simeq$reduced product of $n \times n$ intervals, $*$ $\simeq$reduction)

Interaction between closure and widening (illustration)

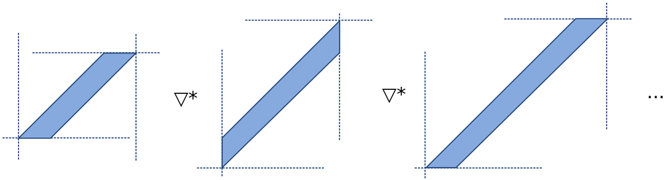
```
X ← 0; Y ← [-1,1];
while ● 1 = 1 do
  R ← [-1,1];
  if X = Y then Y ← X + R
  else X ← Y + R fi
done
```

iter.	X	Y	$X - Y$
0	0	$[-1, 1]$	$[-1, 1]$
1	$[-2, 2]$	$[-1, 1]$	$[-1, 1]$
2	$[-2, 2]$	$[-3, 3]$	$[-1, 1]$
...	...	...	...
$2j$	$[-2j, 2j]$	$[-2j - 1, 2j + 1]$	$[-1, 1]$
$2j + 1$	$[-2j - 2, 2j + 2]$	$[-2j - 1, 2j + 1]$	$[-1, 1]$

widening without closure



widening with closure



Octagon domain

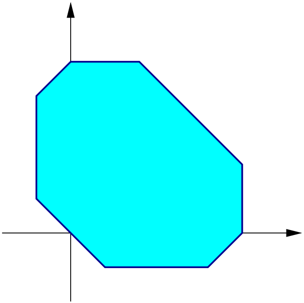
The octagon domain

Now, $\mathbb{I} \in \{\mathbb{Q}, \mathbb{R}\}$.

We look for invariants of the form: $\bigwedge \pm V_i \pm V_j \leq c, \quad c \in \mathbb{I}$

A subset of $\mathbb{I}^n$ defined by such constraints is called an octagon.

It is a generalisation of zones (more symmetric).



[Mine01b]

Machine representation

Idea: use a variable change to get back to potential constraints.

Let $\mathbb{V}' \stackrel{\text{def}}{=} \{V'_1, \ldots, V'_{2n}\}$.

the constraint:	is encoded as:
$V_i - V_j \leq c \quad (i \neq j)$	$V'_{2i-1} - V'_{2j-1} \leq c$ and $V'_{2j} - V'_{2i} \leq c$
$V_i + V_j \leq c \quad (i \neq j)$	$V'_{2i-1} - V'_{2j} \leq c$ and $V'_{2j-1} - V'_{2i} \leq c$
$-V_i - V_j \leq c \quad (i \neq j)$	$V'_{2j} - V'_{2i-1} \leq c$ and $V'_{2i} - V'_{2j-1} \leq c$
$V_i \leq c$	$V'_{2i-1} - V'_{2i} \leq 2c$
$V_i \geq c$	$V'_{2i} - V'_{2i-1} \leq -2c$

We use a matrix $\mathbf{m}$ of size $(2n) \times (2n)$ with elements in $\mathbb{I} \cup \{+\infty\}$ and $\gamma_\pm(\mathbf{m}) \stackrel{\text{def}}{=} \{\ (v_1, \ldots, v_n) \mid (v_1, -v_1, \ldots, v_n, -v_n) \in \gamma(\mathbf{m})\ \}$.

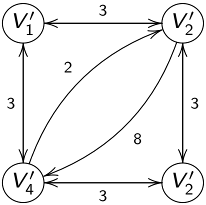
Note:
Two distinct $\mathbf{m}$ elements can represent the same constraint on $\mathbb{V}$.

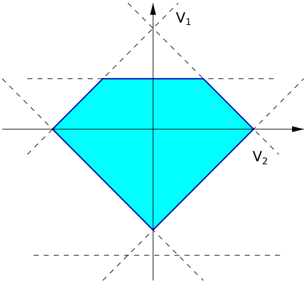
To avoid this, we impose that $\forall i, j,\ m_{ij} = m_{\bar{\jmath}\,\bar{\imath}}$ where $\bar{\imath} = i \oplus 1$.

Machine representation (cont.)

Example:

$$\begin{cases} V_1 + V_2 \leq 3 \\ V_2 - V_1 \leq 3 \\ V_1 - V_2 \leq 3 \\ -V_1 - V_2 \leq -3 \\ 2V_2 \leq 2 \\ -2V_2 \leq 8 \end{cases}$$





Lattice

Constructed by point-wise extension of $\leq$ on $\mathbb{I} \cup \{+\infty\}$.

Algorithms

$\mathbf{m}^*$ is not a normal form for $\gamma_\pm$.

Idea use two local transformations instead of one:

$$\left\{ \begin{array}{l} V_i' - V_k' \leq c \\ V_k' - V_j' \leq d \end{array} \right. \implies V_i' - V_j' \leq c + d$$

and

$$\left\{ \begin{array}{l} V_i' - V_{\bar{\imath}}' \leq c \\ V_{\bar{\jmath}}' - V_j' \leq d \end{array} \right. \implies V_i' - V_j' \leq (c + d)/2$$

Modified Floyd–Warshall algorithm

$\mathbf{m}^\bullet \stackrel{\text{def}}{=} S(\mathbf{m}^{2n+1})$

where:

(A) $\left\{ \begin{array}{l} \mathbf{m}^1 \stackrel{\text{def}}{=} \mathbf{m} \\ [\mathbf{m}^{k+1}]_{ij} \stackrel{\text{def}}{=} \min(n_{ij}, n_{ik} + n_{kj}),\ 1 \leq k \leq 2n \end{array} \right.$

(B) $[S(\mathbf{n})]_{ij} \stackrel{\text{def}}{=} \min(n_{ij}, (n_{i\,\bar{\imath}} + n_{\bar{\jmath}j})/2)$

Algorithms (cont.)

Applications

- $\gamma_\pm(\mathbf{m}) = \emptyset \iff \exists i,\ \mathbf{m}^\bullet_{ii} < 0$,
- if $\gamma_\pm(\mathbf{m}) \neq \emptyset$, $\mathbf{m}^\bullet$ is a normal form:
  $$\mathbf{m}^\bullet = \min_{\subseteq^\sharp} \{\ \mathbf{n} \mid \gamma_\pm(\mathbf{n}) = \gamma_\pm(\mathbf{m})\ \},$$
- $(\mathbf{m}^\bullet) \cup^\sharp (\mathbf{n}^\bullet)$ is the best abstraction for the set-union $\gamma_\pm(\mathbf{m}) \cup \gamma_\pm(\mathbf{n})$.

Widening and narrowing

- The zone widening and narrowing can be used on octagons.
- The widened iterates should not be closed.
  (prevents convergence)

Abstract transfer functions are similar to the case of the zone domain.

Analysis example

Rate limiter

```
Y ← 0; while • 1=1 do
  X ← [-128,128]; D ← [0,16];
  S ← Y; Y ← X; R ← X - S;
  if R ≤ -D then Y ← S - D fi;
  if R ≥ D then Y ← S + D fi
done
```

X:	input signal
Y:	output signal
S:	last output
R:	delta $Y - S$
D:	max. allowed for $\|R\|$

Analysis using:

- the octagon domain,
- an abstract operator for $V_{j_0} \leftarrow [a_0, b_0] + \sum_k [a_k, b_k] \times V_k$ similar to the one we defined on zones,
- a widening with thresholds T.

Result: we prove that $|Y|$ is bounded by: $\min \{ t \in T \mid t \geq 144 \}$.

Note: the polyhedron domain would find $|Y| \leq 128$ and does not require thresholds, but it is more costly.

Summary

Summary of numerical domains

domain	invariants	memory cost	time cost (per operation)
intervals	$V \in [\ell, h]$	$\mathcal{O}(\lvert n\rvert)$	$\mathcal{O}(\lvert n\rvert)$
linear equalities	$\sum_i \alpha_i V_i = \beta_i$	$\mathcal{O}(\lvert n\rvert^2)$	$\mathcal{O}(\lvert n\rvert^3)$
zones	$V_i - V_j \leq c$	$\mathcal{O}(\lvert n\rvert^2)$	$\mathcal{O}(\lvert n\rvert^3)$
polyhedra	$\sum_i \alpha_i V_i \geq \beta_i$	unbounded, exponential in practice	

- abstract domains provide trade-offs between cost and precision
- relational invariants are often necessary

  even to prove non-relational properties
- an abstract domain is defined by the choice of:
  - some properties of interest and operators *(semantic part)*
  - data-structures and algorithms *(algorithmic part)*
- an analysis mixes two kinds of approximations:
  - static approximations *(choice of abstract properties)*
  - dynamic approximations *(widening)*

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