Shape analysis abstractions

MPRI — Cours 2.6 “Interprétation abstraite : application à la vérification et à l’analyse statique”

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Shape analysis

Shape analyses aim at discovering structural invariants of programs that manipulate complex unbounded data-structures

Applications:

- establish **memory safety**
- verify the preservation of **structural properties**
  e.g., list, doubly-linked lists, trees, ...
- reason about programs that manipulate **unbounded** memory states

Previous course: **separation logic based shape analyses**

- **separating conjunction connector** $*$: ties properties that characterize **disjoint memory regions**
- also **many other connectors**:
  disjunctions, classical conjunctions, separating implication...
- can be turned into **an abstract domain**

Properties to verify: examples

A program closing a list of file descriptors

```
//l points to a list
c = l;
while(c ≠ NULL){
  close(c → FD);
  c = c → next;
}
```

Correctness properties

1. memory safety
2. `l` is supposed to store all file descriptors at all times
   will its structure be preserved ?
   yes, no breakage of a `next` link
3. closure of all the descriptors

Examples of structure preservation properties

- algorithms manipulating **trees**, **lists**...
- libraries of algorithms on **balanced trees**
- **not guaranteed by the language** !
  e.g., the balancing of `Maps` in the OCaml standard library was **incorrect** for years (performance bug)

On today's agenda

Another important family of shape analysis abstractions:

- **three valued logic** based abstraction
  maps predicates into "true", "false", "maybe" logical values
- can describe **memory states** (in this course)
  but also **other objects** (not in this course)
- useful **comparison** with separation logic based abstraction

Combination with value abstraction:

- so far, we have considered **pointer information only**
- real states also include **numerical** and **boolean values**, but also **strings** and others...
- **issue 1**: shape abstractions are very **dynamic**
  e.g., the scope of summaries varies during the analysis
- **issue 2**: exchange information between shape and value

Outline

Assumptions: syntax of programs

$\mathtt{l}$	$::=$	**l-valules**	
	$\mid$	$\mathtt{x}$	$(\mathtt{x} \in \mathbb{X})$
	$\mid$	$*\mathtt{e}$	pointer dereference
	$\mid$	$\mathtt{l} \cdot \mathtt{f}$	field read
			pointers, array dereference...
$\mathtt{e}$	$::=$	**expressions**	
	$\mid$	c	$(c \in \mathbb{V})$
	$\mid$	$\mathtt{l}$	(l-value)
	$\mid$	$\mathtt{e} \oplus \mathtt{e}$	(arith operation, comparison)
	$\mid$	$\&\mathtt{l}$	"address of" operator
$\mathtt{s}$	$::=$	**statements**	
	$\mid$	$\mathtt{l} = \mathtt{e}$	(assignment)
	$\mid$	$\mathtt{s}; \ldots \mathtt{s};$	(sequence)
	$\mid$	**if**$(\mathtt{e})\{\mathtt{s}\}$	(condition)
	$\mid$	**while**$(\mathtt{e})\{\mathtt{s}\}$	(loop)
	$\mid$	$\mathtt{x} =$ **malloc**(c)	allocation of c bytes
	$\mid$	**free**$(\mathtt{x})$	deallocation of the block pointed to by

Semantic domains

No one-to-one relation betwee memory cells and program variables

- a variable may correspond to **several cells** (structures...)
- **dynamically allocated cells** correspond to no variable at all...

Thus, we distinguish memory contents and variable addresses:

Environment + Heap

- **Addresses** are values: $\mathbb{V}_{\text{addr}} \subseteq \mathbb{V}$
- **Environments** $e \in \mathbb{E}$ map variables into their addresses
- **Heaps** ($h \in \mathbb{H}$) map addresses into values

$$\begin{array}{rcl} \mathbb{E} & = & \mathbb{X} \rightarrow \mathbb{V}_{\text{addr}} \\ \mathbb{H} & = & \mathbb{V}_{\text{addr}} \rightarrow \mathbb{V} \end{array}$$

  h is actually only a partial function
- **Memory states** (or **memories**): $\mathbb{M} = \mathbb{E} \times \mathbb{H}$

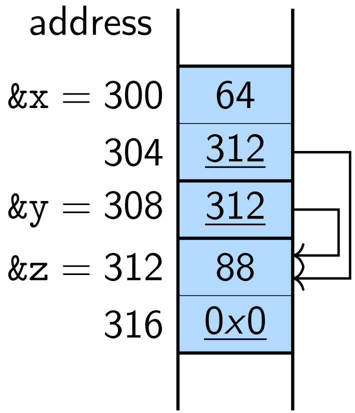
Note: **Avoid confusion between heap (function from addresses to values) and dynamic allocation space (often referred to as "heap")**

Example of a concrete memory state (variables)

- x and z are two list elements containing values 64 and 88, and where the former points to the latter
- y stores a pointer to z

Memory layout

(pointer values underlined)

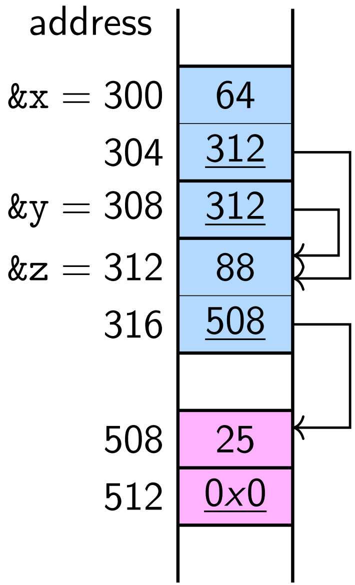


$$
\begin{array}{rlcl}
e : & \mathtt{x} & \mapsto & 300 \\
& \mathtt{y} & \mapsto & 308 \\
& \mathtt{z} & \mapsto & 312 \\
\\
h : & 300 & \mapsto & 64 \\
& 304 & \mapsto & 312 \\
& 308 & \mapsto & 312 \\
& 312 & \mapsto & 88 \\
& 316 & \mapsto & 0
\end{array}
$$

Example of a concrete memory state (variables + dyn. cell)

- same configuration
- + z points to a dynamically allocated list element (in purple)

Memory layout



$$
\begin{array}{rlcl}
e: & \mathtt{x} & \mapsto & 300 \\
 & \mathtt{y} & \mapsto & 308 \\
 & \mathtt{z} & \mapsto & 312 \\
 & & & \\
h: & 300 & \mapsto & 64 \\
 & 304 & \mapsto & 312 \\
 & 308 & \mapsto & 312 \\
 & 312 & \mapsto & 88 \\
 & 316 & \mapsto & 508 \\
 & 508 & \mapsto & 25 \\
 & 512 & \mapsto & 0
\end{array}
$$

Semantics of the pointer operations

Case of l-values: $[\![\mathtt{l}]\!] : \mathbb{M} \to \mathbb{V}_{\mathrm{addr}}$

$$
\begin{array}{rcl}
[\![\mathtt{x}]\!](e, h) & = & e(\mathtt{x}) \\
[\![*\mathtt{e}]\!](e, h) & = & \left\{ \begin{array}{ll} h([\![\mathtt{e}]\!](e, h)) & \text{if } [\![\mathtt{e}]\!](e, h) \neq 0 \wedge [\![\mathtt{e}]\!](e, h) \in \mathrm{Dom}(h) \\ \Omega & \text{otherwise} \end{array} \right. \\
[\![\mathtt{l} \cdot \mathtt{f}]\!](e, h) & = & [\![\mathtt{l}]\!](e, h) + \mathbf{offset}(\mathtt{f}) \text{ (numeric offset)}
\end{array}
$$

Case of expressions: $[\![\mathtt{e}]\!] : \mathbb{M} \to \mathbb{V}$, mostly unchanged

$$
\begin{array}{rcll}
[\![\mathtt{l}]\!](e, h) & = & h([\![\mathtt{l}]\!](e, h)) & \text{(evaluates into the contents)} \\
[\![\&\mathtt{l}]\!](e, h) & = & [\![\mathtt{l}]\!](e, h) & \text{(evaluates into the address)}
\end{array}
$$

Case of statements that are specific to memory operations:

- **memory allocation** $\mathtt{x} = \mathbf{malloc}(c)$: $(e, h) \to (e, h')$ where $h' = h[e(\mathtt{x}) \leftarrow k] \uplus \{k \mapsto v_k, k+1 \mapsto v_{k+1}, \ldots, k+c-1 \mapsto v_{k+c-1}\}$ and $k, \ldots, k+c-1$ are fresh and unused in h
- **memory deallocation** $\mathbf{free}(\mathtt{x})$: $(e, h) \to (e, h')$ where $k = e(\mathtt{x})$ and $h = h' \uplus \{k \mapsto v_k, k+1 \mapsto v_{k+1}, \ldots, k+c-1 \mapsto v_{k+c-1}\}$

Outline

Pointer non relational abstractions

Assumption on the set of values:

- $\mathbb{V} = \mathbb{V}_{\mathrm{addr}} \uplus \ldots$ and $\mathbb{X} = \mathbb{X}_{\mathrm{addr}} \uplus \ldots$
- pointer values ($\mathbb{V}_{\mathrm{addr}}$) describe (either symbolic or numerical) memory addresses
- base values may include integers and other base types
- **abstract cells** $\mathbb{C}^\sharp$ finitely summarize concrete cells, through a **fixed**

$$\phi : \mathbb{V}_{\mathrm{addr}} \longrightarrow \mathbb{C}^\sharp$$

- we apply a **non relational abstraction**:

Non relational pointer abstraction

- Set of **pointer abstract values** $\mathbb{D}^\sharp_{\mathrm{ptr}}$
- **Concretization** $\gamma_{\mathrm{ptr}} : \mathbb{D}^\sharp_{\mathrm{ptr}} \to \mathcal{P}(\mathbb{V}_{\mathrm{addr}})$ into pointer sets
- **Abstract memory states** of the form $\mathbb{C}^\sharp \to \mathbb{D}^\sharp_{\mathrm{ptr}}$ with $\gamma(m^\sharp) = \{(e, h) \mid \forall \mathrm{p} \in \mathbb{X}_{\mathrm{addr}},\ h(e(\mathrm{p})) \in \gamma_{\mathrm{ptr}} \circ m^\sharp \circ \phi(e(\mathrm{p})))\}$

Pointer non relational abstraction: null pointers

The dereference of a null pointer will cause a crash

To establish **safety**: compute **which pointers may be null**

Null pointer analysis

Abstract domain for addresses:

- $\gamma_{\mathrm{ptr}}(\bot) = \emptyset$
- $\gamma_{\mathrm{ptr}}(\top) = \mathbb{V}_{\mathrm{addr}}$
- $\gamma_{\mathrm{ptr}}(\neq \mathtt{NULL}) = \mathbb{V}_{\mathrm{addr}} \setminus \{0\}$

$$\begin{array}{c}\top\\ |\\ \neq \mathtt{NULL}\\ |\\ \bot\end{array}$$

- we may also use a lattice with a fourth element $= \mathtt{NULL}$
  exercise: what do we gain using this lattice ?
- very **lightweight**, can typically resolve rather trivial cases
- useful for **C**, but also for **Java**
- we can define very similar abstractions to deal with dangling or invalid pointers

Pointer non relational abstraction: points-to sets

Determine where a pointer may store a reference to

```
1 :   int x, y;
2 :   int * p;
3 :   y = 9;
4 :   p = &x;
5 :   *p = 0;
```

- what is the final value for x ?
  0, since **it is modified at line 5**...
- what is the final value for y ?
  9, since **it is not modified at line 5**...

Basic pointer abstraction

- We assume a set of **abstract memory locations** $\mathbb{A}^\sharp$ is fixed:
  $$\mathbb{A}^\sharp = \{\&\mathtt{x}, \&\mathtt{y}, \ldots, \&\mathtt{t}, a_0, a_1, \ldots, a_N\}$$
- **Concrete addresses** are **abstracted into** $\mathbb{A}^\sharp$ by $\phi_{\mathbb{A}} : \mathbb{A} \to \mathbb{A}^\sharp \uplus \{\top\}$
- A pointer value is abstracted by the abstraction of the addresses it may point to, *i.e.*, $\mathbb{D}^\sharp_{\mathrm{ptr}} = \mathcal{P}(\mathbb{A}^\sharp)$
  and $\gamma_{\mathrm{ptr}}(a^\sharp) = \{a \in \mathbb{A} \mid \phi_{\mathbb{A}}(a) = a^\sharp\}$

- **example:** p may point to $\{\&\mathtt{x}\}$

Points-to sets computation example

Example code:

```
1 :  int x, y;
2 :  int * p;
3 :  y = 9;
4 :  p = &x;
5 :  *p = 0;
6 :  ...
```

Abstract locations: $\{\&x, \&y, \&p\}$

Analysis results:

	&x	&y	&p
1	$\top$	$\top$	$\top$
2	$\top$	$\top$	$\top$
3	$\top$	$\top$	$\top$
4	$\top$	$[9, 9]$	$\top$
5	$\top$	$[9, 9]$	$\{\&x\}$
6	$[0, 0]$	$[9, 9]$	$\{\&x\}$

Points-to sets computation and imprecision

```
   x ∈ [−10, −5]; y ∈ [5, 10]
1 : int * p;
2 : if(?){
3 :       p = &x;
4 : } else {
5 :       p = &y;
6 : }
7 : *p = 0;
8 : ...
```

	&x	&y	&p
1	$[-10,-5]$	$[5,10]$	$\top$
2	$[-10,-5]$	$[5,10]$	$\top$
3	$[-10,-5]$	$[5,10]$	$\top$
4	$[-10,-5]$	$[5,10]$	$\{\&x\}$
5	$[-10,-5]$	$[5,10]$	$\top$
6	$[-10,-5]$	$[5,10]$	$\{\&y\}$
7	$[-10,-5]$	$[5,10]$	$\{\&x, \&y\}$
8	$[-10,0]$	$[0,10]$	$\{\&x, \&y\}$

- What is the final range for x ?
- What is the final range for y ?

Abstract locations: $\{\&x, \&y, \&p\}$

Imprecise results

- The abstract information about both x and y are weakened
- The fact that $x \neq y$ is lost

Weak-updates

As in array analysis, we encounter:

Weak updates

- **The modified concrete cell cannot be uniquely mapped into a well identified abstract cell that describes only it**
- The resulting abstract information is obtained by **joining the new value and the old information**

Effect in pointer analysis, in the case of an **assignment**:

- if the points-to set contains **exactly one element**, the analysis can perform a **strong update**
- if the points-to set may contain **more than one element**, the analysis needs to perform a **weak-update**

Consequence: **weak updates cause severe losses in precision**

Previous course about memory abstraction: separation logic

Key idea:

Avoid weak updates by localizing memory accesses (read or write) in a very precise manner, and with no ambiguity

Logical items:

- **separating conjunction connector**:
  logically, splits the memory into two **disjoint** regions
- **basic predicates**, to describe individual cells
- **inductive summary predicates**, that describe unbouned memory regions

Main algorithms:

- **unfolding**: to refine summary predicates
- **folding**: to synthesize summary predicates

Today: compare separation logic with another shape abstraction
and augment shape analysis to describe value properties

Outline

1. Introduction
2. Setup (reminder)
3. Shape analysis in Three-Valued Logic (TVL)
   - Principles of Three-Valued Logic based abstraction
   - Comparing and concretizing Three-Valued Logic abstractions
   - Weakening Three-Valued Logic abstractions
   - Transfer functions
   - Focusing
   - Comparing Separation Logic and Three-Valued logic abstractions
4. Combining shape and value abstractions
5. Conclusion

TVLA Analysis → 1995 ≃ before SL

Representation of memory states: memory graphs

Observation: representation of memory states by graphs

- **Nodes** (aka, atoms) denote **variables, memory locations**
- **Edges** denote **properties of addresses / pointers**, such as:
  - "field `f` of location u points to v"
  - "variable `x` is stored at location u"

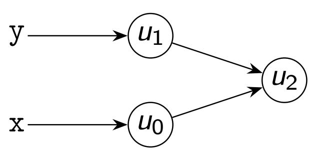
graphs

table of predicates

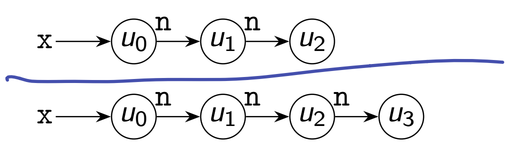
- This representation is also relevant in the case of **separation logic** based shape abstraction

A couple of examples:

Two alias pointers:



A list of length 2 or 3:



We need to over-approximate **sets of shape graphs**

Memory graphs and predicates: variables

Before we apply some abstraction, we **formalize memory graphs** using some **predicates**, such as:

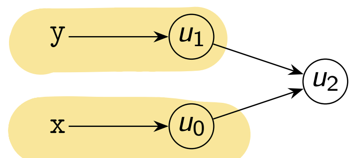
0: False
1: true

"Variable content" predicate

We note $\mathtt{x}(u) = 1$ if node u represents the contents of $\mathtt{x}$.

Examples:

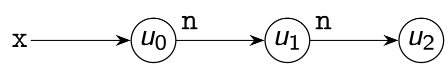
- **Two alias pointers**:



$x(u_2) = 0$
$x(u_0) = 0$

  Then, we have $\mathtt{x}(u_0) = 1$ and $\mathtt{y}(u_1) = 1$, and $\mathtt{x}(u) = 0$ (*resp.*, $\mathtt{y}(u) = 0$) in all the other cases

- **A list of length 2**:



  Then, we have $\mathtt{x}(u_0) = 1$ and $\mathtt{x}(u) = 0$ in all the other cases

Memory graphs and predicates: (field) pointers

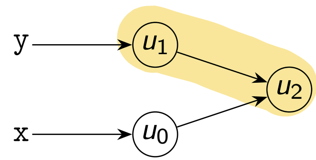
"Field content pointer" predicate

- We note $\mathtt{n}(u, v)$ if the field $\mathtt{n}$ of u stores a pointer to v
- We note $\underline{0}(u, v)$ if u stores a pointer to v (base address field is at offset 0)

Examples:

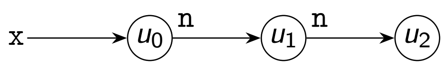
$\underline{0}(u_0, u_1) \overset{?}{=} 0$

- **Two alias pointers**:



Then, we have $\underline{0}(u_0, u_2) = 1$ and $\underline{0}(u_1, u_2) = 1$, and $\underline{0}(u, v) = 0$ in all the other cases

- **A list of length 2**:



Then, we have $\mathtt{n}(u_0, u_1) = 1$ and $\mathtt{n}(u_1, u_2) = 1$, and $\mathtt{n}(u, v) = 0$ in all the other cases

2-structures and conretization

We can represent the memory graphs using **tables of predicate values**:

Two-structures and concretization

We assume a set $\mathcal{P} = \{p_0, p_1, \ldots, p_n\}$ of **predicates** (we write k_i for the arity of predicate p_i). A formal representation of a memory graph is a **2-structure** $(\mathcal{U}, \phi) \in \mathbb{D}_2^\sharp$ defined by:

arity

- a set $\mathcal{U} = \{u_0, u_1, \ldots, u_m\}$ of **atoms**
- a **truth table** ϕ such that $\phi(p_i, u_{l_1}, \ldots, u_{l_{k_i}})$ denotes the truth value of p_i for $u_{l_1}, \ldots, u_{l_{k_i}}$

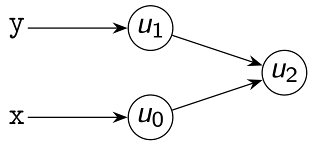
Then, $\gamma_2(\mathcal{U}, \phi)$ is the set of (e, h, ν) where $\nu : \mathcal{U} \to \mathbb{V}_{\mathrm{addr}}$ and that satisfy exactly the truth tables defined by ϕ:

- (e, h, ν) satisfies $\mathtt{x}(u)$ iff $e(\mathtt{x}) = \nu(u)$
- (e, h, ν) satisfies $\mathtt{f}(u, v)$ iff $h(\nu(u), \mathtt{f}) = \nu(v)$

- the name "two-structure" will become clear (very) soon
- the set of two-structures is parameterized by the data of a set of predicates $\mathtt{x}(.), \mathtt{y}(.), \underline{0}(., .), \mathtt{n}(., .)$ (additional predicates will be added soon...)

Examples of two-structures

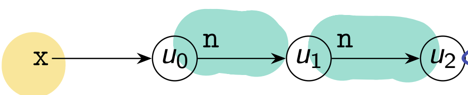
Two alias pointers:



	x	y
u_0	1	0
u_1	0	1
u_2	0	0

$\mapsto$	u_0	u_1	u_2
u_0	0	0	1
u_1	0	0	1
u_2	0	0	0

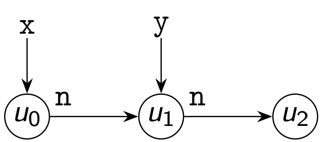
A list of length 2:



	x
u_0	1
u_1	0
u_2	0

$\cdot\mathrm{n} \mapsto$	u_0	u_1	u_2
u_0	0	1	0
u_1	0	0	1
u_2	0	0	0

A list of length 2:



	x	y
u_0	1	0
u_1	0	1
u_2	0	0

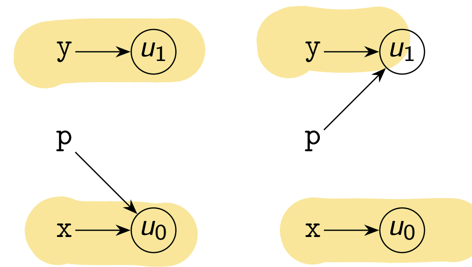
$\cdot\mathrm{n} \mapsto$	u_0	u_1	u_2
u_0	0	1	0
u_1	0	0	1
u_2	0	0	0

Lists of **arbitrary length** ? More on this **later**

Unknown value: three valued logic

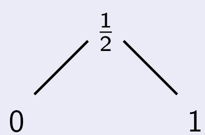
How to abstract away some information ?

i.e., how to abstract several graphs into one ?
Example: pointer variable p alias with x or y

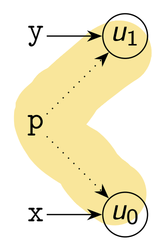


A boolean lattice

- Use **predicate tables**
- Add a $\top$ boolean value;
  (denoted to by $\frac{1}{2}$ in TVLA papers)



- **Graph representation**: **dotted edges**
- **Abstract graph**:



Summary nodes

At this point, we cannot talk about **unbounded memory states** with **finitely many** nodes, since one node represents at most one memory cell

An idea

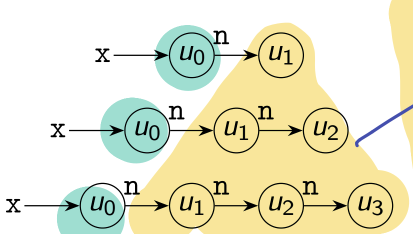
- Choose a node to represent **several** concrete nodes
- Similar to **smashing** of arrays using segments

Definition: summary node

A **summary node** is an atom that may denote several concrete atoms

- intuition: we are using a **non injective function** $\phi_{\mathbb{A}} : \mathbb{A} \longrightarrow \mathbb{A}^\sharp$
- representation: double circled nodes

Lists of lengths 1, 2, 3:



$n(u_0,u_1)=1$

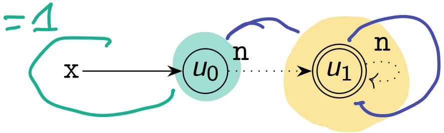
$n(u_0,u_2)=0$

$u_1 \to u_2$

$u_1 \not\to u_3$

Attempt at a **summary** graph:

$x(u_0)=1$



$\frac{1}{2}$

- Edges to u_1 are dotted

Additional graph predicate: sharing

We now define a few **higher level predicates** based on the previously seen **atomic predicates** describing the graphs.

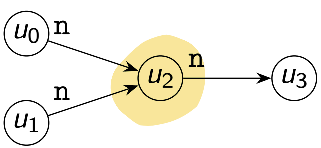
Example: a cell is **shared** if and only if there exists several distinct pointers to it

"Is shared" predicate

The predicate $\underline{\mathrm{sh}}(u)$ holds if and only if

$$\exists v_0, v_1, \left\{ \begin{array}{ll} & v_0 \neq v_1 \\ \wedge & \mathtt{n}(v_0, u) \\ \wedge & \mathtt{n}(v_1, u) \end{array} \right.$$

(for concision, we assume only $\mathtt{n}$ pointers)



- $\underline{\mathrm{sh}}(u_0) = \underline{\mathrm{sh}}(u_1) = \underline{\mathrm{sh}}(u_3) = 0$
- $\underline{\mathrm{sh}}(u_2) = 1$

Additional graph predicate: reachability

We can also define higher level predicates **using induction**:

For instance, a cell is **reachable** from u if and only it is u or it is reachable from a cell pointed to by u.

"Reachability" predicate

The predicate $\underline{\mathrm{r}}(u, v)$ holds if and only if:

$$\begin{array}{rl} & u = v \\ \vee & \exists u_0,\ \mathtt{n}(u, u_0) \wedge \underline{\mathrm{r}}(u_0, v) \end{array}$$

(for concision, we assume only $\mathtt{n}$ pointers)

$$\mathtt{x} \longrightarrow u_0 \xrightarrow{\mathtt{n}} u_1 \xrightarrow{\mathtt{n}} u_2 \xrightarrow{\mathtt{n}} u_3$$

- $\underline{\mathrm{r}}(u_1, u_0) = \underline{\mathrm{r}}(u_2, u_0) = \underline{\mathrm{r}}(u_3, u_1) = 0$
- $\underline{\mathrm{r}}(u_0, u_0) = \underline{\mathrm{r}}(u_0, u_2) = \underline{\mathrm{r}}(u_0, u_3) = 1$

"Acyclicity" predicate

Exercise

The predicate $\underline{\mathrm{acy}}(u)$ holds if and only if $\underline{\mathrm{r}}(u, u)$ does not hold

Three structures

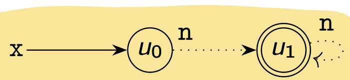
As for 2-structures, we assume a set $\mathcal{P} = \{p_0, p_1, \ldots, p_n\}$ of **predicates** fixed and write k_i for the arity of predicate p_i.

Definition: 3-structures

A **3-structure** is a tuple $(\mathcal{U}, \phi)$ defined by:

- a set $\mathcal{U} = \{u_0, u_1, \ldots, u_m\}$ of **atoms**
- a **truth table** ϕ such that $\phi(p_i, u_{l_1}, \ldots, u_{l_{k_i}})$ denotes the truth value of p_i for $u_{l_1}, \ldots, u_{l_{k_i}}$
  note: truth values are elements of the lattice $\{0, \frac{1}{2}, 1\}$

We write $\mathbb{D}_3^\sharp$ for the set of three-structures.



$$\left\{ \begin{array}{l} \mathcal{U} = \{u_0, u_1\} \\ \mathcal{P} = \{\mathrm{x}(\cdot), \mathrm{n}(\cdot,\cdot), \underline{\mathrm{sum}}(\cdot)\} \end{array} \right.$$

	x	$\underline{\text{sum}}$
u_0	1	0
u_1	0	$\frac{1}{2}$

n	u_0	u_1
u_0	0	1
u_1	0	0

$\frac{1}{2}$

$\frac{1}{2}$

In the following we build up an abstract domain of 3-structures (but a bit more work is need for the definition of the concretization)

Main predicates and concretization

We have already seen:

$\mathtt{x}(u)$	variable $\mathtt{x}$ contains the address of u
$\mathtt{n}(u, v)$	field of u points to v
$\underline{\mathrm{sum}}(u)$	whether u is a summary node (convention: either 0 or $\frac{1}{2}$)
$\underline{\mathrm{sh}}(u)$	whether there exists several distinct pointers to u
$\underline{\mathrm{r}}(u, v)$	whether v is reachable starting from u
$\underline{\mathrm{acy}}(v)$	v may not be on a cycle

Concretization for 2 structures:

$$(e, \mathcal{h}, \nu) \in \gamma_2(\mathcal{U}, \phi) \quad \Longleftrightarrow \quad \bigwedge_{p \in \mathcal{P}} (env, \mathcal{h}, \nu) \text{ evaluates } p \text{ as specified in } \phi$$

Concretization for 3 structures:

- predicates with value $\frac{1}{2}$ may concretize either to true or to false
- but the concretization of summary nodes is still unclear...

Outline

Embedding

Reasons why we need to set up a **relation among structures**:

- learn how to **compare** two 3-structures
- describe the **concretization** of 3-structures into 2-structures

The embedding principle

Let $\mathcal{S}_0 = (\mathcal{U}_0, \phi_0)$ and $\mathcal{S}_1 = (\mathcal{U}_1, \phi_1)$ be two three structures, with the same sets of predicates $\mathcal{P}$. Let $f : \mathcal{U}_0 \to \mathcal{U}_1$, surjective.

We say that f **embeds** $\mathcal{S}_0$ **into** $\mathcal{S}_1$ iff

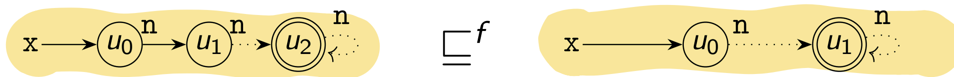
$$\text{for all predicate } p \in \mathcal{P} \text{ of arity } k, \quad \text{for all } u_{l_1}, \ldots, u_{l_{k_i}} \in \mathcal{U}_0,$$
$$\phi_0(u_{l_1}, \ldots, u_{l_{k_i}}) \sqsubseteq \phi_1(f(u_{l_1}), \ldots, f(u_{l_{k_i}}))$$

Then, **we write** $\mathcal{S}_0 \sqsubseteq^f \mathcal{S}_1$ $\sqsubseteq$

- Note: we use the order $\sqsubseteq$ of the lattice $\{0, \frac{1}{2}, 1\}$
- Intuition: **embedding** defines an **abstract pre-order**
  i.e., when $\mathcal{S}_0 \sqsubseteq^f \mathcal{S}_1$, any property that is satsfied by $\mathcal{S}_0$ is also satisfied by $\mathcal{S}_1$

Embedding examples

A few examples of the embedding relation:

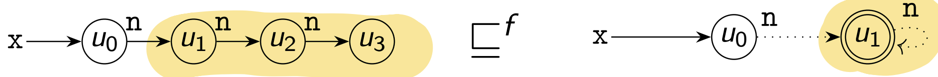


where $f : u_0 \mapsto u_0; u_1 \mapsto u_1; u_2 \mapsto u_1$

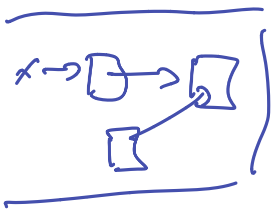
$n(u_1, u_2) = 1$

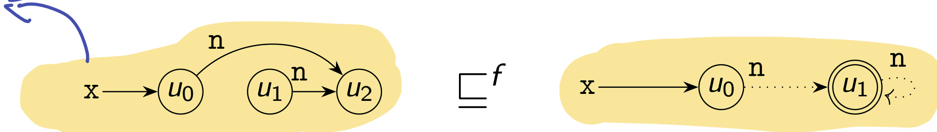
$n(u_1, u_1) = \frac{1}{2}$

$\exists 1$



where $f : u_0 \mapsto u_0; u_1 \mapsto u_1; u_2 \mapsto u_1; u_3 \mapsto u_1$





where $f : u_0 \mapsto u_0; u_1 \mapsto u_1; u_2 \mapsto u_1$

The last example shows summary nodes are not enough to capture just lists:

- **reachability** would be necessary to constrain it be a list
- alternatively: list cells **should not be shared**

Concretization of three-structures

Intuitions:

- concrete memory states correspond to 2-structures
- embedding applies uniformally to 2-structures and 3-structures (in fact, 2-structures are a subset of 3-structures)
- 2-structures can be embedded into 3-structures, that abstract them

This suggests **a concretization of 3-structures in two steps**:

1. turn it into a set of 2-structures that can be embedded into it
2. concretize these 2-structures

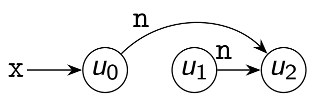
Concretization of 3-structures

Let $\mathcal{S}$ be a 3-structure. Then:

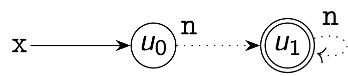
$$\gamma_3(\mathcal{S}) = \bigcup \{\gamma_2(\mathcal{S}') \mid \mathcal{S}' \text{ **2-structure s.t.** } \exists f, \mathcal{S}' \sqsubseteq^f \mathcal{S}\}$$

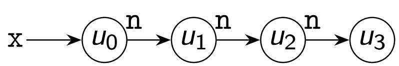
Concretization examples

Without reachability:

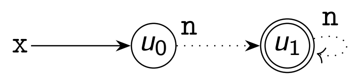


$\sqsubseteq^f$



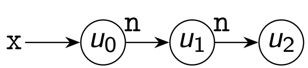


$\sqsubseteq^f$

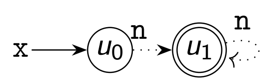


where $f : u_0 \mapsto u_0; u_1 \mapsto u_1; u_2 \mapsto u_1; u_3 \mapsto u_1$

With reachability:



$\sqsubseteq^f$



$\underline{\mathtt{r}}(u_0, u_1)$

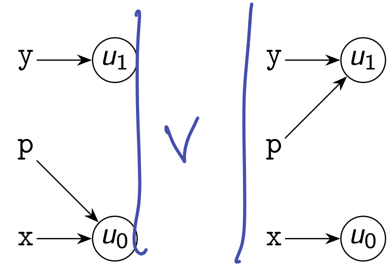
where $f : u_0 \mapsto u_0; u_1 \mapsto u_1; u_2 \mapsto u_1$

Disjunctive completion

- Do 3-structures allow for a **sufficient level of precision** ?
- How to **over-approximate a set of 2-structures** ?

```
int * x; int * y; ...
int * p = NULL;
if(...){
    p = x;
}else{
    p = y;
}
printf("%d", *p);
*p = ...;
```

After the if statement:
abstracting would be imprecise



Abstraction based on disjunctive completion

- In the following, we use **partial disjunctive completion**
  i.e., TVLA manipulates **finite disjunctions** of 3-structures
  We write $\mathbb{D}^\sharp_{\mathcal{P}(3)}$ for the abstract domain made of finite sets of 3-structures in $\mathbb{D}^\sharp_3$
- How to ensure disjunctions **will not grow infinite** ?
  the set of atoms is **unbounded**, so it is not necessarily true!

Outline

Canonical abstraction

To **prevent disjunctions** from growing infinite, we propose to **normalize (in a precision losing manner)** abstract states:

- the analysis may use all 3-structures at most points
- at selected points (including loop heads), only 3-structures in a finite set $\mathbb{D}^{\sharp}_{\mathsf{can}(3)}$ are allowed
- there is a function to coarsen 3-structures into elements of $\mathbb{D}^{\sharp}_{\mathsf{can}(3)}$

Canonicalization function

Let $\mathcal{L}$ be a lattice, $\mathcal{L}' \subseteq \mathcal{L}$ be a finite sub-lattice and $\mathsf{can} : \mathcal{L} \to \mathcal{L}'$:

- operator can is called **canonicalization** if and only if it defines an **upper closure operator**
- then it extends into a **canonicalization operator** $\mathsf{can} : \mathcal{P}(\mathcal{L}) \to \mathcal{P}(\mathcal{L}')$ for the disjunctive completion domain:

$$\mathsf{can}(\mathcal{E}) = \{\mathsf{can}(x) \mid x \in \mathcal{E}\}$$

- proof of the extension two disjunctive completion domains: left as an exercise
- to make the powerset domain work, we simply need a can over 3-structures

Canonical abstraction

Definition of a finite lattice $\mathbb{D}^{\sharp}_{\mathrm{can}(3)}$

We partition the set of predicates $\mathcal{P}$ into two subsets $\mathcal{P}_a$ and $\mathcal{P}_o$:

- $\mathcal{P}_a$ and defines **abstraction predicates** and should contains only unary predicates and have a finite truth table whatever the number of atoms
- $\mathcal{P}_o$ denotes **non-abstraction predicates**, and may define truth tables of unbounded size

Then, we let $\mathbb{D}^{\sharp}_{\mathrm{can}(3)}$ be the set of 3-structures such that **no pair of atoms have the same value of the $\mathcal{P}_a$ predicates**. It defines a finite set of 3-structures.

This sub-lattice defines a clear "canonicalization" algorithm:

Canonical abstraction by truth blurring

1. **Identify** nodes that **have different abstraction predicates**
2. When several nodes have the **same abstraction predicate introduce a summary node**
3. **Compute new predicate values** by doing a **join over truth values**

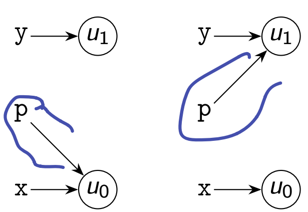
Canonical abstraction examples

Most common TVLA instantiation:

- ae assume there are n variables $\mathtt{x}_1, \ldots, \mathtt{x}_n$ **thus the number of unary predicates is finite**, and provides a good choice for $\mathcal{P}_a$
- **sub-lattice**: structures with atoms **distinguished by the values of the unary predicates** $\mathtt{x}_1, \ldots, \mathtt{x}_n$

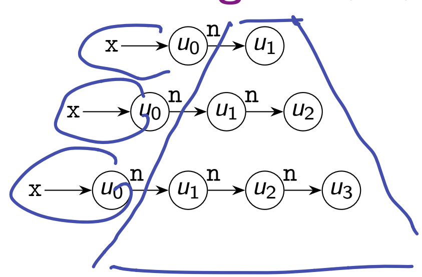
Examples:

Elements not merged:

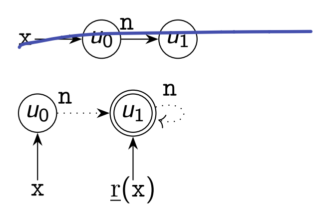


Elements merged:

Lists of lengths 1, 2, 3:



Abstract into:



Outline

Principle for the design of sound transfer functions

- Intuitively, **concrete states** correspond to **2-structures**
- The **analysis** should track **3-structures**, thus the analysis and its soundness proof need to **rely on the embedding relation**

Embedding theorem

We assume that

- $\mathcal{S}_0 = (\mathcal{U}_0, \phi_0)$ and $\mathcal{S}_1 = (\mathcal{U}_1, \phi_1)$ define a pair of 3-structures
- $f : \mathcal{U}_0 \rightarrow \mathcal{U}_1$, is such that $\mathcal{S}_0 \sqsubseteq^f \mathcal{S}_1$ (embedding)
- Ψ is a logical formula, with variables in X
- $g : X \rightarrow \mathcal{U}_0$ is an assignment for the variables of Ψ

Then, the semantics (evaluation) of logical formulae is such that

$$[\![\Psi_{|g}]\!](\mathcal{S}_0) \sqsubseteq [\![\Psi_{|f \circ g}]\!](\mathcal{S}_1)$$

Intuition: this theorem ties the evaluation of conditions in the concrete and in the abstract in a general manner

Principle for the design of sound transfer functions

Transfer functions for static analysis

- **Semantics of concrete statements is encoded into boolean formulas**
- **Evaluation in the abstract is sound (embedding theorem)**

Example: analysis of an assignment $\mathrm{y} := \mathrm{x}$

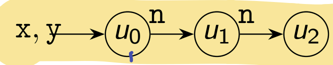
1. let y' be a new predicate that denotes the *new* value of y
2. then we can add the constraint $\mathrm{y}'(u) = \mathrm{x}(u)$
   (using the embedding theorem to prove soundness)
3. rename y' into y

Advantages:

- **abstract transfer functions** derive directly from the concrete transfer functions **(intuition: $\alpha \circ f \circ \gamma$...)**
- the same solution works for **weakest pre-conditions**

Disadvantage: precision will require some care, more on this later!

Assignment: a simple case

Statement $\ell_0 : \mathtt{y = y -> n}; \ell_1 : \ldots$	**Pre-condition** $\mathcal{S}$ 

Transfer function computation:

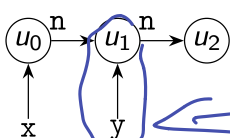
- it should produce an over-approximation of $\{m_1 \in \mathbb{M} \mid (\ell_0, m_0) \rightarrow (\ell_1, m_1)\}$
- **encoding** using **"primed predicates"** to denote predicates **after** the evaluation of the assignment, to evaluate them in the same structure (non primed variables are removed afterwards and primed variables renamed):

x id

new y   y'

n id

$$\boxed{\begin{array}{rcl} \mathtt{x}'(u) & = & \mathtt{x}(u) \\ \mathtt{y}'(u) & = & \exists v,\ \mathtt{y}(v) \wedge \mathtt{n}(v, u) \\ \mathtt{n}'(u, v) & = & \mathtt{n}(u, v) \end{array}}$$

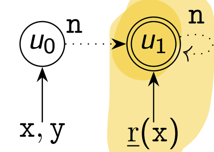
- **resulting structure:** 

$y'(u_1) = \exists v,$ ——
$= 1$
$v = u_0$

This is exactly the expected result

Outline

Assignment: a more involved case

Statement $\ell_0 : \mathtt{y} = \mathtt{y} \mathtt{->} \mathtt{n};\ \ell_1 : \ldots$	**Pre-condition** $\mathcal{S}$ 

- Let us try to **resolve the update in the same way as before**:

$$\begin{array}{rcl} \mathtt{x}'(u) & = & \mathtt{x}(u) \\ \mathtt{y}'(u) & = & \exists v,\ \mathtt{y}(v) \wedge \mathtt{n}(v, u) \\ \mathtt{n}'(u, v) & = & \mathtt{n}(u, v) \end{array}$$

- We **cannot resolve** $\mathtt{y}'$:

$$\left\{ \begin{array}{rcl} \mathtt{y}'(u_0) & = & 0 \\ \mathtt{y}'(u_1) & = & \frac{1}{2} \end{array} \right.$$

  Imprecision: after the statement, y may point to anywhere in the list, save for the first element...

- The assignment transfer function **cannot be computed immediately**
- **We need to refine the 3-structure first**

Focus

Focusing on a formula

We assume a 3-structure $\mathcal{S}$ and a boolean formula f are given, we call a **focusing $\mathcal{S}$ on f** the generation of a set $\hat{\mathcal{S}}$ of 3-structures such that:

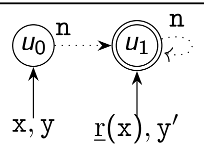
- f **evaluates to** 0 **or** 1 on all elements of $\hat{\mathcal{S}}$
- **precision was gained**: $\forall \mathcal{S}' \in \hat{\mathcal{S}},\ \mathcal{S}' \sqsubseteq \mathcal{S}$ (embedding)
- **soundness is preserved**: $\gamma(\mathcal{S}) = \bigcup\{\gamma(\mathcal{S}') \mid \mathcal{S}' \in \hat{\mathcal{S}}\}$

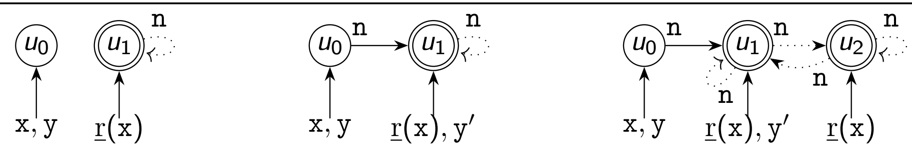
- Details of focusing algorithms are rather complex: not detailed here
- They involve splitting of summary nodes, solving of boolean constraints

Example: focusing on

$$\mathrm{y}'(u) \;=\; \exists v,\ \mathrm{y}(v) \wedge \mathrm{n}(v, u)$$

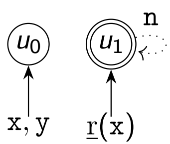
We obtain (we show y and y'):

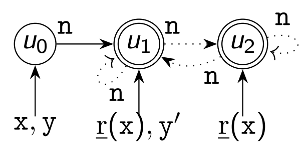




Focus and coerce

Some of the 3-structures generated by focus are not precise





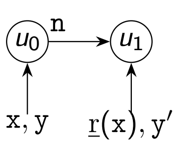
u_1 is reachable from x, but there is no sequence of n fields: this structure has **empty concretization**

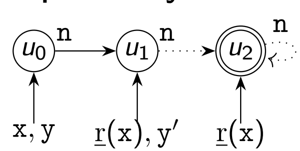
u_0 has an n-field to u_1 so u_1 denotes a unique atom and **cannot be a summary node**

Coerce operation

It **enforces logical constraints** among predicates and discards 3-structures with an empty concretization

Result: one case removed (bottom), two possibly summary nodes non summary





Focus, transfer, abstract...

Computation of a transfer function

We consider a transfer function encoded into boolean formula f

$$
\begin{tikzcd}
\mathcal{S}^\sharp_{\mathrm{pre}} \arrow[d, "\text{focus}\ \text{coerce}"'] & & \mathcal{S}^\sharp_{\mathrm{post}} \\
\hat{\mathcal{S}}_{\mathrm{pre}} \arrow[rr, "f"] & & \hat{\mathcal{S}}_{\mathrm{post}} \arrow[u, "\text{can}"']
\end{tikzcd}
$$

SL unfolding

SL Folding as part $\sqcup$ $\triangledown$

embedding → SL log. comp

Soundness proof steps:

1. **sound encoding of the semantics of program statements into formulas** (typically, no loss of precision at this stage)
2. **focusing** produces a **refined** over-approximation (disjunction)
3. **canonicalization over-approximates graphs** (truth blurring)

A common picture in shape analysis

Shape analysis with three valued logic

Abstract states; **two abstract domains** are used:

- **infinite domain** $\mathbb{D}^\sharp_{\mathcal{P}(3)}$: finite disjunctions of 3-structures in $\mathbb{D}^\sharp_3$ for general abstract computations
- **finite domain** $\mathbb{D}^\sharp_{\mathcal{P}(\mathsf{can}(3))}$: disjunctions of finite domain $\mathbb{D}^\sharp_{\mathsf{can}(3)}$ to simplify abstract states and for loop iteration
- **concretization** via $\mathbb{D}^\sharp_2$

Abstract post-conditions:

1. start from $\mathbb{D}^\sharp_{\mathcal{P}(3)}$ or $\mathbb{D}^\sharp_{\mathsf{can}(3)}$
2. focus and coerce when needed
3. apply the concrete transformation
4. apply can to weaken abstract states; result in $\mathbb{D}^\sharp_{\mathcal{P}(\mathsf{can}(3))}$

Analysis of loops:

- iterations in $\mathbb{D}^\sharp_{\mathcal{P}(\mathsf{can}(3))}$ terminate, as it is finite

Outline

Separation logic

Separation logic formulas (main connectors only)

$$
\begin{array}{rcl}
\mathtt{F} & ::= & \mathsf{emp} \\
& | & \mathtt{TRUE} \\
& | & \mathtt{l} \mapsto \mathtt{l} \\
& | & \mathtt{F}_0 * \mathtt{F}_1 \\
& | & \mathtt{F}_0 \wedge \mathtt{F}_1 \\
& | & \mathtt{F}_0 -\!\!* \mathtt{F}_1
\end{array}
$$

Concretization:

$$
\begin{array}{rcl}
\gamma(\mathsf{emp}) & = & \mathbb{E} \times \{[]\} \\
\gamma(\mathtt{TRUE}) & = & \mathbb{E} \times \mathbb{H} \\
\gamma(\mathtt{l} \mapsto \mathtt{v}) & = & \{(e, [[\![\mathtt{l}]\!](e, h) \mapsto \mathtt{v}]) \mid e \in \mathbb{E}\} \\
\gamma(\mathtt{F}_0 * \mathtt{F}_1) & = & \{(e, h_0 \circledast h_1) \mid (e, h_0) \in \gamma(\mathtt{F}_0) \wedge (e, h_1) \in \gamma(\mathtt{F}_1)\} \\
\gamma(\mathtt{F}_0 \wedge \mathtt{F}_1) & = & \gamma(\mathtt{F}_0) \cap \gamma(\mathtt{F}_1) \\
\gamma(\mathtt{F}_0 -\!\!* \mathtt{F}_1) & = & \text{exercise}
\end{array}
$$

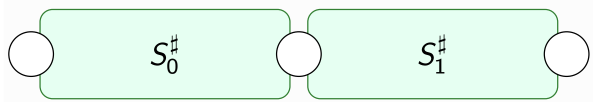
Program reasoning: **frame rule** and **strong updates**

Shape graphs and separation logic

Shape graphs: provide an efficient data-structure to describe a **subset** of separation logic predicates, and do static analysis with them.

Important addition: **inductive predicates**.

Semantic preserving translation Π of graphs into separation logic formulas:

Graph $S^\sharp \in \mathbb{D}^\sharp_{\mathrm{sh}}$	Translated formula $\Pi(S^\sharp)$
$\alpha \xrightarrow{\mathtt{f}} \beta$	$\alpha \cdot \mathtt{f} \mapsto \beta$
	$\Pi(S^\sharp_0) * \Pi(S^\sharp_1)$
$\alpha \xrightarrow{\text{list}}$	$\alpha \cdot \mathsf{list}$
$\alpha \xrightarrow[\text{list}]{\text{list}} \delta$	$\alpha \cdot \mathsf{list\_endp}(\delta)$
other inductives and segments	similar

Note that:

- shape graphs can be encoded into separation logic formula
- the opposite is usually not true

Comparing the structure of abstract formulae

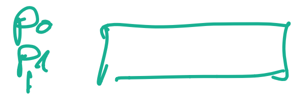
Separation logic:

$$\mathtt{F}_0 * \mathtt{F}_1 * \ldots * \mathtt{F}_n$$

- first the heap is partitioned
- each region is described separately
- some of the $\mathtt{F}_i$ components may be summary predicates, describing unbounded regions
- reachability is implicit
- allows local reasoning

Three valued logic:

$$p_0 \wedge p_1 \wedge \ldots \wedge p_n$$



- first a conjunction of properties
- each predicate p_i may talk about any heap region
- no direct heap partitioning
- reachability can be expressed (natively)
- no local reasoning

Two very different sets of predicates

- one allows local reasoning, the other not
- the other way for reachability predicates

Summarization: one abstract cell, many concrete cells

Large / unbounded numbers of concrete cells need to be abstracted

- **Dynamic structures** (lists, trees) have an unknown and unbounded number of cells, hence require summarization
- We also needed summaries to deal with **arrays**

Summary

A **summary predicate** allows to describe an **unbounded number** of memory locations using a fixed, finite set of predicates

Principles underlying summarization:

- in **separation logic**:
  using inductive definitions for lists, trees...
  unbounded size of the summarized region is hidden in the **recursion**
- in **three-valued logic**:
  summary nodes + high level predicates (such as reachability)
  one summary node **carries the properties** of an unbounded number of cells

Concretize partially, update, abstract

For precise analysis, summaries need to be (temporarily) refined

Separation logic:

Local (partial) concretization

For **materialization**:

$$\begin{array}{ccc} \mathcal{S}^\sharp_{\mathrm{pre}} & & \\ \text{unfold (materialize)} \downarrow & & \\ \mathcal{S}^\sharp_{\mathrm{pre,ref}} & \xrightarrow{f} & \mathcal{S}^\sharp_{\mathrm{post}} \end{array}$$

Global abstraction: widening

$$\begin{array}{ccccc} & & \mathcal{S}^\sharp_0 \triangledown \mathcal{S}^\sharp_1 & & \\ & & \uparrow & & \\ \mathcal{S}^\sharp_0 & \longrightarrow & \triangledown & \longleftarrow & \mathcal{S}^\sharp_1 \end{array}$$

In both cases, two mechanisms are needed:

1. **refine** summaries
2. **synthesize** summaries

TVLA:

Focus, analyze, canonicalize

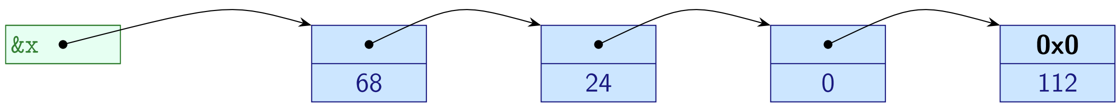
$$\begin{array}{ccc} \mathcal{S}^\sharp_{\mathrm{pre}} & & \mathcal{S}^\sharp_{\mathrm{post}} \\ \text{partially concretize} \downarrow & & \uparrow \text{abstract} \\ \mathcal{S}_{\mathrm{pre}} & \xrightarrow{f} & \mathcal{S}_{\mathrm{post}} \end{array}$$

Outline

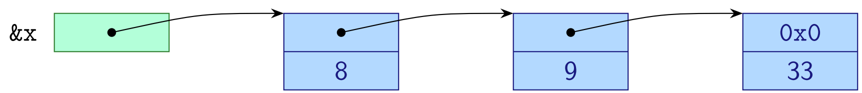
Shape and value properties

Common data-structures require to reason both about shape and data:

- **hybrid stores:** data stored next to inductive structures
- **list of even elements:**



- **sorted list:**



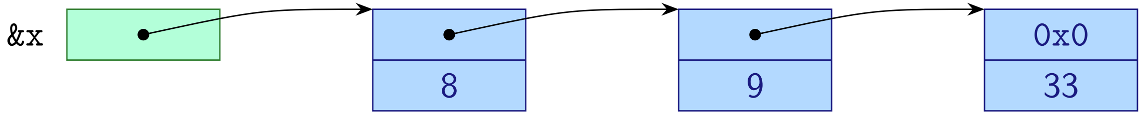
- **list with a length constraint**
- **tries**: binary trees with paths labelled with sequences of "0" and "1"
- **balanced trees**: red-black, AVL...

This part of the course:

- how to **express both shape and numerical properties ?**
- how to **extend shape analysis algorithms**

Description of a sorted list

- **Example:** sorted list



Inductive definition

- Each element should be greater than the previous one
- The first element simply needs be greater than $-\infty$...
- We need to propagate **the lower bound**, using a scalar parameter

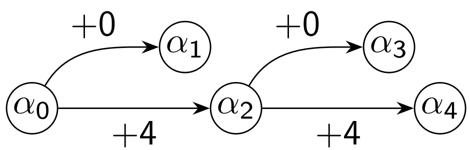
$$\begin{array}{rcl}\alpha \cdot \mathsf{lsort}_{\mathrm{aux}}(n) & := & \alpha = 0 \wedge \mathsf{emp} \\ & \vee & \alpha \neq 0 \wedge n \leq \beta \wedge \alpha \cdot \mathtt{next} \mapsto \delta \\ & & * \, \alpha \cdot \mathtt{data} \mapsto \beta * \delta \cdot \mathsf{lsort}_{\mathrm{aux}}(\beta) \\ \\ \alpha \cdot \mathsf{lsort}() & := & \alpha \cdot \mathsf{lsort}_{\mathrm{aux}}(-\infty)\end{array}$$

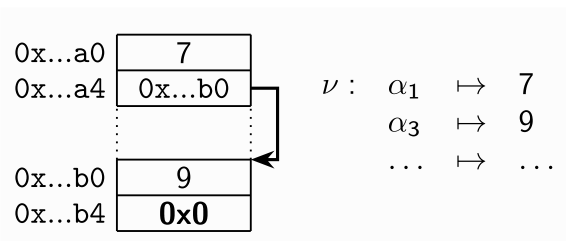
Adding value information (here, numeric)

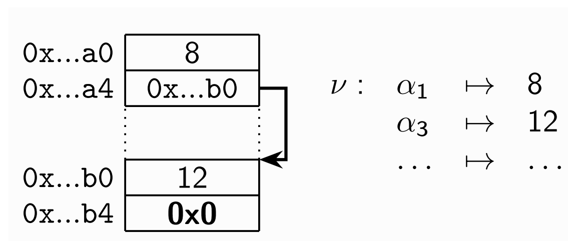
Concrete numeric values appear in the valuation thus the abstracting contents boils down to abstracting ν !

Example: all lists of length 2, sorted in the increasing order of data fields

Memory abstraction:







Abstraction of valuations: $\nu(\alpha_1) < \nu(\alpha_3)$, can be described by the constraint $\alpha_1 < \alpha_3$

A first step towards a combined domain

Domains and their concretization:

- **shape abstract domain** $\mathbb{D}^{\sharp}_{\mathsf{sh}}$ of graphs
  abstract stores together with a **physical mapping** of nodes
  $$\gamma_{\mathsf{sh}} : \mathbb{D}^{\sharp}_{\mathsf{sh}} \rightarrow \mathcal{P}((\mathbb{D}^{\sharp}_{\mathsf{sh}} \rightarrow \mathbb{M}) \times (\mathbb{V}^{\sharp} \rightarrow \mathbb{V}))$$
- **numerical abstract domain** $\mathbb{D}^{\sharp}_{\mathsf{num}}$, abstracts physical mapping of nodes
  $$\gamma_{\mathsf{num}} : \mathbb{D}^{\sharp}_{\mathsf{num}} \rightarrow \mathcal{P}(\mathbb{V}^{\sharp} \rightarrow \mathbb{V})$$

Combined domain [CR]

- **Set of abstract values:** $\mathbb{D}^{\sharp} = \mathbb{D}^{\sharp}_{\mathsf{sh}} \times \mathbb{D}^{\sharp}_{\mathsf{num}}$
- **Concretization:**

$$\gamma(S^{\sharp}, N^{\sharp}) = \{(\hbar, \nu) \in \mathbb{M} \mid \nu \in \gamma_{\mathsf{num}}(N^{\sharp}) \wedge (\hbar, \nu) \in \gamma_{\mathsf{sh}}(S^{\sharp})\}$$

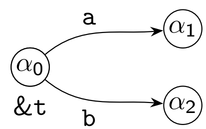
Can it be described as a reduced product ?

- **product abstraction**: $\mathbb{D}^{\sharp} = \mathbb{D}^{\sharp}_0 \times \mathbb{D}^{\sharp}_1$ (componentwise ordering)
- **concretization**: $\gamma(x_0, x_1) = \gamma(x_0) \cap \gamma(x_1)$
- **reduction**: $\mathbb{D}^{\sharp}_r$ is the quotient of $\mathbb{D}^{\sharp}$ by the equivalence relation $\equiv$ defined by
  $(x_0, x_1) \equiv (x'_0, x'_1) \iff \gamma(x_0, x_1) = \gamma(x'_0, x'_1)$

Formalizing the product domain

The use of a simple reduced product raises several issues

Elements without a clear meaning:

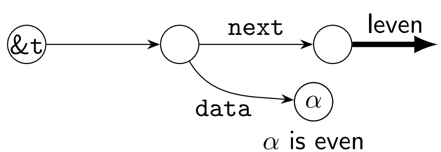


$$\alpha_1 \leq \alpha_2$$
$$\alpha_2 \leq \alpha_1 + \alpha_3$$

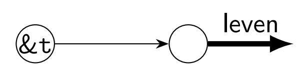
- this element exists in the reduced product domain (independent components)
- but, **... what is α_3 ?**

Unclear comparison:

How can we compare the two elements below ?



and



- in the reduced product domain, they are **not comparable**:
  nodes do not match, so componentwise comparison does not make sense
- when concretizing them, there is **clear inclusion**

Towards a more adapted combination operator

Reason why the reduced product construction does not work well:

- the set of nodes / symbolic variables **is not fixed**
- the set of dimensions in the numerical domain depends on the shape abstraction

$\Rightarrow$ **thus the product is not symmetric**
however, the reduced product construction is symmetric

Intuitions

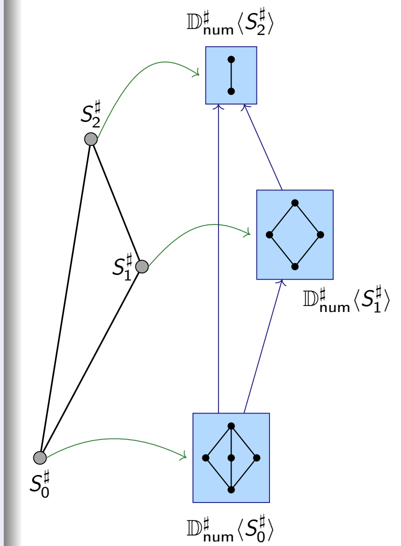
- Graphs form a **shape domain** $\mathbb{D}^\sharp_{\mathsf{sh}}$
- For **each** graph $S^\sharp \in \mathbb{D}^\sharp_{\mathsf{sh}}$, we have a **numerical lattice** $\mathbb{D}^\sharp_{\mathsf{num}\langle S^\sharp \rangle}$
  - example: if graph $S^\sharp$ contains nodes $\alpha_0, \alpha_1, \alpha_2$, $\mathbb{D}^\sharp_{\mathsf{num}\langle S^\sharp \rangle}$ should abstract $\{\alpha_0, \alpha_1, \alpha_2\} \to \mathbb{V}$
- **An abstract value is a pair** $(S^\sharp, N^\sharp)$**, such that** $N^\sharp \in \mathbb{D}^\sharp_{\mathsf{num}\langle N^\sharp \rangle}$

Outline

Cofibered domain

Definition, for shape + num

- **Basis:** abstract domain $(\mathbb{D}^\sharp_{\mathrm{sh}}, \sqsubseteq^\sharp)$, with concretization $\gamma_{\mathrm{sh}} : \mathbb{D}^\sharp_{\mathrm{sh}} \rightarrow \mathbb{D}$
- **Function:** $\phi : \mathbb{D}^\sharp_{\mathrm{sh}} \rightarrow \mathcal{D}$, where each element of $\mathcal{D}$ is an abstract domain instance $(\mathbb{D}^\sharp_{\mathrm{num}}, \sqsubseteq^\sharp_{\mathrm{num}})$, with a concretization $\gamma_{\mathrm{num}} : \mathbb{D}^\sharp_{\mathrm{num}} \rightarrow \mathbb{D}$ (**tied to a shape graph**)
- **Domain** $\mathbb{D}^\sharp$: set of **pairs** $(S^\sharp, N^\sharp)$ **where** $N^\sharp \in \phi(S^\sharp)$
- **Concretization**: $\gamma(S^\sharp, N^\sharp) = \gamma(S^\sharp) \cap \gamma(N^\sharp)$
- **Lift functions:** $\forall S^\sharp_0, S^\sharp_1 \in \mathbb{D}^\sharp_{\mathrm{sh}}$, such that $S^\sharp_0 \sqsubseteq^\sharp S^\sharp_1$, there exists a function $\Pi_{S^\sharp_0, S^\sharp_1} : \phi(S^\sharp_0) \rightarrow \phi(S^\sharp_1)$, that is monotone for $\gamma_{S^\sharp_0}$ and $\gamma_{S^\sharp_1}$

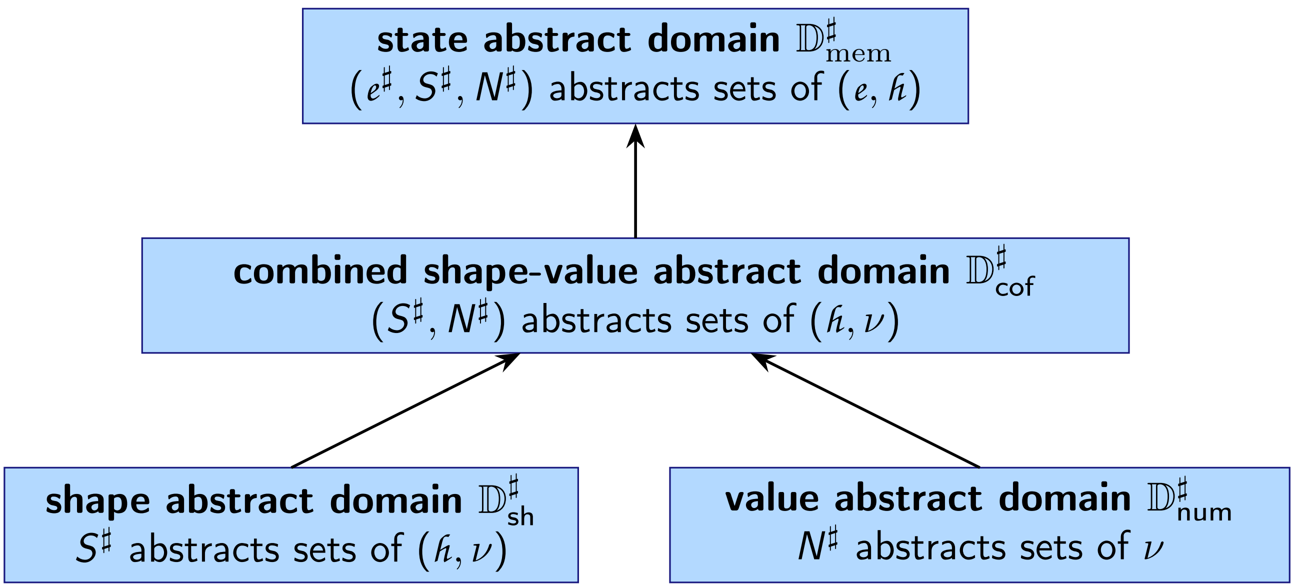


- General construction presented in **[AV]**(Arnaud Venet)
- Intuition: a **dependent domain product**

Overall abstract domain structure

Implementation exploiting the modular structure

- **Each layer** accounts for one **aspect of the concrete states**
- **Each layer** boils down to a **module or functor in ML**



How about operations, transfer functions ? Also to be modularly defined

Domain operations

The cofibered structure allows to define **standard domain operations**:

- **ift functions** allow to **switch domain when needed**
- computations first done in the basis, then in the numerical domains, after lifting, when needed

Comparison of $(S^\sharp_0, N^\sharp_0)$ and $(S^\sharp_1, N^\sharp_1)$

1. First, **compare** $S^\sharp_0$ **and** $S^\sharp_1$ in $\mathbb{D}^\sharp_{\mathsf{sh}}$
2. If $S^\sharp_0 \sqsubseteq^\sharp S^\sharp_1$, **compare** $\Pi_{S^\sharp_0, S^\sharp_1}(N^\sharp_0)$ **and** $N^\sharp_1$

Widening of $(S^\sharp_0, N^\sharp_0)$ and $(S^\sharp_1, N^\sharp_1)$

1. First, compute the **widening in the basis** $S^\sharp = S^\sharp_0 \triangledown S^\sharp_1$
2. Then **move to** $\phi(S^\sharp)$, by computing $N^\sharp_{0c} = \Pi_{S^\sharp_0, S^\sharp}(N^\sharp_0)$ and $N^\sharp_{1c} = \Pi_{S^\sharp_1, S^\sharp}(N^\sharp_1)$
3. Last **widen in** $\phi(S^\sharp)$: $N^\sharp = N^\sharp_{0c} \triangledown_{S^\sharp} N^\sharp_{1c}$
4. Return $(S^\sharp_0, N^\sharp_0) \triangledown (S^\sharp_A, N^\sharp_1) = (S^\sharp, N^\sharp)$

Outline

Domain operations and transfer functions

Abstract assignments, condition tests:

- need to modify both the shape abstraction and the value abstraction
- both modification are interdependent

Typical process to compute abstract post-conditions

1. compute the post in the shape abstract domain and update the basis
2. update the value abstraction (numerics) to model dimensions additions and removals
3. compute the post in the value abstract domain

Proofs of soundness of transfer functions rely on:

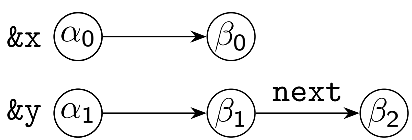
- the soundness of the lift functions
- the soundness of both domain transfer functions

Analysis of an assignment in the graph domain

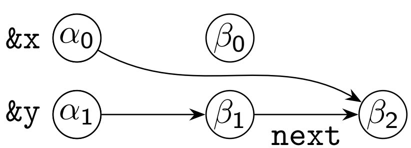
Steps for analyzing $\mathtt{x} = \mathtt{y} \texttt{->} \mathtt{next}$ (local reasoning)

1. Evaluate **l-value** `x` into **points-to edge** $\alpha \mapsto \beta$
2. Evaluate **r-value** `y -> next` into **node** β'
3. Replace points-to edge $\alpha \mapsto \beta$ with **points-to edge** $\alpha \mapsto \beta'$

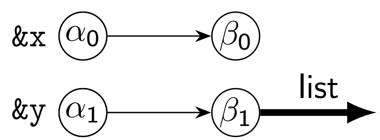
With pre-condition:



- Step 1 produces $\alpha_0 \mapsto \beta_0$
- Step 2 produces β_2
- End result:

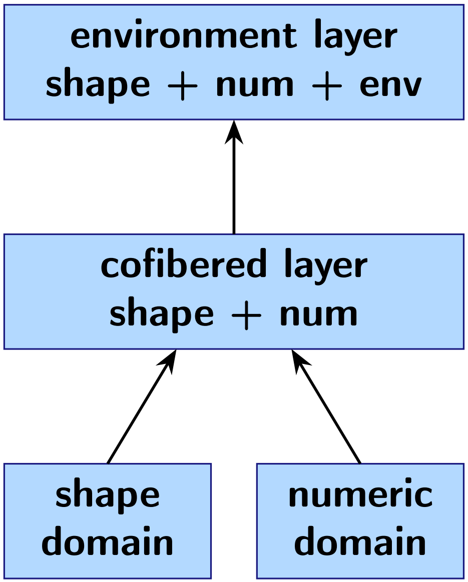


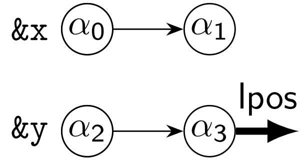
With pre-condition:



- Step 1 produces $\alpha_0 \mapsto \beta_0$
- **Step 2 can succeed only after unfolding is performed**

Analysis of an assignment in the combined domain



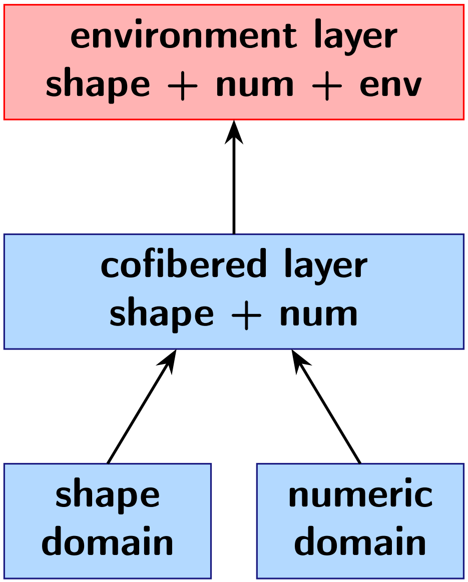


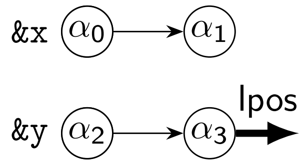
$$N^\sharp = \alpha_1 \geq 0 \wedge \alpha_3 \neq \mathbf{0x0}$$

```
y -> d = x + 1
```

Abstract post-condition ?

Analysis of an assignment in the combined domain





$$N^\sharp = \alpha_1 \geq 0 \wedge \alpha_3 \neq \mathbf{0x0}$$

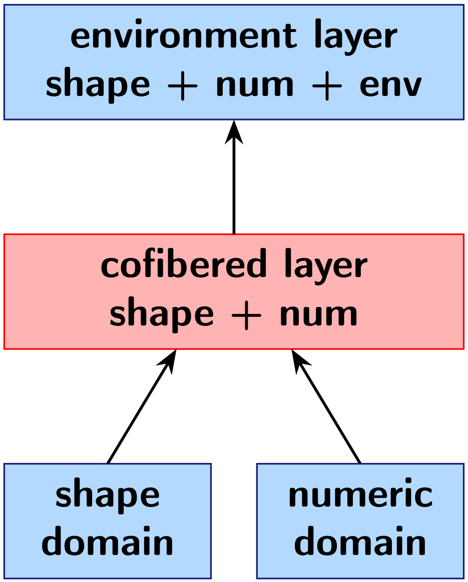
$$\texttt{y -> d} = \texttt{x} + 1 \quad \Rightarrow \quad (*\alpha_2) \cdot \texttt{d} = (*\alpha_0) + 1$$

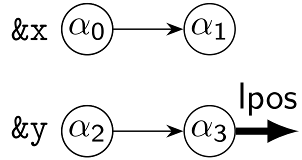
Abstract post-condition ?

Stage 1: environment resolution

- replaces $\texttt{x}$ with $*e^\sharp(\texttt{x})$

Analysis of an assignment in the combined domain





$$N^\sharp = \alpha_1 \geq 0 \wedge \alpha_3 \neq \mathbf{0x0}$$

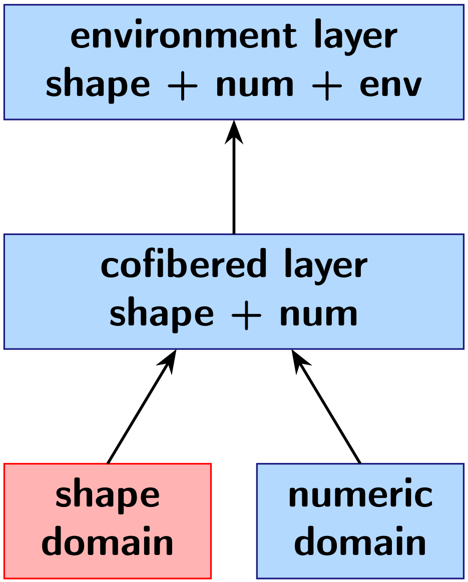
$$(*\alpha_2) \cdot \mathtt{d} = (*\alpha_0) + 1$$

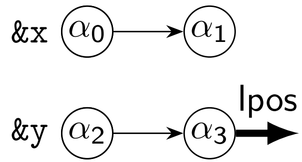
Abstract post-condition ?

Stage 2: propagate into the shape + numerics domain

- only symbolic nodes appear

Analysis of an assignment in the combined domain





$$N^\sharp = \alpha_1 \geq 0 \wedge \alpha_3 \neq \mathbf{0x0}$$

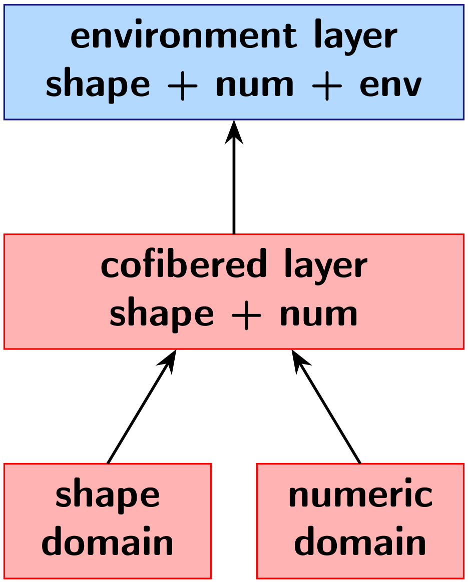
$$(*\alpha_2) \cdot \mathtt{d} = (*\alpha_0) + 1$$

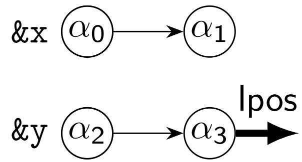
Abstract post-condition ?

Stage 3: resolve cells in the shape graph abstract domain

- $*\alpha_0$ evaluates to α_1; $*\alpha_2$ evaluates to α_3
- $(*\alpha_2) \cdot \mathtt{d}$ **fails to evaluate**: **no points-to out of** α_3

Analysis of an assignment in the combined domain





$$N^\sharp = \alpha_1 \geq 0 \wedge \alpha_3 \neq \mathbf{0x0}$$

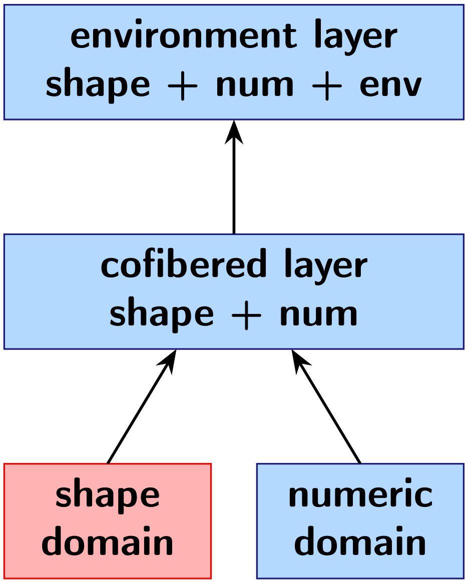
$$(*\alpha_2) \cdot \mathtt{d} = (*\alpha_0) + 1$$

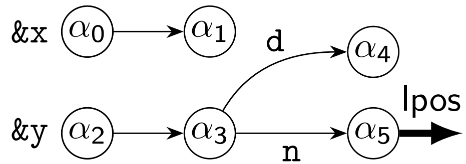
Abstract post-condition ?

Stage 4 (a): unfolding triggered

- the analysis needs to locally materialize $\alpha_3 \cdot \mathsf{lpos}$...
- thus, unfolding starts at symbolic variable α_3

Analysis of an assignment in the combined domain





$$N^\sharp = \alpha_1 \geq 0 \wedge \alpha_3 \neq \mathbf{0x0}$$

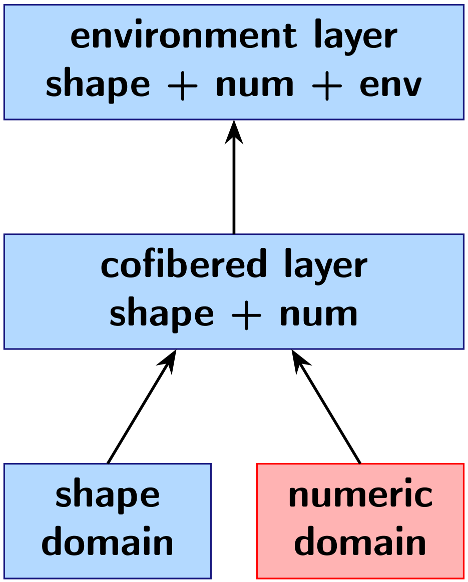
$$(*\alpha_2) \cdot \mathtt{d} = (*\alpha_0) + 1$$

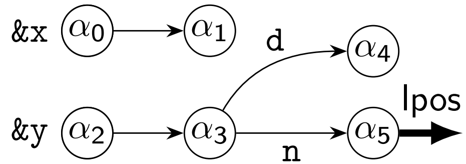
Abstract post-condition ?

Stage 4 (b): unfolding, shape part

- unfolding of the memory predicate part
- numerical predicates still need be taken into account

Analysis of an assignment in the combined domain





$$N^\sharp = \alpha_1 \geq 0 \wedge \alpha_3 \neq \mathbf{0x0} \wedge \alpha_4 \geq 0$$

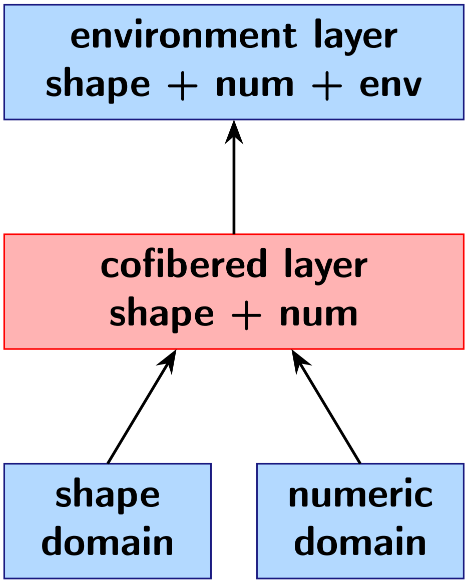
$$(*\alpha_2) \cdot \mathtt{d} = (*\alpha_0) + 1$$

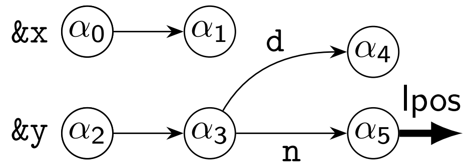
Abstract post-condition ?

Stage 4 (c): unfolding, numeric part

- numerical predicates taken into account
- l-value $\alpha_3 \cdot \mathtt{d}$ **now evaluates into edge** $\alpha_3 \cdot \mathtt{d} \mapsto \alpha_4$

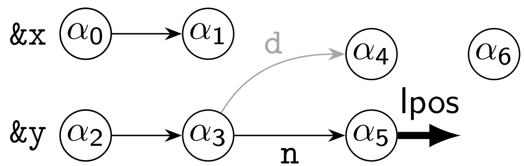
Analysis of an assignment in the combined domain





$$N^\sharp = \alpha_1 \geq 0 \wedge \alpha_3 \neq \mathbf{0x0} \wedge \alpha_4 \geq 0$$

create node α_6

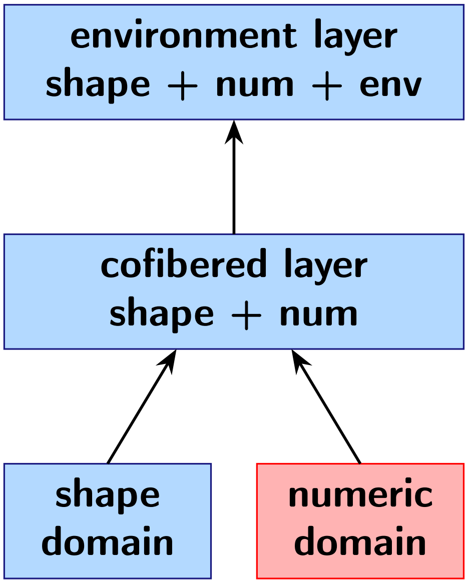


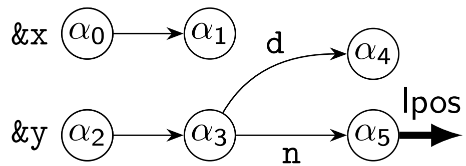
$$N^\sharp = \alpha_1 \geq 0 \wedge \alpha_3 \neq \mathbf{0x0} \wedge \alpha_4 \geq 0$$

Stage 5: create a new node

- new node α_6 denotes a new value
  will store the new value

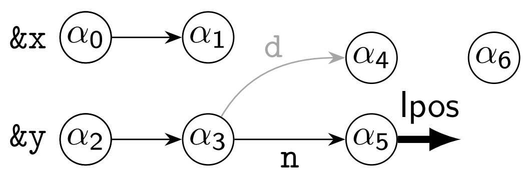
Analysis of an assignment in the combined domain





$$N^\sharp = \alpha_1 \geq 0 \wedge \alpha_3 \neq \mathbf{0x0} \wedge \alpha_4 \geq 0$$

$\alpha_6 \leftarrow \alpha_1 + 1$ in numerics

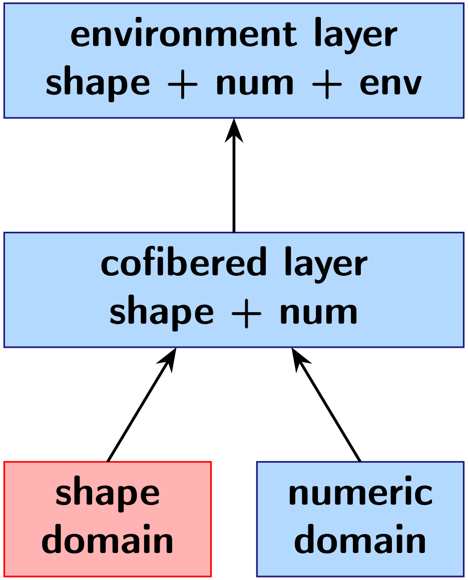


$$N^\sharp = \alpha_1 \geq 0 \wedge \alpha_3 \neq \mathbf{0x0} \wedge \alpha_4 \geq 0 \wedge \alpha_6 \geq 1$$

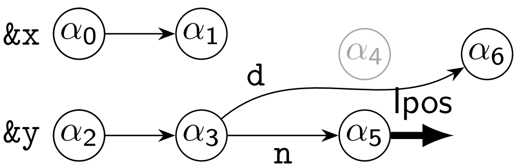
Stage 6: perform numeric assignment

- numeric assignment **completely ignores pointer structures** to the new node

Analysis of an assignment in the combined domain



mutate $(\alpha_3 \cdot \mathrm{d}) \mapsto \alpha_4$ into α_6



$$N^\sharp = \alpha_1 \geq 0 \wedge \alpha_3 \neq \mathbf{0x0} \wedge \alpha_4 \geq 0 \wedge \alpha_6 \geq 1$$

Stage 7: perform the update in the graph

- classic **strong update** in a pointer aware domain
- symbolic node α_4 becomes redundant and can be removed

Shape graph weakening: definition (reminder)

To design **inclusion test**, **join** and **widening** algorithms, we first study a more general notion of **weakening**:

Weakening

We say that $S_0^\sharp$ **can be weakened into** $S_1^\sharp$ if and only if

$$\forall(\hbar, \nu) \in \gamma_{\mathsf{sh}}(S_0^\sharp),\ \exists \nu' \in \mathsf{Val},\ (\hbar, \nu') \in \gamma_{\mathsf{sh}}(S_1^\sharp)$$

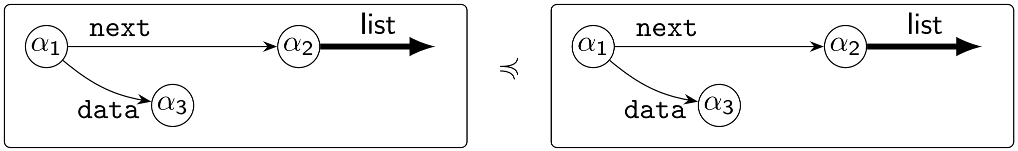
We then note $S_0^\sharp \preccurlyeq S_1^\sharp$

Applications:

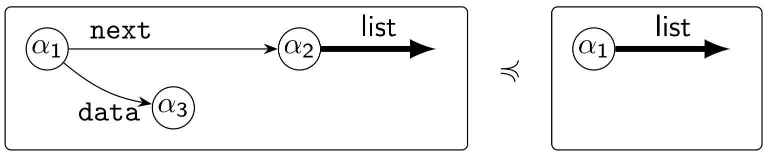
- **inclusion test** (comparison) inputs $S_0^\sharp, S_1^\sharp$; if returns true $S_0^\sharp \preccurlyeq S_1^\sharp$
- **canonicalization** (unary weakening) inputs $S_0^\sharp$ and returns $\rho(S_0^\sharp)$ such that $S_0^\sharp \preccurlyeq \rho(S_0^\sharp)$
- **widening** / **join** (binary weakening ensuring termination or not) inputs $S_0^\sharp, S_1^\sharp$ and returns $S_{\mathrm{up}}^\sharp$ such that $S_i^\sharp \preccurlyeq S_{\mathrm{up}}^\sharp$

Shape graph weakening weakening based on local rules (reminder)

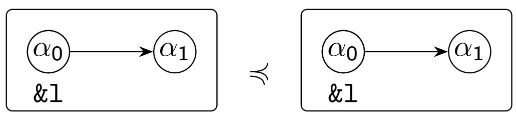
By **rule** $(\preccurlyeq_{\mathbf{Id}})$:



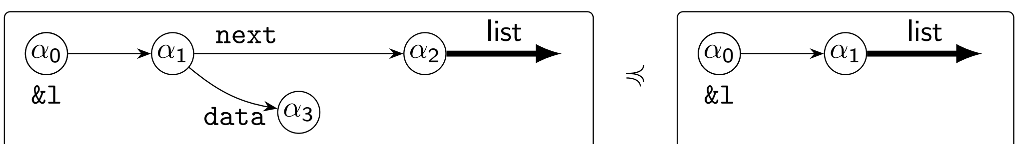
Thus, by **rule** $(\preccurlyeq_{\mathcal{U}})$:



Additionally, by **rule** $(\preccurlyeq_{\mathbf{Id}})$:



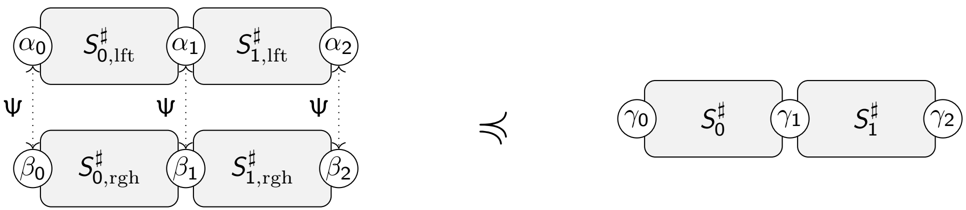
Thus, by **rule** $(\preccurlyeq_{*})$:



Shpae graph abstract union

The principle of join and widening algorithm **is similar to that of $\sqsubseteq^\sharp$:**

- It can be computed **region by region**, as for weakening in general:
  If $\forall i \in \{0, 1\}$, $\forall s \in \{\mathrm{lft}, \mathrm{rgh}\}$, $S^\sharp_{i,s} \preccurlyeq S^\sharp_s$,

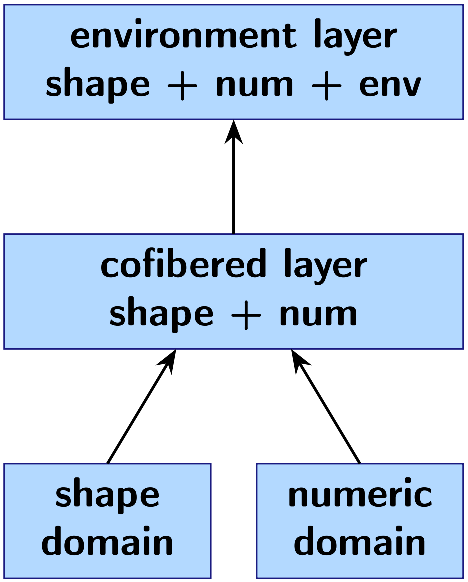


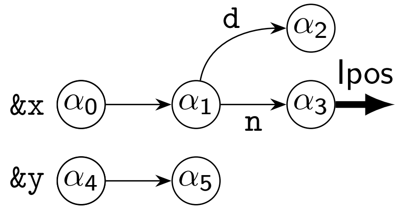
  The partitioning of inputs / different nodes sets requires a **node correspondence function**

$$\Psi : \mathbb{V}^\sharp(S^\sharp_{\mathrm{lft}}) \times \mathbb{V}^\sharp(S^\sharp_{\mathrm{rgh}}) \longrightarrow \mathbb{V}^\sharp(S^\sharp)$$

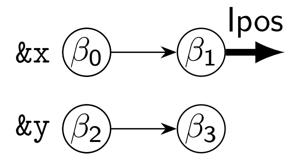
- The computation of the shape join progresses by the application of **local join rules**, that produce a **new (output) shape graph, that weakens both inputs**

Widening / join in the combined domain



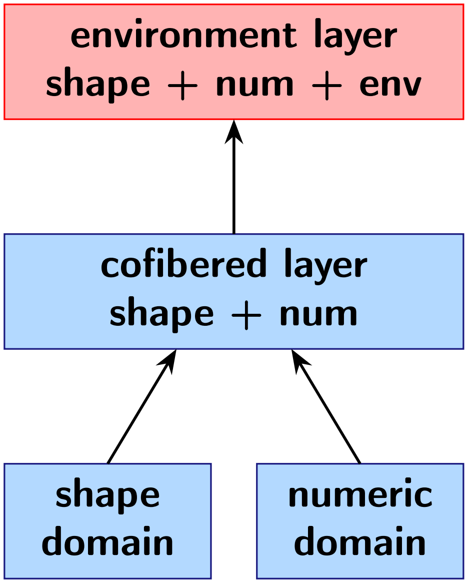


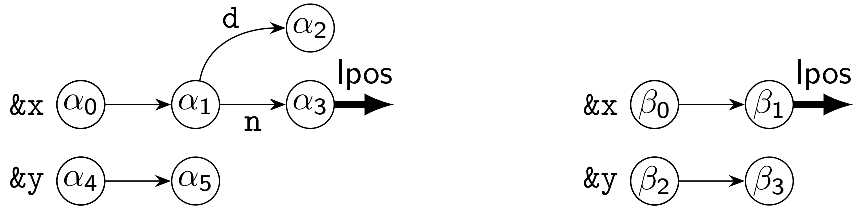
$$N^\sharp_{\mathrm{lft}} = \alpha_2 \geq \alpha_5 \geq 2$$



$$N^\sharp_{\mathrm{rgh}} = \beta_3 \geq 1$$

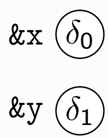
Widening / join in the combined domain





$N^\sharp_{\mathrm{lft}} = \alpha_2 \geq \alpha_5 \geq 2$

$N^\sharp_{\mathrm{rgh}} = \beta_3 \geq 1$

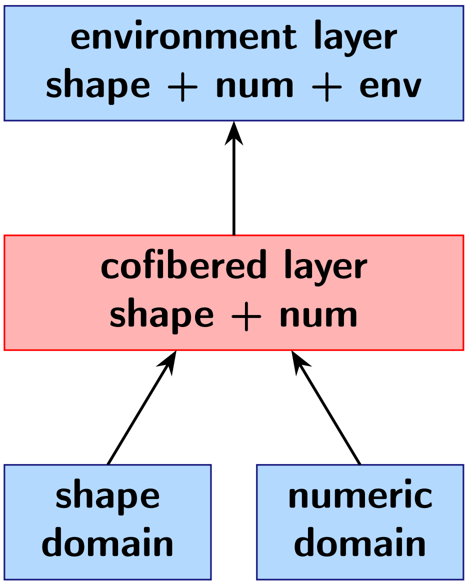


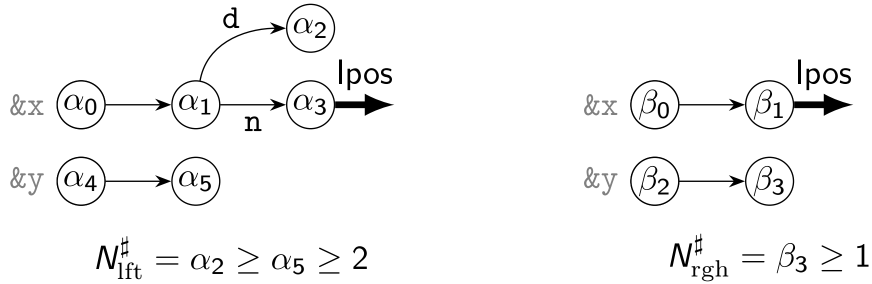
$$\Psi(\alpha_0, \beta_0) = \delta_0$$
$$\Psi(\alpha_4, \beta_2) = \delta_1$$

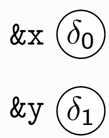
Stage 1: abstract environment

- compute new abstract environment and initial node relation
  e.g., α_0, β_0 both denote &x

Widening / join in the combined domain







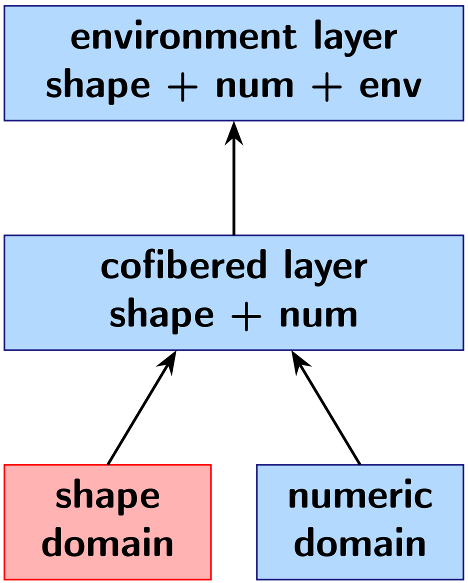
$\Psi(\alpha_0, \beta_0) = \delta_0$
$\Psi(\alpha_4, \beta_2) = \delta_1$

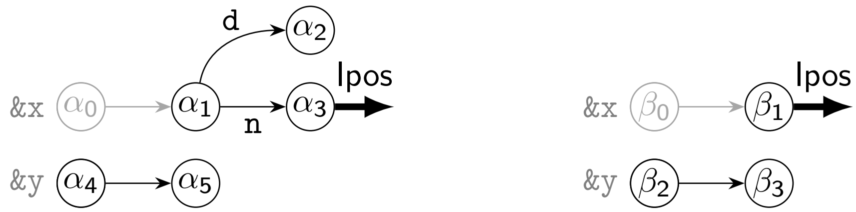
Stage 2: join in the "cofibered" layer

operations to perform:

1. compute the join in the graph
2. convert value abstractions, and join the resulting lattice

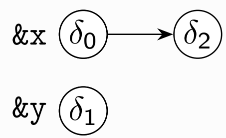
Widening / join in the combined domain





$N^{\sharp}_{\mathrm{lft}} = \alpha_2 \geq \alpha_5 \geq 2$

$N^{\sharp}_{\mathrm{rgh}} = \beta_3 \geq 1$

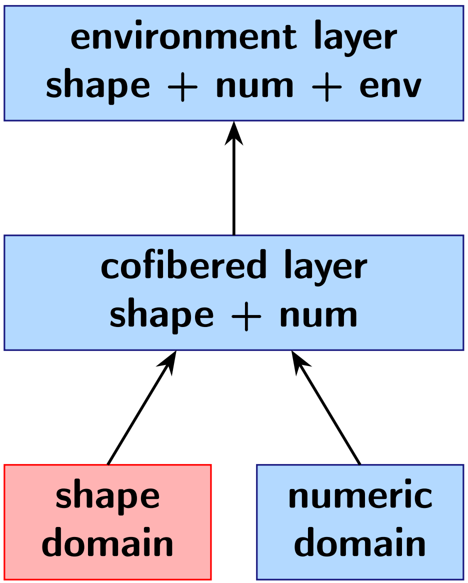


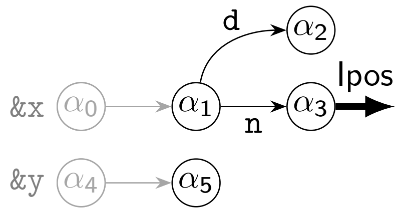
$$\Psi(\alpha_0, \beta_0) = \delta_0$$
$$\Psi(\alpha_4, \beta_2) = \delta_1$$
$$\Psi(\alpha_1, \beta_1) = \delta_2$$

Stage 2: graph join

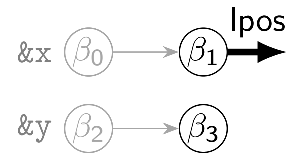
- apply local join rules
  ex: **points-to matching**, **weakening to inductive**...
- incremental algorithm

Widening / join in the combined domain

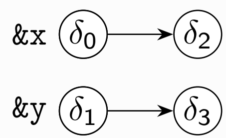




$N^\sharp_{\mathrm{lft}} = \alpha_2 \geq \alpha_5 \geq 2$



$N^\sharp_{\mathrm{rgh}} = \beta_3 \geq 1$

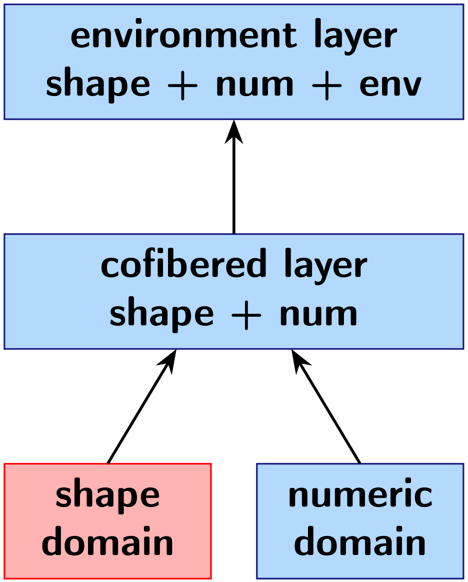


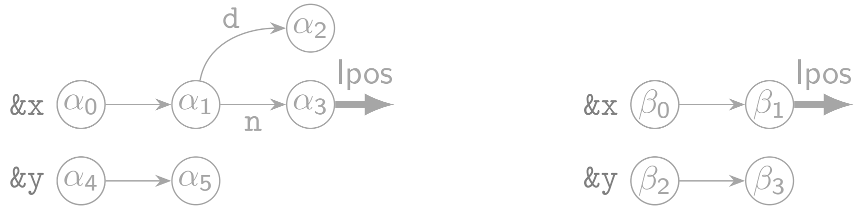
$$\Psi(\alpha_0, \beta_0) = \delta_0$$
$$\Psi(\alpha_4, \beta_2) = \delta_1$$
$$\Psi(\alpha_1, \beta_1) = \delta_2$$
$$\Psi(\alpha_5, \beta_3) = \delta_3$$

Stage 2: graph join

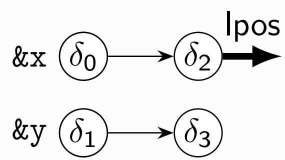
- apply local join rules
  ex: **points-to matching**, **weakening to inductive**...
- incremental algorithm

Widening / join in the combined domain





$$N^\sharp_{\mathrm{lft}} = \alpha_2 \geq \alpha_5 \geq 2 \qquad N^\sharp_{\mathrm{rgh}} = \beta_3 \geq 1$$

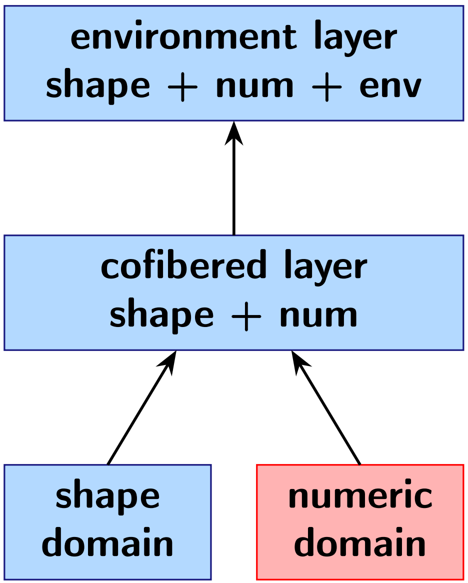


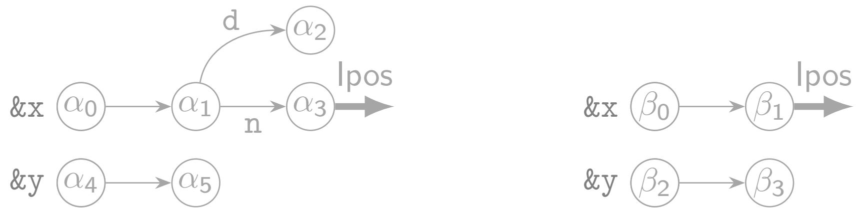
$$\begin{aligned}\Psi(\alpha_0, \beta_0) &= \delta_0\\ \Psi(\alpha_4, \beta_2) &= \delta_1\\ \Psi(\alpha_1, \beta_1) &= \delta_2\\ \Psi(\alpha_5, \beta_3) &= \delta_3\end{aligned}$$

Stage 2: graph join

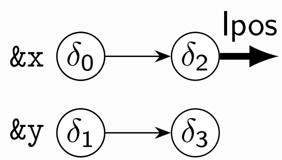
- apply local join rules
  ex: **points-to matching**, **weakening to inductive**...
- incremental algorithm

Widening / join in the combined domain





$$N^\sharp_{\mathrm{lft}} = \alpha_2 \geq \alpha_5 \geq 2 \qquad N^\sharp_{\mathrm{rgh}} = \beta_3 \geq 1$$



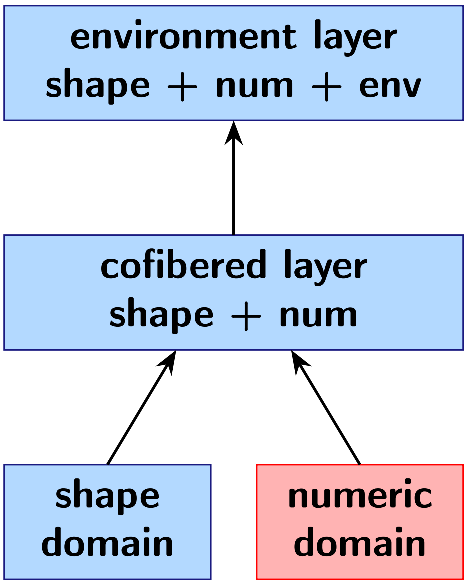
$$\Psi(\alpha_0, \beta_0) = \delta_0$$
$$\Psi(\alpha_4, \beta_2) = \delta_1$$
$$\Psi(\alpha_1, \beta_1) = \delta_2$$
$$\Psi(\alpha_5, \beta_3) = \delta_3$$

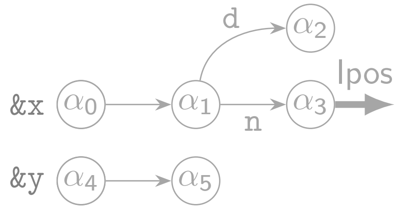
$$N^\sharp_{\sqcup} = [\delta_3 \geq 2] \sqcup [\delta_3 \geq 1]$$

Stage 3: conversion function application in numerics

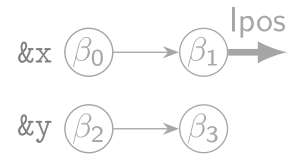
- remove nodes that were abstracted away
- rename other nodes

Widening / join in the combined domain

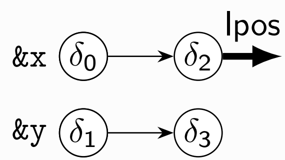




$N^\sharp_{\mathrm{lft}} = \alpha_2 \geq \alpha_5 \geq 2$



$N^\sharp_{\mathrm{rgh}} = \beta_3 \geq 1$



$$\Psi(\alpha_0, \beta_0) = \delta_0$$
$$\Psi(\alpha_4, \beta_2) = \delta_1$$
$$\Psi(\alpha_1, \beta_1) = \delta_2$$
$$\Psi(\alpha_5, \beta_3) = \delta_3$$

$N^\sharp_{\sqcup} = [\delta_3 \geq 1]$

Stage 4: join in the numeric domain

- apply $\sqcup$ for regular join, $\triangledown$ for a widening

Outline

Shape analysis and summarization

Summaries:

- describe **unbounded** memory regions, with general predicates
  e.g., list or tree structures, local and global sharing (doubly-linked lists)
- **summary nodes** + associated predicates in TVLA,
  inductive predicates in separation logic

Local refinement (concretization):

- **focus** in TVLA, **unfolding** in separation logic based aanlysis
- required to **analyze precisely post-conditions** that touch summaries

Global abstraction:

- **ensure termination** despite unbounded, infinite domain
- in TVLA, **canonical abstraction** into a finite domain

In all cases, analysis algorithms aim at avoiding **weak updates**
(that would cause a severe precision loss over the whole memory)

Shape analysis and value abstraction

Main issue: the support of the shape abstraction is **always changing**

- summaries appear at canonicalization/widening points
- new atoms/nodes appear at focus/materialization points

Cofibered domain

an abstract form of dependent product

assymetric version of $\mathbb{D}^{\sharp}_{\mathsf{sh}} \times \mathbb{D}^{\sharp}_{\mathsf{num}}$

- the shape abstraction "controls" the value abstraction
- information can still be exchanged in both directions (reduction)
- slightly more complex lattice structure
  but standard definitions for widening, inclusion test...

Bibliography

- **[SRW]**: Parametric Shape Analysis via 3-Valued Logic.
  Shmuel Sagiv, Thomas W. Reps et Reinhard Wilhelm. In POPL'99, pages 105–118, 1999.
- **[AV]**: Abstract Cofibered Domains: Application to the Alias Analysis of Untyped Programs.
  Arnaud Venet.
  In SAS'96, pages 366–382.
- **[CR]**: Relational inductive shape analysis.
  Bor-Yuh Evan Chang et Xavier Rival.
  In POPL'08, pages 247–260, 2008.

Assignment: formalization and paper reading

Formalization of the concretization of 2-structures:

- describe the concretization formula, assuming that we consider the predicates discussed in the course
- run it on the list abstraction example (from the 3-structure to a few select 2-structures, and down to memory states)
- prove the correctness and termination of the widening of the cofibered abstract domain

Reading:

Parametric Shape Analysis via 3-Valued Logic.
Shmuel Sagiv, Thomas W. Reps et Reinhard Wilhelm.
In POPL'99, pages 105–118, 1999.