Order Theory

MPRI 2–6: Abstract Interpretation,
application to verification and static analysis

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Course 1
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Plan

- Partially ordered structures
  - (complete) partial orders
  - (complete) lattices
- Fixpoints
- Abstractions
  - Galois connections, upper closure operators
    (first-class citizens)
  - Concretization-only framework
  - Operator abstraction
  - Fixpoint abstraction

Partial orders

Partial orders

Given a set X, a relation $\sqsubseteq \in X \times X$ is a partial order if it is:

1. reflexive: $\forall x \in X,\ x \sqsubseteq x$
2. antisymmetric: $\forall x, y \in X,\ (x \sqsubseteq y) \wedge (y \sqsubseteq x) \implies x = y$
3. transitive: $\forall x, y, z \in X,\ (x \sqsubseteq y) \wedge (y \sqsubseteq z) \implies x \sqsubseteq z$

$(X, \sqsubseteq)$ is a poset (partially ordered set).

If we drop antisymmetry, we have a preorder instead.

Examples: partial orders

Partial orders:

- $(\mathbb{Z}, \leq)$
  (completely ordered)
- $(\mathcal{P}(X), \subseteq)$
  (not completely ordered: $\{1\} \not\subseteq \{2\}$, $\{2\} \not\subseteq \{1\}$)
- $(S, =)$ is a poset for any S
- $(\mathbb{Z}^2, \sqsubseteq)$, where $(a, b) \sqsubseteq (a', b') \iff (a \geq a') \wedge (b \leq b')$
  (ordering of interval bounds that implies inclusion)

Examples: preorders

Preorders:

- $(\mathcal{P}(X), \sqsubseteq)$, where $a \sqsubseteq b \iff |a| \leq |b|$

  (ordered by cardinal)

- $(\mathbb{Z}^2, \sqsubseteq)$, where
  $(a, b) \sqsubseteq (a', b') \iff \{ x \mid a \leq x \leq b \} \subseteq \{ x \mid a' \leq x \leq b' \}$
  (inclusion of intervals represented by pairs of bounds)

  not antisymmetric: $[1, 0] \neq [2, 0]$ but $[1, 0] \sqsubseteq [2, 0] \sqsubseteq [1, 0]$

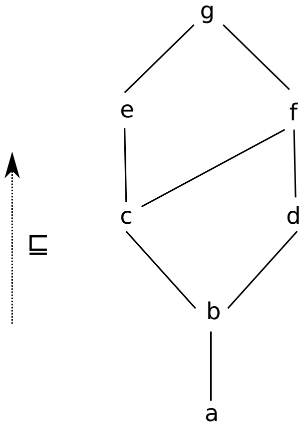
Equivalence: $\equiv$

$X \equiv Y \iff (X \sqsubseteq Y) \wedge (Y \sqsubseteq X)$

We obtain a partial order by quotienting by $\equiv$.

Examples of posets (cont.)

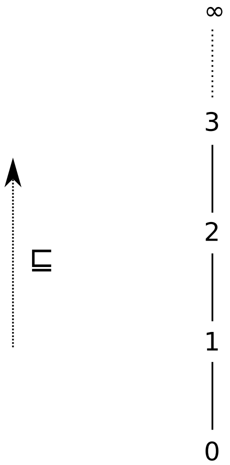
- Given by a Hasse diagram, e.g.:



$g \sqsubseteq g$
$f \sqsubseteq f, g$
$e \sqsubseteq e, g$
$d \sqsubseteq d, f, g$
$c \sqsubseteq c, e, f, g$
$b \sqsubseteq b, c, d, e, f, g$
$a \sqsubseteq a, b, c, d, e, f, g$

Examples of posets (cont.)

- Infinite Hasse diagram for $(\mathbb{N} \cup \{\, \infty \,\}, \leq)$:



$\infty \sqsubseteq \infty$

$\cdots$

$1 \sqsubseteq 1, 2, \ldots, \infty$

$0 \sqsubseteq 0, 1, 2, \ldots, \infty$

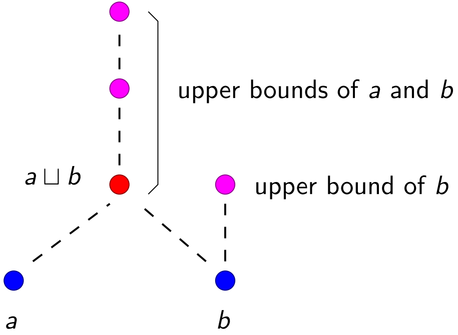
Use of posets (informally)

Posets are a very useful notion to discuss about:

- logic: formulas ordered by implication $\implies$
- program verification: program semantics $\sqsubseteq$ specification
  (e.g.: behaviors of program $\subseteq$ accepted behaviors)
- approximation: $\sqsubseteq$ is an information order
  ("$a \sqsubseteq b$" means: "a caries more information than b")
- iteration: fixpoint computation
  (e.g., a computation is directed, with a limit: $X_1 \sqsubseteq X_2 \sqsubseteq \cdots \sqsubseteq X_n$)

(Least) Upper bounds

- c is an upper bound of a and b if: $a \sqsubseteq c$ and $b \sqsubseteq c$
- c is a least upper bound (lub or join) of a and b if
  - c is an upper bound of a and b
  - for every upper bound d of a and b, $c \sqsubseteq d$



(Least) Upper bounds

If it exists, the lub of a and b is unique, and denoted as $a \sqcup b$.

(proof: assume that c and d are both lubs of a and b; by definition of lubs, $c \sqsubseteq d$ and $d \sqsubseteq c$; by antisymmetry of $\sqsubseteq$, $c = d$)

Generalized to upper bounds of arbitrary (even infinite) sets $\sqcup Y$, $Y \subseteq X$

(well-defined, as $\sqcup$ is commutative and associative).

Similarly, we define greatest lower bounds (glb, meet) $a \sqcap b$, $\sqcap Y$.

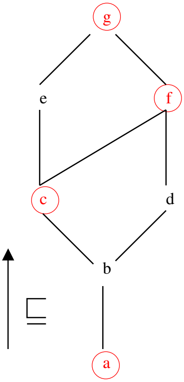
$(a \sqcap b \sqsubseteq a) \wedge (a \sqcap b \sqsubseteq b)$ and $\forall c$, $(c \sqsubseteq a) \wedge (c \sqsubseteq b) \implies (c \sqsubseteq a \sqcap b)$

Note: not all posets have lubs, glbs

(e.g.: $a \sqcup b$ not defined on $(\{ a, b \}, =)$)

Chains

$C \subseteq X$ is a chain in $(X, \sqsubseteq)$ if it is totally ordered by $\sqsubseteq$:
$\forall x, y \in C, (x \sqsubseteq y) \vee (y \sqsubseteq x)$.



$a \sqsubseteq c \sqsubseteq f \sqsubseteq g$

Complete partial orders (CPO)

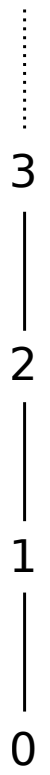
A poset $(X, \sqsubseteq)$ is a complete partial order (CPO)
if every chain C (including $\emptyset$) has a least upper bound $\sqcup\, C$.

A CPO has a least element $\sqcup\, \emptyset$, denoted $\bot$.

Examples, Counter-examples:

- $(\mathbb{N}, \leq)$ is not complete, but $(\mathbb{N} \cup \{\, \infty \,\}, \leq)$ is complete.
- $(\{\, x \in \mathbb{Q} \,|\, 0 \leq x \leq 1 \,\}, \leq)$ is not complete, but
  $(\{\, x \in \mathbb{R} \,|\, 0 \leq x \leq 1 \,\}, \leq)$ is complete.
- $(\mathcal{P}(Y), \subseteq)$ is complete for any Y.
- $(X, \sqsubseteq)$ is complete if X is finite.

Complete partial order examples



$(\mathbb{N}, \leq)$

non-complete

$(\mathbb{N} \cup \{ \infty \}, \leq)$

complete

Lattices

Lattices

A lattice $(X, \sqsubseteq, \sqcup, \sqcap)$ is a poset with

1. a lub $a \sqcup b$ for every pair of elements a and b;
2. a glb $a \sqcap b$ for every pair of elements a and b.

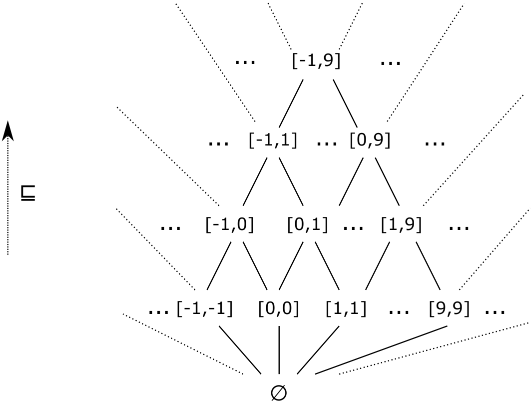
Examples:

- integers $(\mathbb{Z}, \leq, \max, \min)$
- integer intervals (next slide)
- divisibility (in two slides)

If we drop one condition, we have a (join or meet) semilattice.

Reference on lattices: Birkhoff [Birk76].

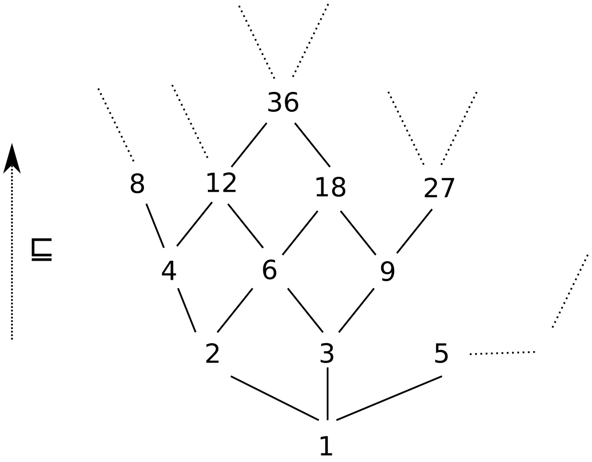
Example: the interval lattice



Integer intervals: $(\{\, [a, b] \mid a, b \in \mathbb{Z},\ a \leq b \,\} \cup \{\, \emptyset \,\}, \subseteq, \sqcup, \cap)$

where $[a, b] \sqcup [a', b'] \overset{\text{def}}{=} [\min(a, a'), \max(b, b')]$.

Example: the divisibility lattice



Divisibility $(\mathbb{N}^*, |, \text{lcm}, \text{gcd})$ where $x|y \overset{\text{def}}{\iff} \exists k \in \mathbb{N},\ kx = y$

Example: the divisibility lattice (cont.)

Let $P \overset{\text{def}}{=} \{ p_1, p_2, \ldots \}$ be the (infinite) set of prime numbers.

We have a correspondence ι between $\mathbb{N}^*$ and $P \to \mathbb{N}$:

- $\alpha = \iota(x)$ is the (unique) decomposition of x into prime factors
- $\iota^{-1}(\alpha) \overset{\text{def}}{=} \prod_{a \in P} a^{\alpha(a)} = x$
- ι is one-to-one on functions $P \to \mathbb{N}$ with finite support

  ($\alpha(a) = 0$ except for finitely many factors a)

We have a correspondence between $(\mathbb{N}^*, |, \text{lcm}, \text{gcd})$
and $(\mathbb{N}, \leq, \max, \min)$.

Assume that $\alpha = \iota(x)$ and $\beta = \iota(y)$ are the decompositions of x and y, then:

- $\prod_{a \in P} a^{\max(\alpha(a), \beta(a))} = \text{lcm}(\prod_{a \in P} a^{\alpha(a)}, \prod_{a \in P} a^{\beta(a)}) = \text{lcm}(x, y)$
- $\prod_{a \in P} a^{\min(\alpha(a), \beta(a))} = \text{gcd}(\prod_{a \in P} a^{\alpha(a)}, \prod_{a \in P} a^{\beta(a)}) = \text{gcd}(x, y)$
- $(\forall a \colon \alpha(a) \leq \beta(a)) \iff (\prod_{a \in P} a^{\alpha(a)}) \mid (\prod_{a \in P} a^{\beta(a)}) \iff x|y$

Complete lattices

A complete lattice $(X, \sqsubseteq, \sqcup, \sqcap, \bot, \top)$ is a poset with

1. a lub $\sqcup S$ for every set $S \subseteq X$
2. a glb $\sqcap S$ for every set $S \subseteq X$
3. a least element $\bot$
4. a greatest element $\top$

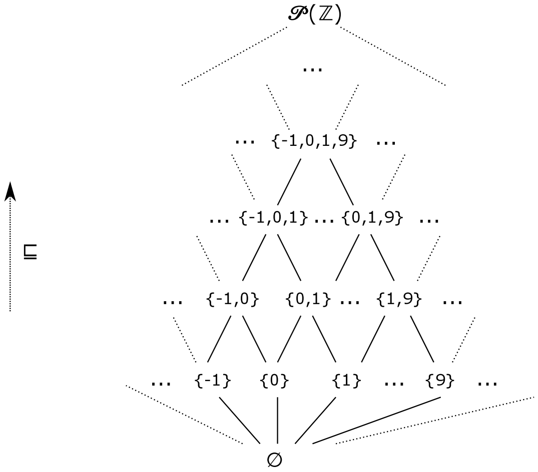
Notes:

- 1 implies 2 as $\sqcap S = \sqcup \{ y \mid \forall x \in S, y \sqsubseteq x \}$ (and 2 implies 1 as well),
- 1 and 2 imply 3 and 4: $\bot = \sqcup \emptyset = \sqcap X$, $\top = \sqcap \emptyset = \sqcup X$,
- a complete lattice is also a CPO.

Complete lattice examples

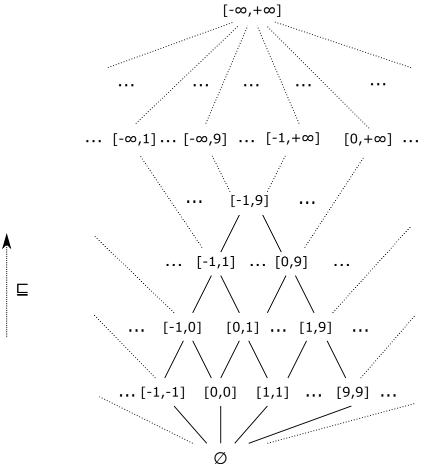
- real segment $[0,1]$: $(\{\, x \in \mathbb{R} \mid 0 \leq x \leq 1 \,\}, \leq, \max, \min, 0, 1)$
- powersets $(\mathcal{P}(S), \subseteq, \cup, \cap, \emptyset, S)$
  (next slide)
- any finite lattice
  ($\sqcup Y$ and $\sqcap Y$ for finite $Y \subseteq X$ are always defined)
- integer intervals with finite and infinite bounds:
  $(\{\, [a,b] \mid a \in \mathbb{Z} \cup \{\, -\infty \,\},\ b \in \mathbb{Z} \cup \{\, +\infty \,\},\ a \leq b \,\} \cup \{\, \emptyset \,\},$
  $\subseteq,\ \sqcup,\ \cap,\ \emptyset,\ [-\infty, +\infty])$
  with $\sqcup_{i \in I} [a_i, b_i] \overset{\text{def}}{=} [\min_{i \in I} a_i,\ \max_{i \in I} b_i]$.
  (in two slides)

Example: the powerset complete lattice



Example: $(\mathcal{P}(\mathbb{Z}), \subseteq, \cup, \cap, \emptyset, \mathbb{Z})$

Example: the intervals complete lattice



The integer intervals with finite and infinite bounds:
$(\{\,[a,b] \mid a \in \mathbb{Z} \cup \{\,-\infty\,\},\ b \in \mathbb{Z} \cup \{\,+\infty\,\},\ a \leq b\,\} \cup \{\,\emptyset\,\},$
$\subseteq,\ \sqcup,\ \cap,\ \emptyset,\ [-\infty, +\infty])$

Derivation

Given a (complete) lattice or partial order $(X, \sqsubseteq, \sqcup, \sqcap, \bot, \top)$ we can derive new (complete) lattices or partial orders by:

- duality
  $(X, \sqsupseteq, \sqcap, \sqcup, \top, \bot)$
  - $\sqsubseteq$ is reversed
  - $\sqcup$ and $\sqcap$ are switched
  - $\bot$ and $\top$ are switched
- lifting (adding a smallest element)
  $(X \cup \{\,\bot'\,\}, \sqsubseteq', \sqcup', \sqcap', \bot', \top)$
  - $a \sqsubseteq' b \iff a = \bot' \vee a \sqsubseteq b$
  - $\bot' \sqcup' a = a \sqcup' \bot' = a$, and $a \sqcup' b = a \sqcup b$ if $a, b \neq \bot'$
  - $\bot' \sqcap' a = a \sqcap' \bot' = \bot'$, and $a \sqcap' b = a \sqcap b$ if $a, b \neq \bot'$
  - $\bot'$ replaces $\bot$
  - $\top$ is unchanged

Derivation (cont.)

Given (complete) lattices or partial orders:
$(X_1, \sqsubseteq_1, \sqcup_1, \sqcap_1, \bot_1, \top_1)$ and $(X_2, \sqsubseteq_2, \sqcup_2, \sqcap_2, \bot_2, \top_2)$

We can combine them by:

- product
  $(X_1 \times X_2, \sqsubseteq, \sqcup, \sqcap, \bot, \top)$ where
  - $(x, y) \sqsubseteq (x', y') \iff x \sqsubseteq_1 x' \land y \sqsubseteq_2 y'$
  - $(x, y) \sqcup (x', y') \overset{\text{def}}{=} (x \sqcup_1 x',\ y \sqcup_2 y')$
  - $(x, y) \sqcap (x', y') \overset{\text{def}}{=} (x \sqcap_1 x',\ y \sqcap_2 y')$
  - $\bot \overset{\text{def}}{=} (\bot_1, \bot_2)$
  - $\top \overset{\text{def}}{=} (\top_1, \top_2)$
- smashed product (coalescent product, merging $\bot_1$ and $\bot_2$)
  $(((X_1 \setminus \{\, \bot_1 \,\}) \times (X_2 \setminus \{\, \bot_2 \,\})) \cup \{\, \bot \,\}, \sqsubseteq, \sqcup, \sqcap, \bot, \top)$

  (as $X_1 \times X_2$, but all elements of the form $(\bot_1, y)$ and $(x, \bot_2)$ are identified to a unique $\bot$ element)

Derivation (cont.)

Given a (complete) lattice or partial order $(X, \sqsubseteq, \sqcup, \sqcap, \bot, \top)$ and a set S:

- point-wise lifting (functions from S to X)
  $(S \to X, \sqsubseteq', \sqcup', \sqcap', \bot', \top')$ where
  - $x \sqsubseteq' y \iff \forall s \in S\colon x(s) \sqsubseteq y(s)$
  - $\forall s \in S\colon (x \sqcup' y)(s) \stackrel{\text{def}}{=} x(s) \sqcup y(s)$
  - $\forall s \in S\colon (x \sqcap' y)(s) \stackrel{\text{def}}{=} x(s) \sqcap y(s)$
  - $\forall s \in S\colon \bot'(s) = \bot$
  - $\forall s \in S\colon \top'(s) = \top$
- smashed point-wise lifting
  $((S \to (X \setminus \{\bot\})) \cup \{\bot'\}, \sqsubseteq', \sqcup', \sqcap', \bot', \top')$

  as $S \to X$, but identify to $\bot'$ any map x where $\exists s \in S\colon x(s) = \bot$

  (e.g. map each program variable in S to an interval in X)

Distributivity

A lattice $(X, \sqsubseteq, \sqcup, \sqcap)$ is distributive if:

- $a \sqcup (b \sqcap c) = (a \sqcup b) \sqcap (a \sqcup c)$ and
- $a \sqcap (b \sqcup c) = (a \sqcap b) \sqcup (a \sqcap c)$

Examples, Counter-examples:

- $(\mathcal{P}(X), \subseteq, \cup, \cap)$ is distributive
- intervals are not distributive
  $([0,0] \sqcup [2,2]) \sqcap [1,1] = [0,2] \sqcap [1,1] = [1,1]$ but
  $([0,0] \sqcap [1,1]) \sqcup ([2,2] \sqcap [1,1]) = \emptyset \sqcup \emptyset = \emptyset$

  common cause of precision loss in static analyses:
  merging abstract information early, at control-flow joins
  vs. merging executions paths late, at the end of the program

Sublattice

Given a lattice $(X, \sqsubseteq, \sqcup, \sqcap)$ and $X' \subseteq X$
$(X', \sqsubseteq, \sqcup, \sqcap)$ is a sublattice of X if X' is closed under $\sqcup$ and $\sqcap$

Example, Counter-examples:

- if $Y \subseteq X$, $(\mathcal{P}(Y), \subseteq, \cup, \cap, \emptyset, Y)$ is a sublattice of $(\mathcal{P}(X), \subseteq, \cup, \cap, \emptyset, X)$
- integer intervals are not a sublattice of $(\mathcal{P}(\mathbb{Z}), \subseteq, \cup, \cap, \emptyset, \mathbb{Z})$
  $[\min(a, a'), \max(b, b')] \neq [a, b] \cup [a', b']$

  another common cause of precision loss in static analyses:
  $\sqcup$ cannot represent the exact union, and loses precision

Functions and Fixpoints

Functions

A function $f : (X_1, \sqsubseteq_1, \sqcup_1, \bot_1) \to (X_2, \sqsubseteq_2, \sqcup_2, \bot_2)$ is

- monotonic if

  $$\forall x, x', \ x \sqsubseteq_1 x' \implies f(x) \sqsubseteq_2 f(x')$$

  (aka: increasing, isotone, order-preserving, morphism)

- strict if $f(\bot_1) = \bot_2$

- continuous between CPO if

  $\forall C$ chain $\subseteq X_1$, $\{ f(c) \mid c \in C \}$ is a chain in X_2
  and $f(\sqcup_1 C) = \sqcup_2 \{ f(c) \mid c \in C \}$

- a (complete) $\sqcup-$morphism between (complete) lattices

  if $\forall S \subseteq X_1,\ f(\sqcup_1 S) = \sqcup_2 \{ f(s) \mid s \in S \}$

- extensive if $X_1 = X_2$ and $\forall x,\ x \sqsubseteq_1 f(x)$
- reductive if $X_1 = X_2$ and $\forall x,\ f(x) \sqsubseteq_1 x$

Fixpoints

Given $f : (X, \sqsubseteq) \to (X, \sqsubseteq)$

- x is a fixpoint of f if $f(x) = x$
- x is a pre-fixpoint of f if $x \sqsubseteq f(x)$
- x is a post-fixpoint of f if $f(x) \sqsubseteq x$

We may have several fixpoints (or none)

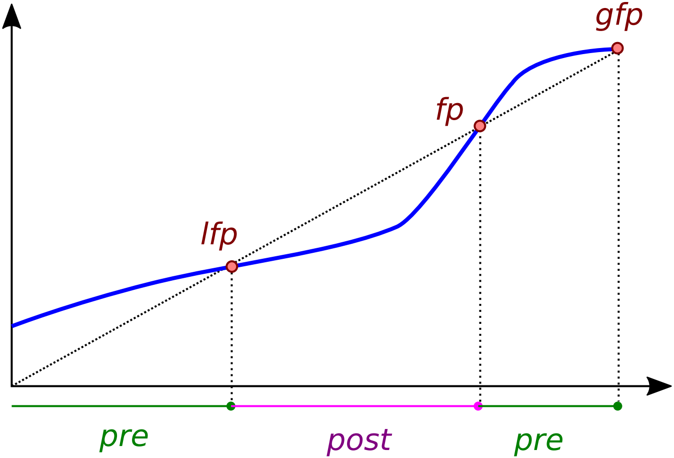
- $\mathrm{fp}(f) \stackrel{\mathrm{def}}{=} \{\, x \in X \,|\, f(x) = x \,\}$
- $\mathrm{lfp}_x\, f \stackrel{\mathrm{def}}{=} \min_{\sqsubseteq} \{\, y \in \mathrm{fp}(f) \,|\, x \sqsubseteq y \,\}$ if it exists

  (least fixpoint greater than x)
- $\mathrm{lfp}\, f \stackrel{\mathrm{def}}{=} \mathrm{lfp}_\perp\, f$

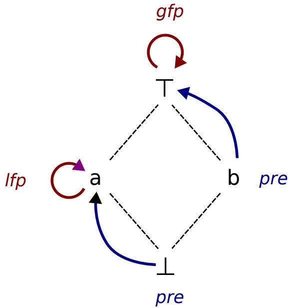
  (least fixpoint)
- dually: $\mathrm{gfp}_x\, f \stackrel{\mathrm{def}}{=} \max_{\sqsubseteq} \{\, y \in \mathrm{fp}(f) \,|\, y \sqsubseteq x \,\}$, $\mathrm{gfp}\, f \stackrel{\mathrm{def}}{=} \mathrm{gfp}_\top\, f$

  (greatest fixpoints)

Fixpoints: illustration

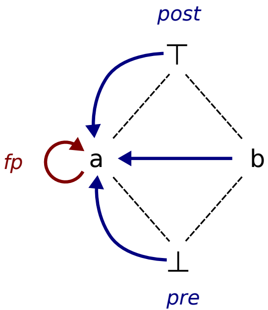


Fixpoints: example



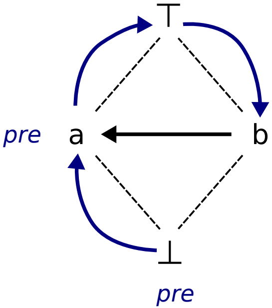
Monotonic function with two distinct fixpoints

Fixpoints: example



Monotonic function with a unique fixpoint

Fixpoints: example



Non-monotonic function with no fixpoint

Uses of fixpoints: examples

- Express solutions of mutually recursive equation systems

Example:

The solutions of $\left\{\begin{array}{l} x_1 = f(x_1, x_2) \\ x_2 = g(x_1, x_2) \end{array}\right.$ with x_1, x_2 in lattice X

are exactly the fixpoint of $\vec{F}$ in lattice $X \times X$, where

$$\vec{F}\left(\begin{array}{c} x_1, \\ x_2 \end{array}\right) = \left(\begin{array}{c} f(x_1, x_2), \\ g(x_1, x_2) \end{array}\right)$$

The least solution of the system is $\text{lfp}\ \vec{F}$.

Uses of fixpoints: examples

- Close (complete) sets to satisfy a given property

Example:

$r \subseteq X \times X$ is transitive if:
$(a, b) \in r \wedge (b, c) \in r \implies (a, c) \in r$

The transitive closure of r is the smallest transitive relation containing r.

Let $f(s) = r \cup \{\, (a, c) \mid (a, b) \in s \wedge (b, c) \in s \,\}$, then $\operatorname{lfp} f$:

- $\operatorname{lfp} f$ contains r
- $\operatorname{lfp} f$ is transitive
- $\operatorname{lfp} f$ is minimal

$\implies$ $\operatorname{lfp} f$ is the transitive closure of r.

Tarski's fixpoint theorem

Tarski's theorem

If $f : X \to X$ is monotonic in a complete lattice X then $\mathrm{fp}(f)$ is a complete lattice.

Proved by Knaster and Tarski [Tars55].

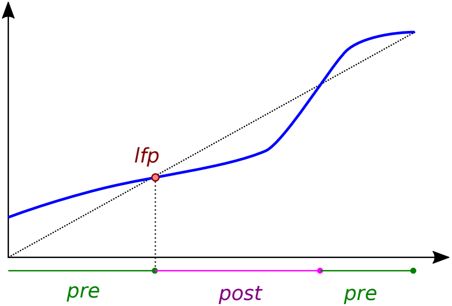
Tarski's fixpoint theorem

Tarski's theorem

If $f : X \to X$ is monotonic in a complete lattice X then $\mathrm{fp}(f)$ is a complete lattice.

Proof:

We prove $\mathrm{lfp}\, f = \sqcap \{\, x \mid f(x) \sqsubseteq x \,\}$ (meet of post-fixpoints).



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Proof:

We prove $\mathrm{lfp}\, f = \sqcap \{\, x \mid f(x) \sqsubseteq x \,\}$ (meet of post-fixpoints).
Let $f^* = \{\, x \mid f(x) \sqsubseteq x \,\}$ and $a = \sqcap f^*$.

$\forall x \in f^*,\ a \sqsubseteq x$ (by definition of $\sqcap$)
so $f(a) \sqsubseteq f(x)$ (as f is monotonic)
so $f(a) \sqsubseteq x$ (as x is a post-fixpoint).
We deduce that $f(a) \sqsubseteq \sqcap f^*$, i.e. $f(a) \sqsubseteq a$.

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If $f : X \to X$ is monotonic in a complete lattice X then $\text{fp}(f)$ is a complete lattice.

Proof:

We prove $\text{lfp}\, f = \sqcap \{ x \mid f(x) \sqsubseteq x \}$ (meet of post-fixpoints).

$f(a) \sqsubseteq a$
so $f(f(a)) \sqsubseteq f(a)$ (as f is monotonic)
so $f(a) \in f^*$ (by definition of f^*)
so $a \sqsubseteq f(a)$.

We deduce that $f(a) = a$, so $a \in \text{fp}(f)$.

Note that $y \in \text{fp}(f)$ implies $y \in f^*$.
As $a = \sqcap f^*$, $a \sqsubseteq y$, and we deduce $a = \text{lfp}\, f$.

Tarski's fixpoint theorem

Tarski's theorem

If $f : X \to X$ is monotonic in a complete lattice X then $\mathrm{fp}(f)$ is a complete lattice.

Proof:

Given $S \subseteq \mathrm{fp}(f)$, we prove that $\mathrm{lfp}_{\sqcup S}\, f$ exists.

Consider $X' = \{\, x \in X \mid \sqcup\, S \sqsubseteq x \,\}$.
X' is a complete lattice.
Moreover $\forall x' \in X'$, $f(x') \in X'$.
f can be restricted to a monotonic function f' on X'.
We apply the preceding result, so that $\mathrm{lfp}\, f' = \mathrm{lfp}_{\sqcup S}\, f$ exists.

By definition, $\mathrm{lfp}_{\sqcup S}\, f \in \mathrm{fp}(f)$ and is smaller than any fixpoint larger than all $s \in S$.

Tarski's fixpoint theorem

Tarski's theorem

If $f : X \rightarrow X$ is monotonic in a complete lattice X then $\mathrm{fp}(f)$ is a complete lattice.

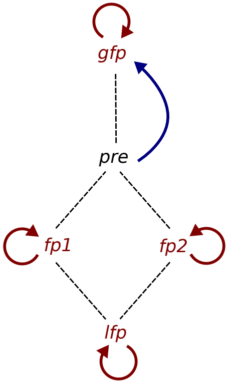
Proof:

By duality, we construct $\mathrm{gfp}\, f$ and $\mathrm{gfp}_{\sqcap S}\, f$.

The complete lattice of fixpoints is:
$(\mathrm{fp}(f),\ \sqsubseteq,\ \lambda S.\mathrm{lfp}_{\sqcup S}\, f,\ \lambda S.\mathrm{gfp}_{\sqcap S}\, f,\ \mathrm{lfp}\, f,\ \mathrm{gfp}\, f)$.

Not necessarily a sublattice of $(X, \sqsubseteq, \sqcup, \sqcap, \bot, \top)$!

Tarski's fixpoint theorem: example



Lattice: $(\{\text{lfp}, \text{fp1}, \text{fp2}, \text{pre}, \text{gfp}\}, \sqcup, \sqcap, \text{lfp}, \text{gfp})$

Fixpoint lattice: $(\{\text{lfp}, \text{fp1}, \text{fp2}, \text{gfp}\}, \sqcup', \sqcap', \text{lfp}, \text{gfp})$

(not a sublattice as $\text{fp1} \sqcup' \text{fp2} = \text{gfp}$ while $\text{fp1} \sqcup \text{fp2} = \text{pre}$,

but gfp is the smallest fixpoint greater than pre)

“Kleene” fixpoint theorem

“Kleene” fixpoint theorem

If $f : X \to X$ is continuous in a CPO X and $a \sqsubseteq f(a)$ then $\mathrm{lfp}_a\, f$ exists.

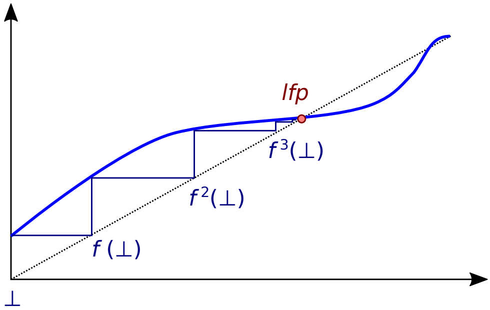
Inspired by Kleene [Klee52].

"Kleene" fixpoint theorem

"Kleene" fixpoint theorem

If $f : X \to X$ is continuous in a CPO X and $a \sqsubseteq f(a)$ then $\mathrm{lfp}_a\, f$ exists.

We prove that $\{\, f^n(a) \mid n \in \mathbb{N} \,\}$ is a chain and $\mathrm{lfp}_a\, f = \sqcup \{\, f^n(a) \mid n \in \mathbb{N} \,\}$.



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We prove that $\{\, f^n(a) \mid n \in \mathbb{N} \,\}$ is a chain and $\mathrm{lfp}_a\, f = \sqcup \{\, f^n(a) \mid n \in \mathbb{N} \,\}$.

$a \sqsubseteq f(a)$ by hypothesis.
$f(a) \sqsubseteq f(f(a))$ by monotony of f.
(Note that any continuous function is monotonic.
Indeed, $x \sqsubseteq y \implies x \sqcup y = y \implies f(x \sqcup y) = f(y)$;
by continuity $f(x) \sqcup f(y) = f(x \sqcup y) = f(y)$, which implies $f(x) \sqsubseteq f(y)$.)
By recurrence $\forall n$, $f^n(a) \sqsubseteq f^{n+1}(a)$.
Thus, $\{\, f^n(a) \mid n \in \mathbb{N} \,\}$ is a chain and $\sqcup \{\, f^n(a) \mid n \in \mathbb{N} \,\}$ exists.

"Kleene" fixpoint theorem

"Kleene" fixpoint theorem

If $f : X \to X$ is continuous in a CPO X and $a \sqsubseteq f(a)$ then $\mathrm{lfp}_a\, f$ exists.

$f(\sqcup \{\, f^n(a) \mid n \in \mathbb{N} \,\})$
$= \sqcup \{\, f^{n+1}(a) \mid n \in \mathbb{N} \,\})$ (by continuity)
$= a \sqcup (\sqcup \{\, f^{n+1}(a) \mid n \in \mathbb{N} \,\})$ (as all $f^{n+1}(a)$ are greater than a)
$= \sqcup \{\, f^n(a) \mid n \in \mathbb{N} \,\}$.
So, $\sqcup \{\, f^n(a) \mid n \in \mathbb{N} \,\} \in \mathrm{fp}(f)$

Moreover, any fixpoint greater than a must also be greater than all $f^n(a)$, $n \in \mathbb{N}$.
So, $\sqcup \{\, f^n(a) \mid n \in \mathbb{N} \,\} = \mathrm{lfp}_a\, f$.

Well-ordered sets

$(S, \sqsubseteq)$ is a well-ordered set if:

- $\sqsubseteq$ is a total order on S
- every $X \subseteq S$ such that $X \neq \emptyset$ has a least element $\sqcap X \in X$

Consequences:

- any element $x \in S$ has a successor $x + 1 \stackrel{\text{def}}{=} \sqcap \{ y \mid x \sqsubset y \}$
  (except the greatest element, if it exists)
- if $\nexists y,\ x = y + 1$, x is a limit and $x = \sqcup \{ y \mid y \sqsubset x \}$
  (every bounded subset $X \subseteq S$ has a lub $\sqcup X = \sqcap \{ y \mid \forall x \in X,\ x \sqsubseteq y \}$)

Examples:

- $(\mathbb{N}, \leq)$ and $(\mathbb{N} \cup \{ \infty \}, \leq)$ are well-ordered
- $(\mathbb{Z}, \leq)$, $(\mathbb{R}, \leq)$, $(\mathbb{R}^+, \leq)$ are not well-ordered
- ordinals $0, 1, 2, \ldots, \omega, \omega + 1, \ldots$ are well-ordered (ω is a limit)
  well-ordered sets are ordinals up to order-isomorphism
  (i.e., bijective functions f such that f and f^{-1} are monotonic)

Constructive Tarski theorem by transfinite iterations

Given a function $f : X \to X$ and $a \in X$,
the transfinite iterates of f from a are:

$$\begin{cases} x_0 \overset{\text{def}}{=} a & \\ x_n \overset{\text{def}}{=} f(x_{n-1}) & \text{if } n \text{ is a successor ordinal} \\ x_n \overset{\text{def}}{=} \sqcup \{ x_m \mid m < n \} & \text{if } n \text{ is a limit ordinal} \end{cases}$$

Constructive Tarski theorem

If $f : X \to X$ is monotonic in a CPO X and $a \sqsubseteq f(a)$,
then $\mathrm{lfp}_a\, f = x_\delta$ for some ordinal δ.

Generalisation of “Kleene” fixpoint theorem, from [Cous79].

Proof

f is monotonic in a CPO X,

$$\begin{cases} x_0 \overset{\text{def}}{=} a \sqsubseteq f(a) & \\ x_n \overset{\text{def}}{=} f(x_{n-1}) & \text{if } n \text{ is a successor ordinal} \\ x_n \overset{\text{def}}{=} \sqcup \{\, x_m \mid m < n \,\} & \text{if } n \text{ is a limit ordinal} \end{cases}$$

Proof:

We prove that $\exists \delta,\, x_\delta = x_{\delta+1}$.

We note that $m \leq n \implies x_m \sqsubseteq x_n$.
Assume by contradiction that $\nexists \delta,\, x_\delta = x_{\delta+1}$.
If n is a successor ordinal, then $x_{n-1} \sqsubset x_n$.
If n is a limit ordinal, then $\forall m < n,\, x_m \sqsubset x_n$.
Thus, all the x_n are distinct.
By choosing $n > |X|$, we arrive at a contradiction.
Thus δ exists.

Proof

f is monotonic in a CPO X,

$$\begin{cases} x_0 \overset{\text{def}}{=} a \sqsubseteq f(a) & \\ x_n \overset{\text{def}}{=} f(x_{n-1}) & \text{if } n \text{ is a successor ordinal} \\ x_n \overset{\text{def}}{=} \sqcup \{\, x_m \mid m < n \,\} & \text{if } n \text{ is a limit ordinal} \end{cases}$$

Proof:

Given δ such that $x_{\delta+1} = x_\delta$, we prove that $x_\delta = \mathrm{lfp}_a\, f$.

$f(x_\delta) = x_{\delta+1} = x_\delta$, so $x_\delta \in \mathrm{fp}(f)$.

Given any $y \in \mathrm{fp}(f)$, $y \sqsupseteq a$, we prove by transfinite induction that $\forall n,\ x_n \sqsubseteq y$.

By definition $x_0 = a \sqsubseteq y$.

If n is a successor ordinal, by monotony,

$x_{n-1} \sqsubseteq y \implies f(x_{n-1}) \sqsubseteq f(y)$, i.e., $x_n \sqsubseteq y$.

If n is a limit ordinal, $\forall m < n$, $x_m \sqsubseteq y$ implies

$x_n = \sqcup \{\, x_m \mid m < n \,\} \sqsubseteq y$.

Hence, $x_\delta \sqsubseteq y$ and $x_\delta = \mathrm{lfp}_a\, f$.

Ascending chain condition (ACC)

An ascending chain C in $(X, \sqsubseteq)$ is a sequence $c_i \in X$ such that $i \leq j \implies c_i \sqsubseteq c_j$.

A poset $(X, \sqsubseteq)$ satisfies the ascending chain condition (ACC) iff for every ascending chain C, $\exists i \in \mathbb{N}, \forall j \geq i, c_i = c_j$.

Similarly, we can define the descending chain condition (DCC).

Examples:

- the powerset poset $(\mathcal{P}(X), \subseteq)$ is ACC when X is finite
- the pointed integer poset $(\mathbb{Z} \cup \{ \perp \}, \sqsubseteq)$ where $x \sqsubseteq y \iff x = \perp \lor x = y$ is ACC and DCC
- the divisibility poset $(\mathbb{N}^*, |)$ is DCC but not ACC.

Kleene fixpoints in ACC posets

"Kleene" finite fixpoint theorem

If $f : X \to X$ is monotonic in an ACC poset X and $a \sqsubseteq f(a)$ then $\mathrm{lfp}_a\, f$ exists.

Proof:

We prove $\exists n \in \mathbb{N},\ \mathrm{lfp}_a\, f = f^n(a)$.

By monotony of f, the sequence $x_n = f^n(a)$ is an increasing chain.
By definition of ACC, $\exists n \in \mathbb{N},\ x_n = x_{n+1} = f(x_n)$.
Thus, $x_n \in \mathrm{fp}(f)$.

Obviously, $a = x_0 \sqsubseteq f(x_n)$.
Moreover, if $y \in \mathrm{fp}(f)$ and $y \sqsupseteq a$, then $\forall i,\ y \sqsupseteq f^i(a) = x_i$.
Hence, $y \sqsupseteq x_n$ and $x_n = \mathrm{lfp}_a\,(f)$.

Comparison of fixpoint theorems

theorem	**function**	**domain**	**fixpoint**	**method**
Tarski	monotonic	complete lattice	fp(f)	meet of post-fixpoints
Kleene	continuous	CPO	$\mathrm{lfp}_a(f)$	countable iterations
constructive Tarski	monotonic	CPO	$\mathrm{lfp}_a(f)$	transfinite iteration
ACC Kleene	monotonic	poset	$\mathrm{lfp}_a(f)$	finite iteration

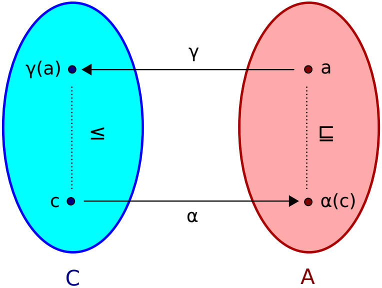
Galois connections

Galois connections

Given two posets $(C,\leq)$ and $(A,\sqsubseteq)$, the pair $(\alpha : C \to A,\ \gamma : A \to C)$ is a Galois connection iff:

$$\forall a \in A,\ c \in C,\ \alpha(c) \sqsubseteq a \iff c \leq \gamma(a)$$

which is noted $(C,\leq) \xleftrightarrow[\alpha]{\gamma} (A,\sqsubseteq)$.



- α is the upper adjoint or abstraction; A is the abstract domain.
- γ is the lower adjoint or concretization; C is the concrete domain.

Galois connection example

Abstract domain of intervals of integers $\mathbb{Z}$
represented as pairs of bounds (a, b).

We have: $(\mathcal{P}(\mathbb{Z}), \subseteq) \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} (I, \sqsubseteq)$

- $I \overset{\text{def}}{=} (\mathbb{Z} \cup \{-\infty\}) \times (\mathbb{Z} \cup \{+\infty\})$
- $(a, b) \sqsubseteq (a', b') \iff (a \geq a') \wedge (b \leq b')$
- $\gamma(a, b) \overset{\text{def}}{=} \{\, x \in \mathbb{Z} \mid a \leq x \leq b \,\}$
- $\alpha(X) \overset{\text{def}}{=} (\min X, \max X)$

proof:

Galois connection example

Abstract domain of intervals of integers $\mathbb{Z}$ represented as pairs of bounds (a, b).

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- $\gamma(a, b) \overset{\text{def}}{=} \{\, x \in \mathbb{Z} \mid a \leq x \leq b \,\}$
- $\alpha(X) \overset{\text{def}}{=} (\min X, \max X)$

proof:

$$\begin{array}{l} \alpha(X) \sqsubseteq (a, b) \\ \iff \min X \geq a \wedge \max X \leq b \\ \iff \forall x \in X \colon a \leq x \leq b \\ \iff \forall x \in X \colon x \in \{\, y \mid a \leq y \leq b \,\} \\ \iff \forall x \in X \colon x \in \gamma(a, b) \\ \iff X \subseteq \gamma(a, b) \end{array}$$

Properties of Galois connections

Assuming $\forall a, c,\ \alpha(c) \sqsubseteq a \iff c \leq \gamma(a)$, we have:

1. $\gamma \circ \alpha$ is extensive: $\forall c,\ c \leq \gamma(\alpha(c))$

   proof: $\alpha(c) \sqsubseteq \alpha(c) \implies c \leq \gamma(\alpha(c))$

2. $\alpha \circ \gamma$ is reductive: $\forall a,\ \alpha(\gamma(a)) \sqsubseteq a$

3. α is monotonic

   proof: $c \leq c' \implies c \leq \gamma(\alpha(c')) \implies \alpha(c) \sqsubseteq \alpha(c')$

4. γ is monotonic

5. $\gamma \circ \alpha \circ \gamma = \gamma$

   proof: $\alpha(\gamma(a)) \sqsubseteq \alpha(\gamma(a)) \implies \gamma(a) \leq \gamma(\alpha(\gamma(a)))$ and
   $a \sqsupseteq \alpha(\gamma(a)) \implies \gamma(a) \geq \gamma(\alpha(\gamma(a)))$

6. $\alpha \circ \gamma \circ \alpha = \alpha$

7. $\alpha \circ \gamma$ is idempotent: $\alpha \circ \gamma \circ \alpha \circ \gamma = \alpha \circ \gamma$

8. $\gamma \circ \alpha$ is idempotent

Alternate characterization

If the pair $(\alpha : C \rightarrow A, \gamma : A \rightarrow C)$ satisfies:

1. γ is monotonic,
2. α is monotonic,
3. $\gamma \circ \alpha$ is extensive
4. $\alpha \circ \gamma$ is reductive

then (α, γ) is a Galois connection.

(proof left as exercise)

Uniqueness of the adjoint

Given $(C, \leq) \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} (A, \sqsubseteq)$,
each adjoint can be uniquely defined in term of the other:

1. $\alpha(c) = \sqcap \{\, a \mid c \leq \gamma(a) \,\}$
2. $\gamma(a) = \vee \{\, c \mid \alpha(c) \sqsubseteq a \,\}$

Proof: of 1

$\forall a,\ c \leq \gamma(a) \implies \alpha(c) \sqsubseteq a$.
Hence, $\alpha(c)$ is a lower bound of $\{\, a \mid c \leq \gamma(a) \,\}$.

Assume that a' is another lower bound.
Then, $\forall a,\ c \leq \gamma(a) \implies a' \sqsubseteq a$.
By Galois connection, we have then $\forall a,\ \alpha(c) \sqsubseteq a \implies a' \sqsubseteq a$.
This implies $a' \sqsubseteq \alpha(c)$.
Hence, the greatest lower bound of $\{\, a \mid c \leq \gamma(a) \,\}$ exists,
and equals $\alpha(c)$.

The proof of 2 is similar (by duality).

Properties of Galois connections (cont.)

If $(\alpha : C \to A, \gamma : A \to C)$, then:

1. $\forall X \subseteq C$, if $\vee X$ exists, then $\alpha(\vee X) = \sqcup \{\, \alpha(x) \mid x \in X \,\}$.
2. $\forall X \subseteq A$, if $\sqcap X$ exists, then $\gamma(\sqcap X) = \wedge \{\, \gamma(x) \mid x \in X \,\}$.

Proof: of 1

By definition of lubs, $\forall x \in X$, $x \leq \vee X$.
By monotony, $\forall x \in X$, $\alpha(x) \sqsubseteq \alpha(\vee X)$.
Hence, $\alpha(\vee X)$ is an upper bound of $\{\, \alpha(x) \mid x \in X \,\}$.

Assume that y is another upper bound of $\{\, \alpha(x) \mid x \in X \,\}$.
Then, $\forall x \in X$, $\alpha(x) \sqsubseteq y$.
By Galois connection $\forall x \in X$, $x \leq \gamma(y)$.
By definition of lubs, $\vee X \leq \gamma(y)$.
By Galois connection, $\alpha(\vee X) \sqsubseteq y$.
Hence, $\{\, \alpha(x) \mid x \in X \,\}$ has a lub, which equals $\alpha(\vee X)$.

The proof of 2 is similar (by duality).

Deriving Galois connections

Given $(C, \leq) \xleftrightarrow[\alpha]{\gamma} (A, \sqsubseteq)$, we have:

- duality: $(A, \sqsupseteq) \xleftrightarrow[\gamma]{\alpha} (C, \geq)$

  $(\alpha(c) \sqsubseteq a \iff c \leq \gamma(a)$ is exactly $\gamma(a) \geq c \iff a \sqsupseteq \alpha(c))$

- point-wise lifting by some set S: $(S \to C, \dot{\leq}) \xleftrightarrow[\dot{\alpha}]{\dot{\gamma}} (S \to A, \dot{\sqsubseteq})$

  where

  $f \dot{\leq} f' \iff \forall s,\ f(s) \leq f'(s),\quad (\dot{\gamma}(f))(s) = \gamma(f(s)),$
  $f \dot{\sqsubseteq} f' \iff \forall s,\ f(s) \sqsubseteq f'(s),\quad (\dot{\alpha}(f))(s) = \alpha(f(s)).$

Given $(X_1, \sqsubseteq_1) \xleftrightarrow[\alpha_1]{\gamma_1} (X_2, \sqsubseteq_2) \xleftrightarrow[\alpha_2]{\gamma_2} (X_3, \sqsubseteq_3)$:

- composition: $(X_1, \sqsubseteq_1) \xleftrightarrow[\alpha_2 \circ \alpha_1]{\gamma_1 \circ \gamma_2} (X_3, \sqsubseteq_3)$

  $((\alpha_2 \circ \alpha_1)(c) \sqsubseteq_3 a \iff \alpha_1(c) \sqsubseteq_2 \gamma_2(a) \iff c \sqsubseteq_1 (\gamma_1 \circ \gamma_2)(a))$

Galois embeddings

If $(C, \leq) \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} (A, \sqsubseteq)$, the following properties are equivalent:

1. α is surjective $(\forall a \in A, \exists c \in C, \alpha(c) = a)$
2. γ is injective $(\forall a, a' \in A, \gamma(a) = \gamma(a') \implies a = a')$
3. $\alpha \circ \gamma = id$ $(\forall a \in A, id(a) = a)$

Such (α, γ) is called a Galois embedding, which is noted
$(C, \leq) \underset{\alpha}{\overset{\gamma}{\leftrightarrows\!\!\rightarrow}} (A, \sqsubseteq)$

Proof:

Galois embeddings

If $(C, \leq) \xleftarrow[\alpha]{\gamma} \xrightarrow{} (A, \sqsubseteq)$, the following properties are equivalent:

1. α is surjective $(\forall a \in A, \exists c \in C, \alpha(c) = a)$
2. γ is injective $(\forall a, a' \in A, \gamma(a) = \gamma(a') \implies a = a')$
3. $\alpha \circ \gamma = id$ $(\forall a \in A, id(a) = a)$

Such (α, γ) is called a Galois embedding, which is noted

$$(C, \leq) \underset{\alpha}{\overset{\gamma}{\leftarrows\!\!\!\twoheadrightarrow}} (A, \sqsubseteq)$$

Proof: $1 \implies 2$

Assume that $\gamma(a) = \gamma(a')$.
By surjectivity, take c, c' such that $a = \alpha(c)$, $a' = \alpha(c')$.
Then $\gamma(\alpha(c)) = \gamma(\alpha(c'))$.
And $\alpha(\gamma(\alpha(c))) = \alpha(\gamma(\alpha(c')))$.
As $\alpha \circ \gamma \circ \alpha = \alpha$, $\alpha(c) = \alpha(c')$.
Hence $a = a'$.

Galois embeddings

If $(C, \leq) \xleftrightarrow[\alpha]{\gamma} (A, \sqsubseteq)$, the following properties are equivalent:

1. α is surjective $(\forall a \in A, \exists c \in C, \alpha(c) = a)$
2. γ is injective $(\forall a, a' \in A, \gamma(a) = \gamma(a') \implies a = a')$
3. $\alpha \circ \gamma = id$ $(\forall a \in A, id(a) = a)$

Such (α, γ) is called a Galois embedding, which is noted
$(C, \leq) \underset{\alpha}{\overset{\gamma}{\leftarrow\!\!\!\twoheadrightarrow}} (A, \sqsubseteq)$

Proof: $2 \implies 3$

Given $a \in A$, we know that $\gamma(\alpha(\gamma(a))) = \gamma(a)$.
By injectivity of γ, $\alpha(\gamma(a)) = a$.

Galois embeddings

If $(C, \leq) \xleftarrow[\alpha]{\gamma} \!\!\rightarrow (A, \sqsubseteq)$, the following properties are equivalent:

1. α is surjective $\qquad (\forall a \in A, \exists c \in C, \alpha(c) = a)$
2. γ is injective $\qquad (\forall a, a' \in A, \gamma(a) = \gamma(a') \implies a = a')$
3. $\alpha \circ \gamma = id$ $\qquad (\forall a \in A, id(a) = a)$

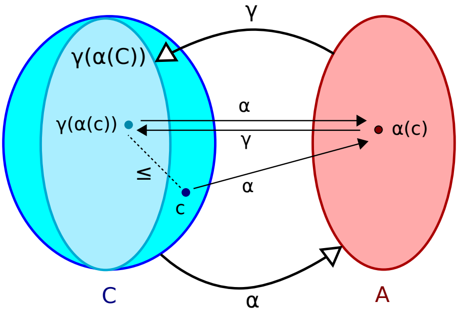
Such (α, γ) is called a Galois embedding, which is noted
$(C, \leq) \xleftarrow[\alpha]{\gamma} \!\!\twoheadrightarrow (A, \sqsubseteq)$

Proof: $3 \implies 1$

Given $a \in A$, we have $\alpha(\gamma(a)) = a$.
Hence, $\exists c \in C$, $\alpha(c) = a$, using $c = \gamma(a)$.

Galois embeddings (cont.)

$$(C, \leq) \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} (A, \sqsubseteq)$$



A Galois connection can be made into an embedding by quotienting A by the equivalence relation $a \equiv a' \iff \gamma(a) = \gamma(a')$.

Galois embedding example

Abstract domain of intervals of integers $\mathbb{Z}$
represented as pairs of ordered bounds (a, b) or $\perp$.

We have: $(\mathcal{P}(\mathbb{Z}), \subseteq) \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} (I, \sqsubseteq)$

- $I \overset{\text{def}}{=} \{\, (a, b) \mid a \in \mathbb{Z} \cup \{-\infty\}, b \in \mathbb{Z} \cup \{+\infty\}, a \leq b \,\} \cup \{\perp\}$
- $(a, b) \sqsubseteq (a', b') \iff (a \geq a') \wedge (b \leq b'), \quad \forall x: \perp \sqsubseteq x$
- $\gamma(a, b) \overset{\text{def}}{=} \{\, x \in \mathbb{Z} \mid a \leq x \leq b \,\}, \quad \gamma(\perp) = \emptyset$
- $\alpha(X) \overset{\text{def}}{=} (\min X, \max X)$, or $\perp$ if $X = \emptyset$

proof:

Galois embedding example

Abstract domain of intervals of integers $\mathbb{Z}$
represented as pairs of ordered bounds (a, b) or $\bot$.

We have: $(\mathcal{P}(\mathbb{Z}), \subseteq) \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} (I, \sqsubseteq)$

- $I \stackrel{\text{def}}{=} \{\, (a, b) \mid a \in \mathbb{Z} \cup \{-\infty\}, b \in \mathbb{Z} \cup \{+\infty\}, a \leq b \,\} \cup \{\bot\}$
- $(a, b) \sqsubseteq (a', b') \iff (a \geq a') \wedge (b \leq b'), \quad \forall x: \bot \sqsubseteq x$
- $\gamma(a, b) \stackrel{\text{def}}{=} \{\, x \in \mathbb{Z} \mid a \leq x \leq b \,\}, \quad \gamma(\bot) = \emptyset$
- $\alpha(X) \stackrel{\text{def}}{=} (\min X, \max X)$, or $\bot$ if $X = \emptyset$

proof:

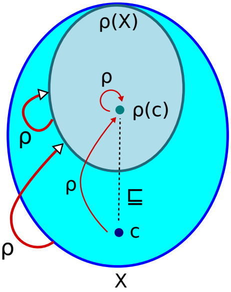
Quotient of the "pair of bounds" domain $(\mathbb{Z} \cup \{-\infty\}) \times (\mathbb{Z} \cup \{+\infty\})$ by the relation
$(a, b) \equiv (a', b') \iff \gamma(a, b) = \gamma(a', b')$

i.e., $(a \leq b \wedge a = a' \wedge b = b') \vee (a > b \wedge a' > b')$.

Upper closures

$\rho : X \to X$ is an upper closure in the poset $(X, \sqsubseteq)$ if it is:

1. monotonic: $x \sqsubseteq x' \implies \rho(x) \sqsubseteq \rho(x')$,
2. extensive: $x \sqsubseteq \rho(x)$, and
3. idempotent: $\rho \circ \rho = \rho$.



Upper closures and Galois connections

Given $(C, \leq) \xleftrightarrow[\alpha]{\gamma} (A, \sqsubseteq)$,
$\gamma \circ \alpha$ is an upper closure on $(C, \leq)$.

Given an upper closure ρ on $(X, \sqsubseteq)$, we have a Galois embedding:
$(X, \sqsubseteq) \xleftrightarrow[\rho]{id} (\rho(X), \sqsubseteq)$

$\Longrightarrow$ we can rephrase abstract interpretation using upper closures instead of Galois connections, but we lose:

- the notion of abstract representation

  (a data-structure A representing elements in $\rho(X)$)

- the ability to have several distinct abstract representations for a single concrete object

  (non-necessarily injective γ versus id)

Operator approximations

Abstractions in the concretization framework

Given a concrete $(C, \leq)$ and an abstract $(A, \sqsubseteq)$ poset and a monotonic concretization $\gamma : A \to C$

($\gamma(a)$ is the "meaning" of a in C; we use intervals in our examples)

- $a \in A$ is a sound abstraction of $c \in C$ if $c \leq \gamma(a)$.

  (e.g.: $[0, 10]$ is a sound abstraction of $\{0, 1, 2, 5\}$ in the integer interval domain)

- $g : A \to A$ is a sound abstraction of $f : C \to C$ if $\forall a \in A$: $(f \circ \gamma)(a) \leq (\gamma \circ g)(a)$.

  (e.g.: $\lambda([a, b].[-\infty, +\infty]$ is a sound abstraction of $\lambda X.\{\, x + 1 \mid x \in X \,\}$ in the interval domain)

- $g : A \to A$ is an exact abstraction of $f : C \to C$ if $f \circ \gamma = \gamma \circ g$.

  (e.g.: $\lambda([a, b].[a + 1, b + 1]$ is an exact abstraction of $\lambda X.\{\, x + 1 \mid x \in X \,\}$ in the interval domain)

Abstractions in the Galois connection framework

Assume now that $(C, \leq) \xleftrightarrow[\alpha]{\gamma} (A, \sqsubseteq)$.

- sound abstractions
  - $c \leq \gamma(a)$ is equivalent to $\alpha(c) \sqsubseteq a$.
  - $(f \circ \gamma)(a) \leq (\gamma \circ g)(a)$ is equivalent to $(\alpha \circ f \circ \gamma)(a) \sqsubseteq g(a)$.

- Given $c \in C$, its best abstraction is $\alpha(c)$.

  (proof: recall that $\alpha(c) = \sqcap \{\, a \mid c \leq \gamma(a) \,\}$, so, $\alpha(c)$ is the smallest sound abstraction of c)

  (e.g.: $\alpha(\{0, 1, 2, 5\}) = [0, 5]$ in the interval domain)

- Given $f : C \to C$, its best abstraction is $\alpha \circ f \circ \gamma$

  (proof: g sound $\iff$ $\forall a$, $(\alpha \circ f \circ \gamma)(a) \sqsubseteq g(a)$, so $\alpha \circ f \circ \gamma$ is the smallest sound abstraction of f)

  (e.g.: $g([a, b]) = [2a, 2b]$ is the best abstraction in the interval domain of $f(X) = \{\, 2x \mid x \in X \,\}$; it is not an exact abstraction as $\gamma(g([0, 1])) = \{0, 1, 2\} \supsetneq \{0, 2\} = f(\gamma([0, 1]))$)

Composition of sound, best, and exact abstractions

If g and g' soundly abstract respectively f and f' then:

- if f is monotonic,
  then $g \circ g'$ is a sound abstraction of $f \circ f'$,

  (proof: $\forall a,\ (f \circ f' \circ \gamma)(a) \leq (f \circ \gamma \circ g')(a) \leq (\gamma \circ g \circ g')(a)$)

- if g, g' are exact abstractions of f and f',
  then $g \circ g'$ is an exact abstraction,

  (proof: $f \circ f' \circ \gamma = f \circ \gamma \circ g' = \gamma \circ g \circ g'$)

- if g and g' are the best abstractions of f and f',
  then $g \circ g'$ is not always the best abstraction!

  (e.g.: $g([a,b]) = [a, \min(b,1)]$ and $g'([a,b]) = [2a, 2b]$ are the best abstractions of $f(X) = \{\, x \in X \mid x \leq 1 \,\}$ and $f'(X) = \{\, 2x \mid x \in X \,\}$ in the interval domain, but $g \circ g'$ is not the best abstraction of $f \circ f'$ as $(g \circ g')([0,1]) = [0,1]$ while $(\alpha \circ f \circ f' \circ \gamma)([0,1]) = [0,0]$)

Fixpoint approximations

Fixpoint transfer

If we have:

- a Galois connection $(C, \leq) \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} (A, \sqsubseteq)$ between CPOs
- monotonic concrete and abstract functions
  $f : C \to C$, $f^\sharp : A \to A$
- a commutation condition $\alpha \circ f = f^\sharp \circ \alpha$
- an element a and its abstraction $a^\sharp = \alpha(a)$

then $\alpha(\mathrm{lfp}_a\, f) = \mathrm{lfp}_{a^\sharp}\, f^\sharp$.

(proof on next slide)

Fixpoint transfer (proof)

Proof:

By the constructive Tarski theorem, $\mathrm{lfp}_a\, f$ is the limit of transfinite iterations: $a_0 \overset{\text{def}}{=} a$, $a_{n+1} \overset{\text{def}}{=} f(a_n)$, and $a_n \overset{\text{def}}{=} \bigvee \{\, a_m \mid m < n \,\}$ for limit ordinals n.
Likewise, $\mathrm{lfp}_{a^\sharp}\, f^\sharp$ is the limit of a transfinite iteration $a^\sharp_n$.

We prove by transfinite induction that $a^\sharp_n = \alpha(a_n)$ for all ordinals n:

- $a^\sharp_0 = \alpha(a_0)$, by definition;
- $a^\sharp_{n+1} = f^\sharp(a^\sharp_n) = f^\sharp(\alpha(a_n)) = \alpha(f(a_n)) = \alpha(a_{n+1})$ for successor ordinals, by commutation;
- $a^\sharp_n = \bigsqcup \{\, a^\sharp_m \mid m < n \,\} = \bigsqcup \{\, \alpha(a_m) \mid m < n \,\} = \alpha(\bigvee \{\, a_m \mid m < n \,\}) = \alpha(a_n)$ for limit ordinals, because α is always continuous in Galois connections.

Hence, $\mathrm{lfp}_{a^\sharp}\, f^\sharp = \alpha(\mathrm{lfp}_a\, f)$.

Fixpoint approximation

If we have:

- a complete lattice $(C, \leq, \vee, \wedge, \bot, \top)$
- a monotonic concrete function f
- a sound abstraction $f^\sharp : A \to A$ of f
  $(\forall x^\sharp : (f \circ \gamma)(x^\sharp) \leq (\gamma \circ f^\sharp)(x^\sharp))$
- a post-fixpoint $a^\sharp$ of $f^\sharp$ $(f^\sharp(a^\sharp) \sqsubseteq a^\sharp)$

then $a^\sharp$ is a sound abstraction of lfp f: $\text{lfp}\, f \leq \gamma(a^\sharp)$.

Proof:

By definition, $f^\sharp(a^\sharp) \sqsubseteq a^\sharp$.
By monotony, $\gamma(f^\sharp(a^\sharp)) \leq \gamma(a^\sharp)$.
By soundness, $f(\gamma(a^\sharp)) \leq \gamma(a^\sharp)$.
By Tarski's theorem $\text{lfp}\, f = \wedge \{\, x \mid f(x) \leq x \,\}$.

Hence, $\text{lfp}\, f \leq \gamma(a^\sharp)$.

Other fixpoint transfer / approximation theorems can be constructed...

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