

MPRI

Static Analysis of Digital Filters

ESOP 2004, NSAD 2005

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Overview

1. Introduction
2. Case studies
3. Concrete semantics
4. Generic approximation
5. Filter domains
6. Post fixpoint inference of contracting function in floating-point arithmetics
7. Basic simplified filters
8. Higher order simplified filters
9. Bounded expansion
10. Filter detection
11. Conclusion

Context

We want to **prove run time error absence**, in **critical embedded software**.
Filter behaviour is implemented at the software level, using hardware floating point numbers.



Full certification requires special care about these filters.

Issues

- **Detection**: to locate **filter resets** and **filter iterations**.
- **Invariant inference**: we are not interested in functional properties.
We seek precise bounds on the output, using information inferred about the input.
(**Linear invariants do not yield accurate bounds**).
- To take into account **floating-point rounding**:
 - in **the semantics**,
 - when implementing **the abstract domain**.

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The high band-pass filter

```
 $V \in \mathbb{R};$   
 $E_1 := 0; S := 0;$   
while  $(V \geq 0)$  {  
   $V \in \mathbb{R}; T \in \mathbb{R};$   
   $E_0 \in [-1;1];$   
  if  $(T \geq 0)$  { $S := 0$ }  
  else { $S := 0.999 \times S + E_0 - E_1$ }  
   $E_1 := E_0;$   
}
```

Interval approximation (simplified)

The analyzer infers the following sound counterpart $\mathbb{F}^\#$:

$$\mathbb{F}^\#(X) = \{0.999s + e_0 + e_1 \mid s \in X, e_0, e_1 \in [-1; 1]\}$$

to the loop body.

Abstract iteration

1. The analyzer starts iterating \mathbb{F}^\sharp :

$$\mathbb{F}^\sharp(\{0\}) = [-2; 2],$$

$$\mathbb{F}^\sharp([-2; 2]) = [-3.998; 3.998],$$

...

2. then it widens the iterates:

$$\mathbb{F}^\sharp([-10; 10]) \not\subseteq [-10; 10],$$

$$\mathbb{F}^\sharp([-100; 100]) \not\subseteq [-100; 100],$$

...

3. until it discovers a stable threshold:

$$\mathbb{F}^\sharp([-10000; 10000]) = [-9992; 9992];$$

4. finally, it keeps iterating to refine the solution:

$$\mathbb{F}^\sharp([-9992; 9992]) = [-9984.008; 9984.008].$$

Driving the analysis

Theorem 1 (**High band-pass filter** (history-insensitive))

Let $D \geq 0$, $m \geq 0$, a , X and Z be real numbers such that:

1. $|X| \leq D$;
2. $aX - m \leq Z \leq aX + m$;

then we have:

1. $|Z| \leq |a|D + m$;
2. $\left[|a| < 1 \text{ and } D \geq \frac{m}{1-|a|} \right] \implies |Z| \leq D.$

□

Theorem 1 implies that **2000** can be used as a widening threshold.

History sensitive approximation

Theorem 2 (**High band-pass filter** (history-sensitive version))

Let $\alpha \in [\frac{1}{2}; 1[$, i and $m > 0$ be real numbers.

Let E_n be a real number sequence, such that $\forall k \in \mathbb{N}$, $E_k \in [-m; m]$.

Let S_n be the following sequence:

$$\begin{cases} S_0 = i \\ S_{n+1} = \alpha S_n + E_{n+1} - E_n. \end{cases}$$

We have:

1. $S_n = \alpha^n i + E_n - \alpha^n E_0 + \sum_{l=1}^{n-1} (\alpha - 1) \alpha^{l-1} E_{n-l}$
2. $|S_n| \leq |\alpha|^n |i| + (1 + |\alpha|^n + |1 - \alpha^{n-1}|)m;$
3. $|S_n| \leq 2m + |i|.$

□

Theorem 2 implies that 2 is a sound bound on $|S|$.

The second order filter

$V \in \mathbb{R};$

$E_1 := 0; E_2 := 0; S_0 := 0; S_1 := 0; S_2 := 0;$

while $(V \geq 0)$ {

$V \in \mathbb{R}; T \in \mathbb{R};$

$E_0 \in [-1; 1];$

if $(T \geq 0)$ { $S_0 := E_0; S_1 := E_0; E_1 := E_0$ }

else { $S_0 := 1.5 \times S_1 - 0.7 \times S_2$
 $+ 0.5 \times E_0 - 0.7 \times E_1 + 0.4 \times E_2$ };

$E_2 := E_1; E_1 := E_0;$

$S_2 := S_1; S_1 := S_0$

}

Quadratic constraints

Theorem 3 (second order filter (history insensitive))

Let $a, b, K \geq 0, m \geq 0, X, Y, Z$ be real numbers such that:

1. $a^2 + 4b < 0,$
2. $X^2 - aXY - bY^2 \leq K,$
3. $aX + bY - m \leq Z \leq aX + bY + m.$

We have:

1. $Z^2 - aZX - bX^2 \leq (\sqrt{-bK} + m)^2;$

2.
$$\begin{cases} \sqrt{-b} < 1 \\ K \geq \left(\frac{m}{1-\sqrt{-b}} \right)^2 \end{cases} \implies Z^2 - aZX - bX^2 \leq K.$$

Proof

We define $Q(X, Y) \triangleq X^2 - aXY - bY^2$ and $Z \triangleq aX + bY + e$.

We have:

$$Q(Z, X) = (aX + bY + e)^2 - a(aX + bY + e)X - bX^2$$

$$Q(Z, X) = -b(X^2 - aXY - bY^2) + e(aX + 2bY + e)$$

$$Q(Z, X) = -bQ(X, Y) + e(aX + 2bY + e)$$

$$Q(Z, X) \leq -bQ(X, Y) + m|aX + 2bY| + m^2 \quad \text{since } |e| \leq m$$

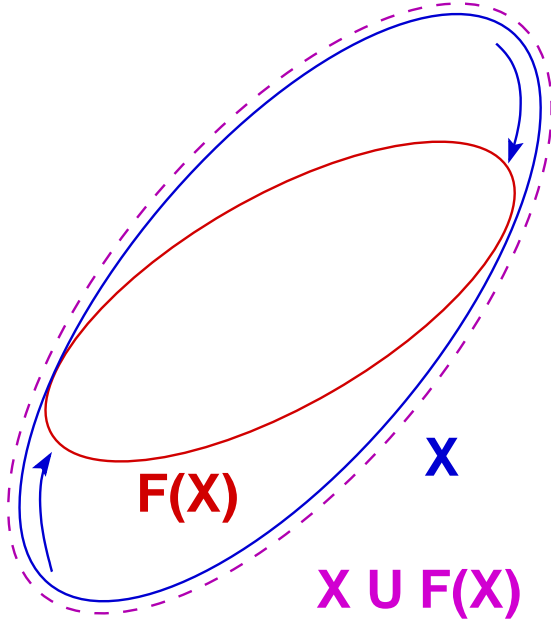
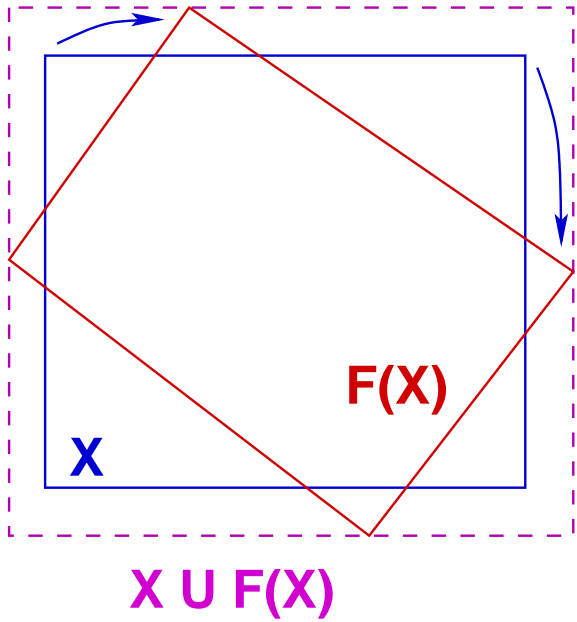
$$(aX + 2bY)^2 = -4b\left(\frac{a^2}{-4b}X^2 - aXY - bY^2\right)$$

$$(aX + 2bY)^2 \leq -4bQ(X, Y) \quad \text{since } a^2 + 4b < 0$$

$$|aX + 2bY| \leq 2\sqrt{-bQ(X, Y)}$$

$$Q(Z, X) \leq \left(\sqrt{-bQ(X, Y)} + m\right)^2$$

Linear versus quadratic invariants



Second order filter approximation

1. without relational domain,
we cannot limit $|S_2|$;
2. with quadratic constraints (history insensitive abstraction),
we can infer that $|S_2| < 22.111$;
3. by formally expanding the output as a sum of all previous inputs,
we can prove that $|S_2| < 1.41824$;

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Syntax

Let \mathcal{V} be a finite set of variables.

Let \mathcal{I} be the set of real intervals (including \mathbb{R}).

Expressions \mathcal{E} are affine forms of variables \mathcal{V} with real interval coefficients:

$$E ::= I + \sum_{j \in J} I_j \cdot V_j$$

Programs are given by the following grammar:

```
 $P ::=$  skip  
|  $P; P$   
|  $V := E$   
| if  $(V \geq 0)$   $\{P\}$  else  $\{P\}$   
| while  $(V \geq 0)$   $\{P\}$ 
```

Semantics

We define the semantics of a program P :

$$\llbracket P \rrbracket : (\mathcal{V} \rightarrow \mathbb{R}) \rightarrow \wp(\mathcal{V} \rightarrow \mathbb{R})$$

by induction over the syntax of P :

$$\llbracket \text{skip} \rrbracket(\rho) = \{\rho\},$$

$$\llbracket P_1; P_2 \rrbracket(\rho) = \{\rho'' \mid \exists \rho' \in \llbracket P_1 \rrbracket(\rho), \rho'' \in \llbracket P_2 \rrbracket(\rho')\},$$

$$\llbracket V := I + \sum_{j \in J} I_j \cdot V_j \rrbracket(\rho) = \left\{ \rho \left[V \mapsto i + \sum_{j \in J} i_j \cdot \rho(V_j) \right] \mid i \in I, \forall j \in J, i_j \in I_j \right\},$$

$$\llbracket \text{if } (V \geq 0) \{P_1\} \text{ else } \{P_2\} \rrbracket(\rho) = \begin{cases} \llbracket P_1 \rrbracket(\rho) & \text{if } \rho(V) \geq 0 \\ \llbracket P_2 \rrbracket(\rho) & \text{otherwise,} \end{cases}$$

$$\llbracket \text{while } (V \geq 0) \{P\} \rrbracket(\rho) = \{\rho' \in \text{Inv} \mid \rho'(V) < 0\}$$

where $\text{Inv} = \text{lfp } (X \mapsto \{\rho\} \cup \{\rho'' \mid \exists \rho' \in X, \rho'(V) \geq 0 \text{ and } \rho'' \in \llbracket P \rrbracket(\rho')\})$.

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Abstract domain

An abstract domain $\text{ENV}^\#$ is a set of environment properties.

A concretization map γ relates each property to the set of its solutions:

$$\gamma : \text{ENV}^\# \rightarrow \wp(\mathcal{V} \rightarrow \mathbb{R}).$$

Some primitives simulate concrete computation steps in the abstract:

- an abstract control path merge \sqcup ;
- an abstract guard GUARD and an abstract assignment ASSIGN ;
- an abstract least fixpoint $\text{lfp}^\#$ operator, which maps sound counterpart $f^\#$ to monotonic function f , to an abstraction of the least fixpoint of f .

$\text{lfp}^\#$ is defined using extrapolation operators (\perp, ∇, Δ) .

Soundness follows from the monotony of the concrete semantics.

Abstract semantics

$$\llbracket \text{skip} \rrbracket^\#(a) = a$$

$$\llbracket P_1; P_2 \rrbracket^\#(\rho^\#) = \llbracket P_2 \rrbracket^\#(\llbracket P_1 \rrbracket^\#(\rho^\#))$$

$$\llbracket V := E \rrbracket^\#(a) = \text{ASSIGN}(V, E, a)$$

$$\llbracket \text{if } (V \geq 0) \{ P_1 \} \text{ else } \{ P_2 \} \rrbracket^\#(a) = a_1 \sqcup a_2,$$

with $\begin{cases} a_1 = \llbracket P_1 \rrbracket^\#(\text{GUARD}(V, [0; +\infty[, a)) \\ a_2 = \llbracket P_2 \rrbracket^\#(\text{GUARD}(V,]-\infty; 0[, a)) \end{cases}$

$$\llbracket \text{while } (V \geq 0) \{ P \} \rrbracket^\#(a) = \text{GUARD}(V,]-\infty; 0[, \text{Inv}^\#)$$

where $\text{Inv}^\# = \text{lfp}^\# \left(X \mapsto a \sqcup \llbracket P \rrbracket^\#(\text{GUARD}(V, [0; +\infty[, X)) \right)$

Soundness

We prove by induction over the syntax:

Theorem 4 (Soundness) *For any program P , environment ρ , abstract element a , we have:*

$$\rho \in \gamma(a) \implies \llbracket P \rrbracket(\rho) \subseteq \gamma(\llbracket P \rrbracket^\#(a)).$$

□

Extrapolation operators

- iteration basis: $\perp \in \text{ENV}^\#$
- a widening operator ∇ such that:
 1. $\nabla \in (\text{ENV}^\# \times \text{ENV}^\#) \rightarrow \text{ENV}^\#$,
 2. $\forall a, b \in \text{ENV}^\#, \gamma(a) \cup \gamma(b) \subseteq \gamma(a \nabla b)$,
 3. $\forall (a_i) \in (\text{ENV}^\#)^\mathbb{N}$, the sequence (a_i^∇) defined by:
$$a_0^\nabla = a_0 \text{ and } a_{n+1}^\nabla = a_n^\nabla \nabla a_{n+1}$$
is eventually stationary;
- a narrowing operator Δ such that:
 1. $\Delta \in (\text{ENV}^\# \times \text{ENV}^\#) \rightarrow \text{ENV}^\#$,
 2. $\forall a, b \in \text{ENV}^\#, \gamma(a) \cap \gamma(b) \subseteq \gamma(a \Delta b)$,
 3. $(a_i) \in (\text{ENV}^\#)^\mathbb{N}$, the sequence (a_i^Δ) defined by:
$$a_0^\Delta = a_0 \text{ and } a_{n+1}^\Delta = a_n^\Delta \Delta a_{n+1}$$
is eventually stationary;

Abstract iterations

Let f^\sharp be a map in $\text{ENV}^\sharp \rightarrow \text{ENV}^\sharp$.

Abstract upward-iterates:

$$\begin{cases} C_0^\nabla = \perp, \\ C_{n+1}^\nabla = C_n^\nabla \nabla f^\sharp(C_n^\nabla), \end{cases}$$

is eventually stationary: We denote by C_ω^∇ its limit.

Abstract downward-iterates:

$$\begin{cases} D_0^\Delta = C_\omega^\nabla, \\ D_{n+1}^\Delta = D_n^\Delta \Delta f^\sharp(D_n^\Delta), \end{cases}$$

is eventually stationary: We define $\text{lfp}^\sharp(f^\sharp)$ as this limit.

Soundness

Let f be a \cup -complete morphism such that:

$$\forall a \in \mathbf{ENV}^\#, f(\gamma(a)) \subseteq \gamma(f^\#(a)).$$

We want to prove that $\mathbf{lfp}(f) \subseteq \gamma(\mathbf{lfp}^\#(f^\#))$.

We know that (kleenean iteration):

$$\forall a \in \wp(\mathcal{V} \rightarrow \mathbb{R}), f(a) \subseteq a \implies \mathbf{lfp}(f) \subseteq a.$$

So, we only have to prove that:

$$\exists b \in \wp(\mathcal{V} \rightarrow \mathbb{R}), f(b) \subseteq b \text{ and } b \subseteq \gamma(\mathbf{lfp}^\#(f^\#)).$$

Soundness proof (continued)

1. $f(\gamma(C_\omega^\nabla)) \subseteq \gamma(C_\omega^\nabla)$ since:

$$f(\gamma(C_\omega^\nabla)) \subseteq \gamma(f^\#(C_\omega^\nabla)),$$

(soundness of $f^\#$)

$$\gamma(f^\#(C_\omega^\nabla)) \subseteq \gamma(C_\omega^\nabla \nabla f^\#(C_\omega^\nabla)),$$

(soundness of ∇)

$$C_\omega^\nabla \nabla f^\#(C_\omega^\nabla) = C_\omega^\nabla,$$

(C_ω^∇ is a limit)

2. $\forall n \in \mathbb{N}, \exists a \in \wp(\mathcal{V} \rightarrow \mathbb{R})$ such that $f(a) \subseteq a$ and $a \subseteq \gamma(D_n^\Delta)$:

(a) $\gamma(D_0^\Delta) = \gamma(C_\omega^\nabla)$ and $f(\gamma(C_\omega^\nabla)) \subseteq \gamma(C_\omega^\nabla)$;

(b) let $a \in \wp(\mathcal{V} \rightarrow \mathbb{R})$ such that $f(a) \subseteq a$ and $a \subseteq \gamma(D_n^\Delta)$,

then

- $f(f(a)) \subseteq f(a)$ (f is monotonic),

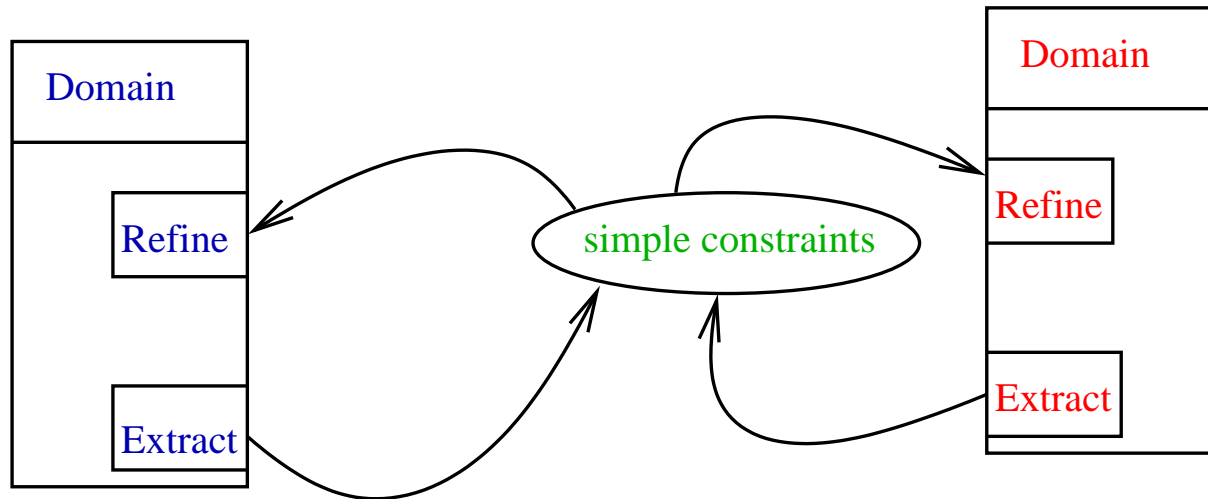
$$f(a) \subseteq f(\gamma(D_n^\Delta)) \subseteq \gamma(f^\#(D_n^\Delta)),$$

- $f(a \cap f(a)) \subseteq f(a) \cap f(f(a)) \subseteq a \cap f(a)$,

$$a \cap f(a) \subseteq \gamma(D_n^\Delta) \cap \gamma(f^\#(D_n^\Delta)) \subseteq \gamma(D_n^\Delta \Delta f^\#(D_n^\Delta)) \subseteq \gamma(D_{n+1}^\Delta)$$

Approximated reduced product

Domains are refined by simple constraints computed in other domains:



Interface with other domains

We only use two kinds of simple constraints:

- $\gamma_{=} : \begin{cases} \wp(\mathcal{V}^2) & \rightarrow \wp(\mathcal{V} \rightarrow \mathbb{R}) \\ \mathcal{R} & \mapsto \{\rho \mid (X, Y) \in \mathcal{R} \implies \rho(X) = \rho(Y)\}; \end{cases}$
- $\gamma_{\mathcal{I}} : \begin{cases} (\mathcal{V} \rightarrow \mathcal{I}) & \rightarrow \wp(\mathcal{V} \rightarrow \mathbb{R}) \\ \rho^{\#} & \mapsto \{\rho \mid \forall X \in \mathcal{V}, \rho(X) \in \rho^{\#}(X)\}. \end{cases}$

We can get such constraints by weakening of abstract properties:

1. $\text{EQU} : \text{ENV}^{\#} \rightarrow \wp(\mathcal{V}^2)$
 $\forall a \in \text{ENV}^{\#}, \gamma(a) \subseteq \gamma_{=}(\text{EQU}(a));$
2. $\text{RANGE} : \text{ENV}^{\#} \rightarrow (\mathcal{V} \rightarrow \mathcal{I})$:
 $\forall a \in \text{ENV}^{\#}, \gamma(a) \subseteq \gamma_{\mathcal{I}}(\text{RANGE}(a)).$

We refine abstract properties by weakening range constraints:

- $\text{REDUCE} : ((\mathcal{V} \rightarrow \mathcal{I}) \times \text{ENV}^{\#}) \rightarrow \text{ENV}^{\#}$
 $\forall a \in \text{ENV}^{\#}, \rho^{\#} \in (\mathcal{V} \rightarrow \mathcal{I}), \gamma(a) \cap \gamma_{\mathcal{I}}(\rho^{\#}) \subseteq \gamma(\text{REDUCE}(\rho^{\#}, a)),$

Reduction policy

We will refine abstract properties when it is necessary:

- after assignments
- after guards
- after extrapolation steps

To ensure termination, we **forbid cyclic reductions** after extrapolation steps:
domains are **ordered** by the relation “**is used to refine**”.

Extrapolation (revisited)

We also require that:

- $\forall k \in \mathbb{N}, \rho_1, \dots, \rho_k \in (\mathcal{V} \rightarrow \mathcal{I}), (a_i) \in (\mathbf{ENV}^\#)^\mathbb{N}$,
the sequence (a_i^∇) defined by:

$$a_0^\nabla = \rho(a_0) \text{ and } a_{n+1}^\nabla = \rho(a_n^\nabla \nabla a_{n+1})$$

$$\text{with } \rho = [X \mapsto \mathbf{REDUCE}(\rho_k, X)] \circ \dots \circ [X \mapsto \mathbf{REDUCE}(\rho_1, X)],$$

is eventually stationary;

- $\forall k \in \mathbb{N}, \rho_1, \dots, \rho_k \in (\mathcal{V} \rightarrow \mathcal{I}), (a_i) \in (\mathbf{ENV}^\#)^\mathbb{N}$,
the sequence (a_i^Δ) defined by:

$$a_0^\Delta = \rho(a_0) \text{ and } a_{n+1}^\Delta = \rho(a_n^\Delta \Delta a_{n+1}),$$

$$\text{with } \rho = [X \mapsto \mathbf{REDUCE}(\rho_k, X)] \circ \dots \circ [X \mapsto \mathbf{REDUCE}(\rho_1, X)],$$

is eventually stationary.

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Filter family

A filter class is given by:

- the number p of outputs and the number q of inputs involved in the computation of the next output;
- a (generic/symbolic) description of F with parameters;
- some conditions over these parameters

In the case of the second order filter:

- $p = 2, q = 3$;
- $F(V, W, X, Y, Z) = a \times V + b \times W + c \times X + d \times Y + e \times Z$;
- $a^2 + 4b < 0$.

Filter domain

A filter constraint is a couple in $\mathcal{T} \times \mathcal{B}$ where:

- $\mathcal{T} \in \wp_{\text{finite}}(\mathcal{V}^m \times \mathbb{R}^n)$ with:
 - m , the **number of variables** that are involved in the computation of the next output. m depends on the abstraction;
 - n , the **number of filter parameters**;
- \mathcal{B} is an **abstract domain** encoding some “ranges”.

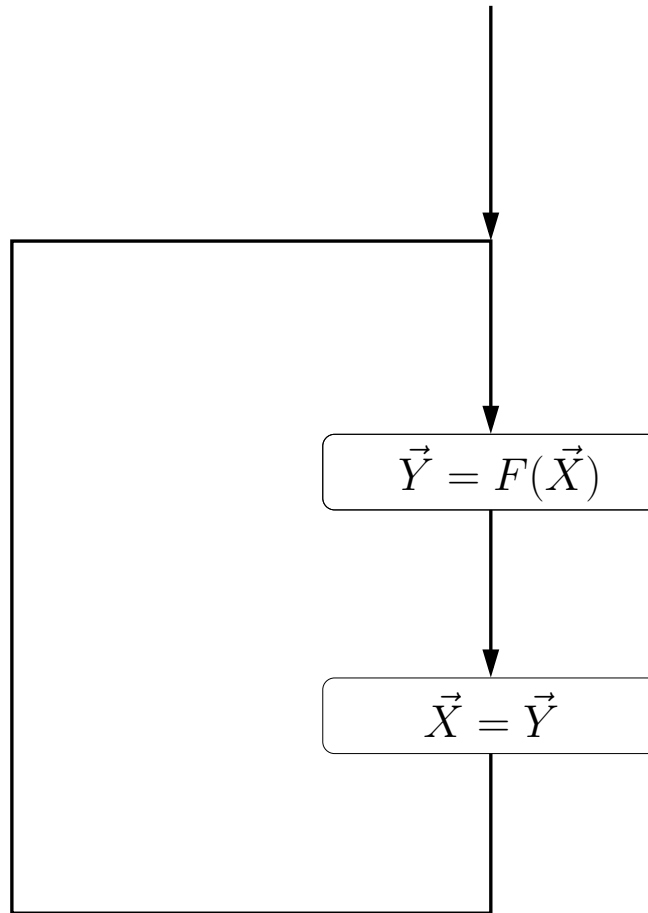
A constraint (t, d) is related to a set of environments:

$$\gamma_{\mathcal{B}} : \mathcal{T} \times \mathcal{B} \rightarrow \wp(\mathcal{V} \rightarrow \mathbb{R}).$$

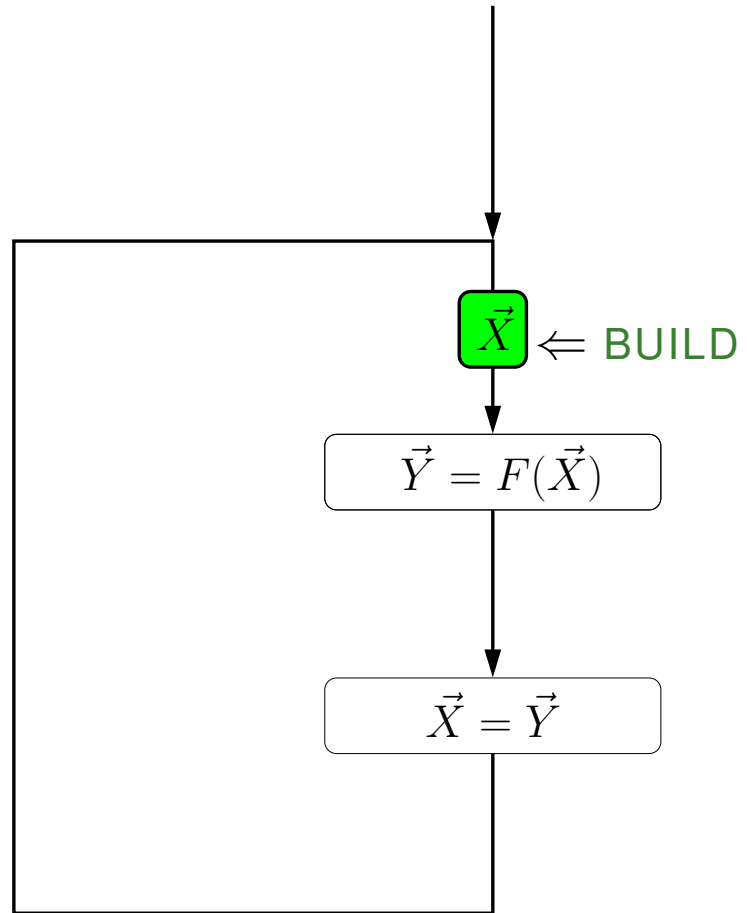
An approximation of second order filter may consist in relating:

- the last two outputs and the first two coefficients of the filter (a and b)
- to the ‘radius’ of an ellipsis.

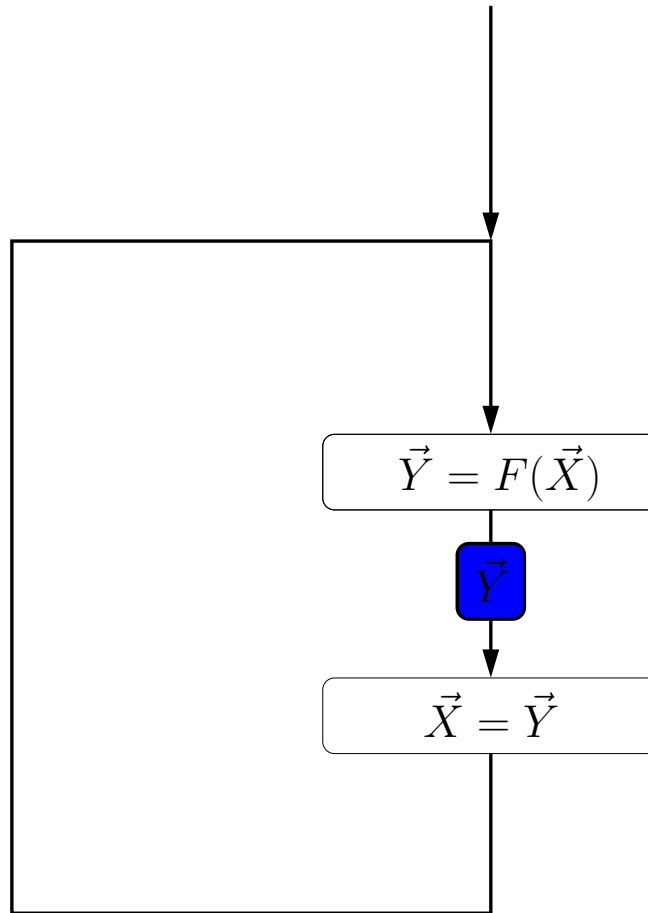
Iterations



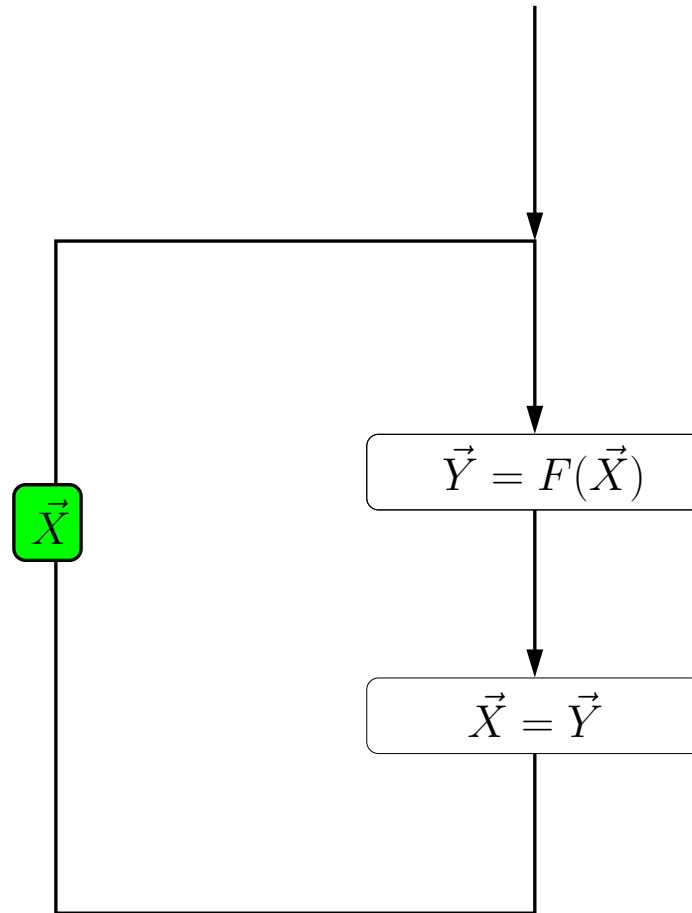
Iterations



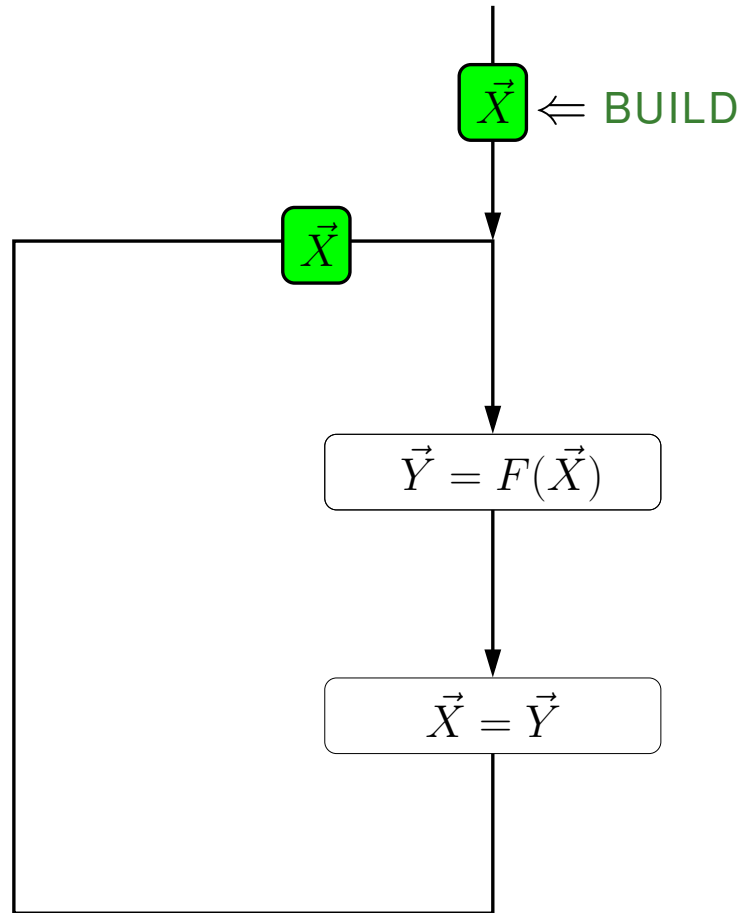
Iterations



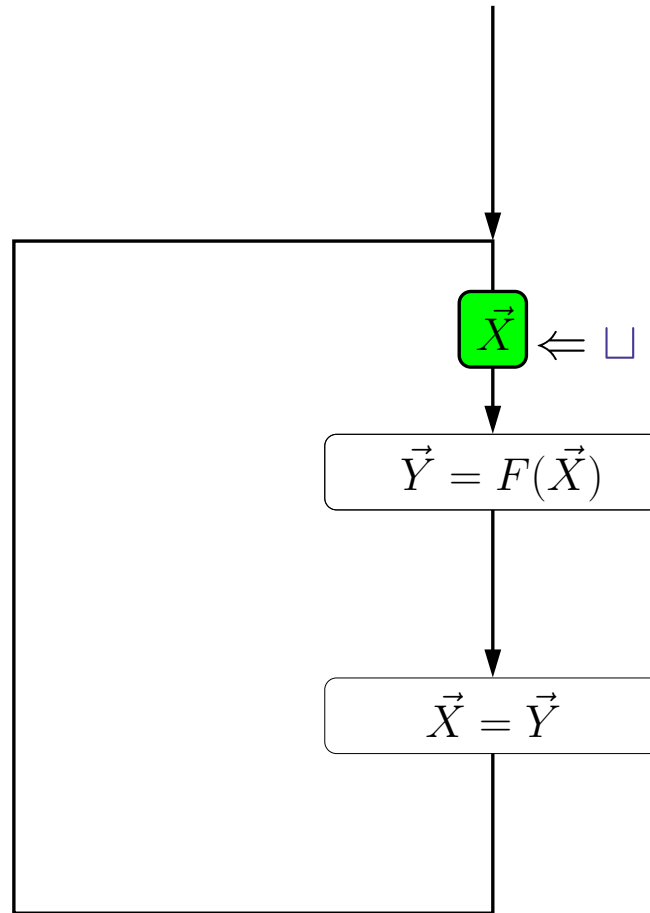
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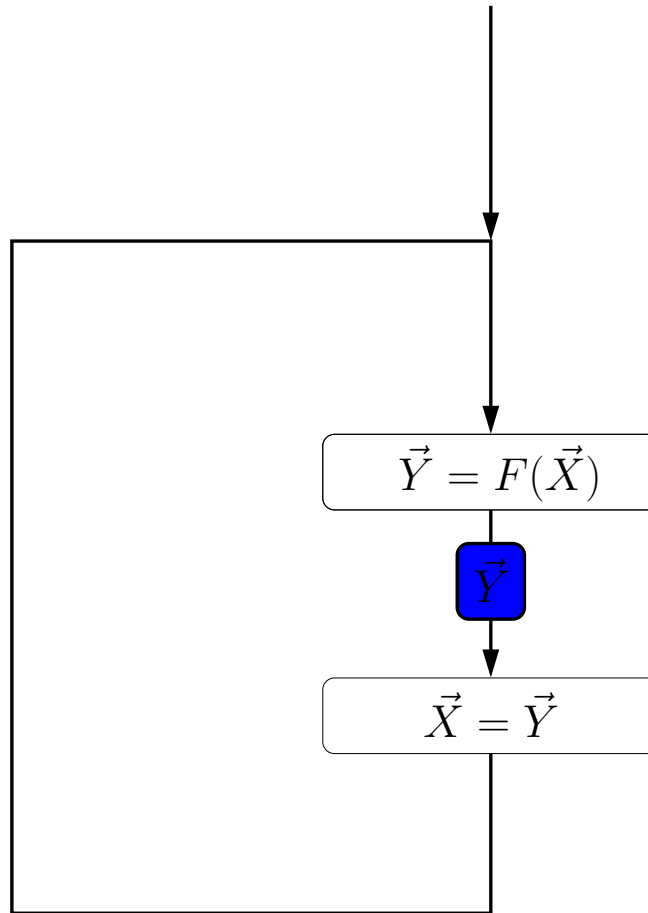
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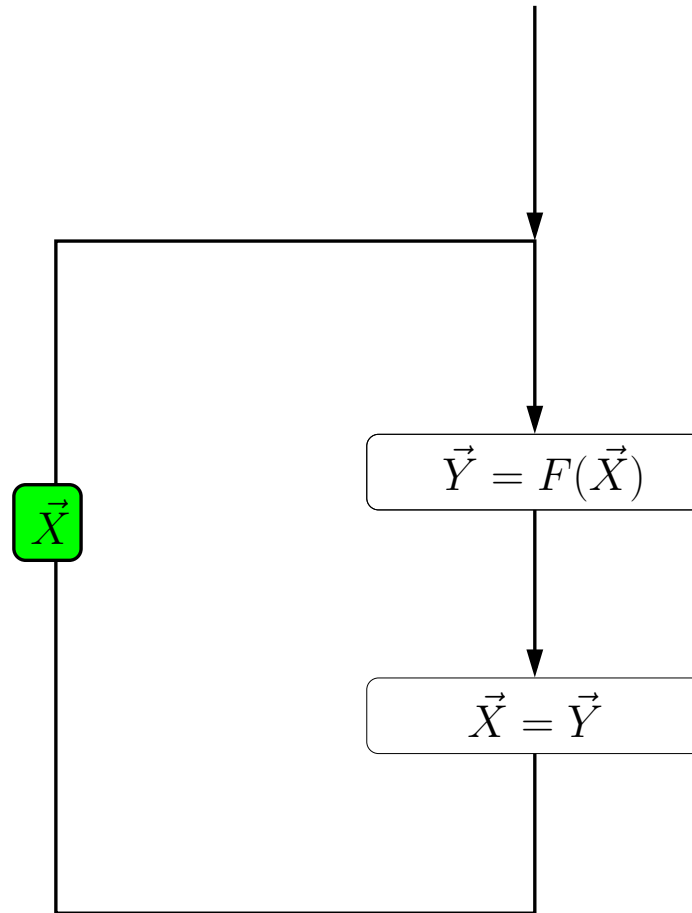
Iterations



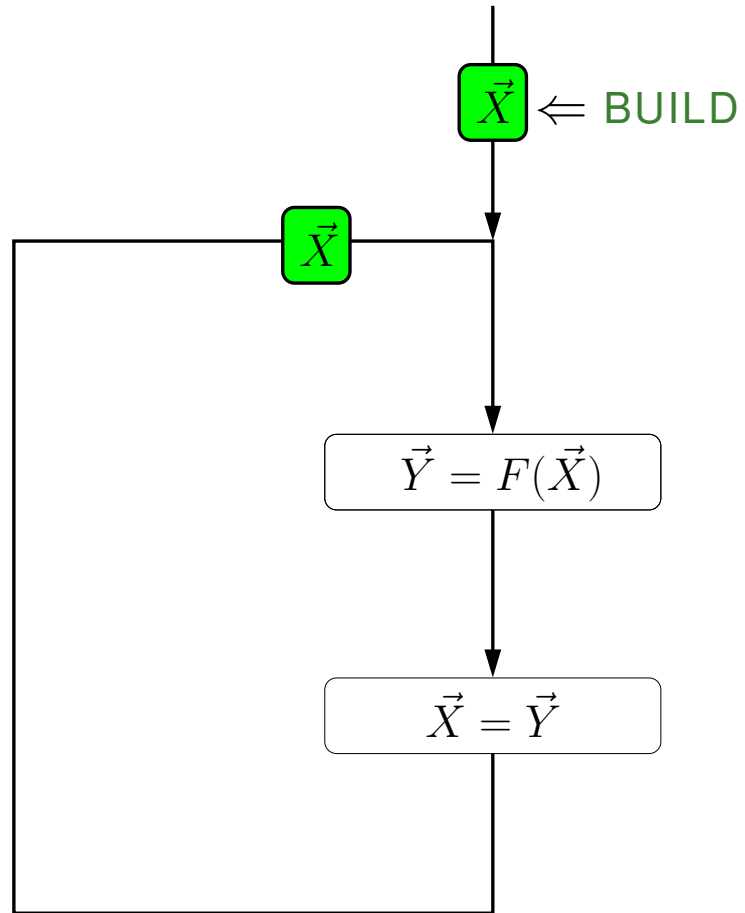
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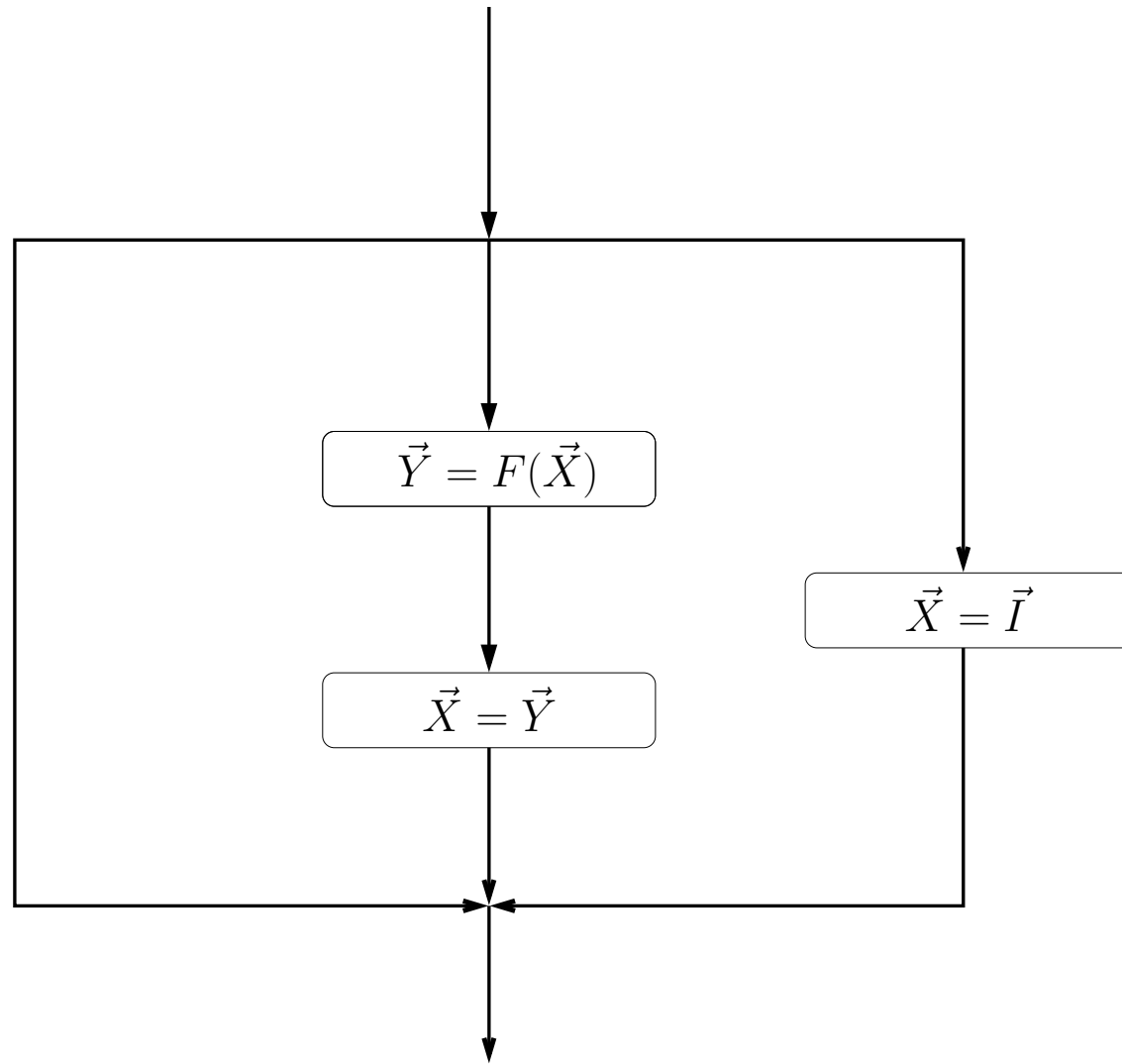
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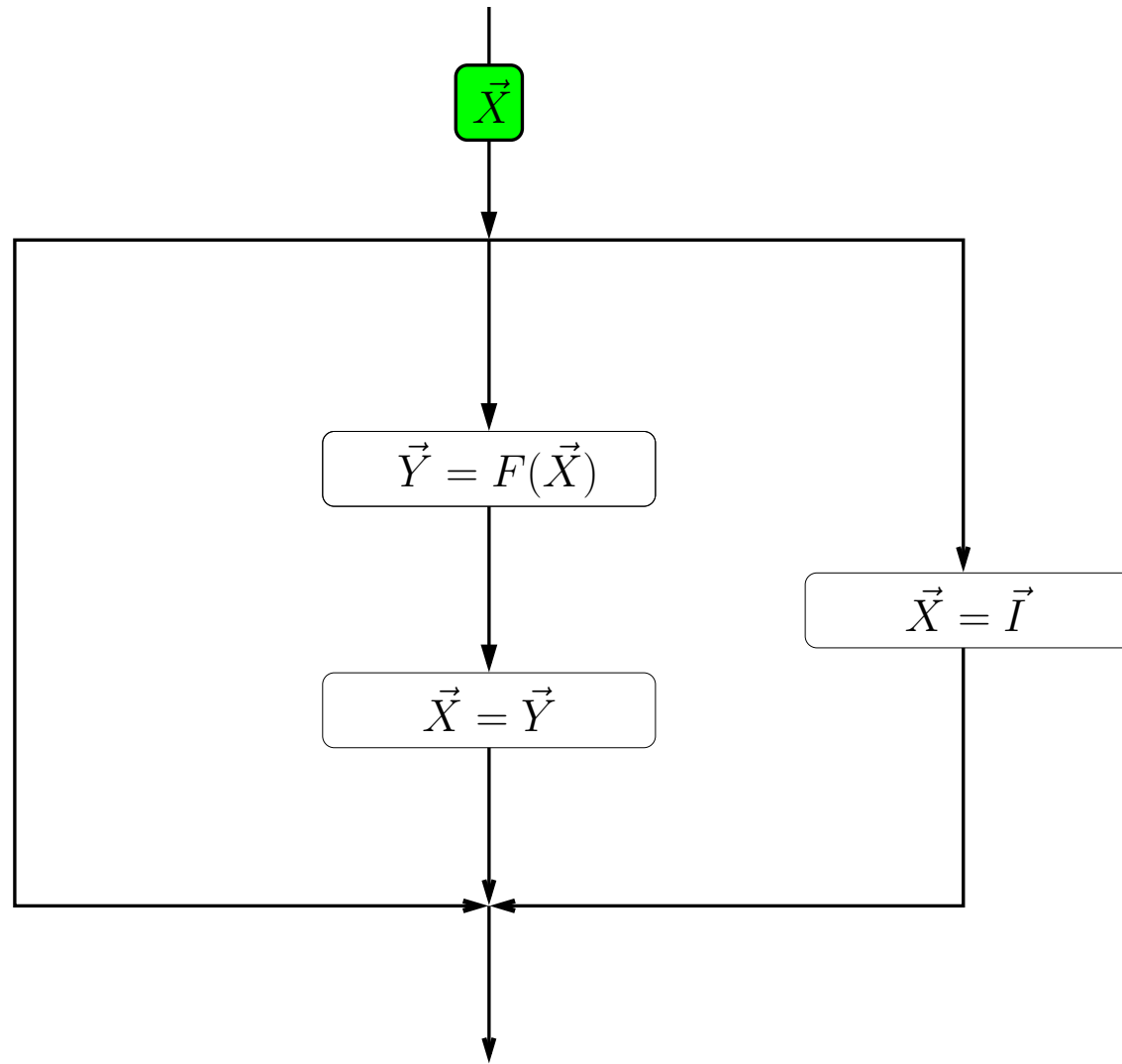
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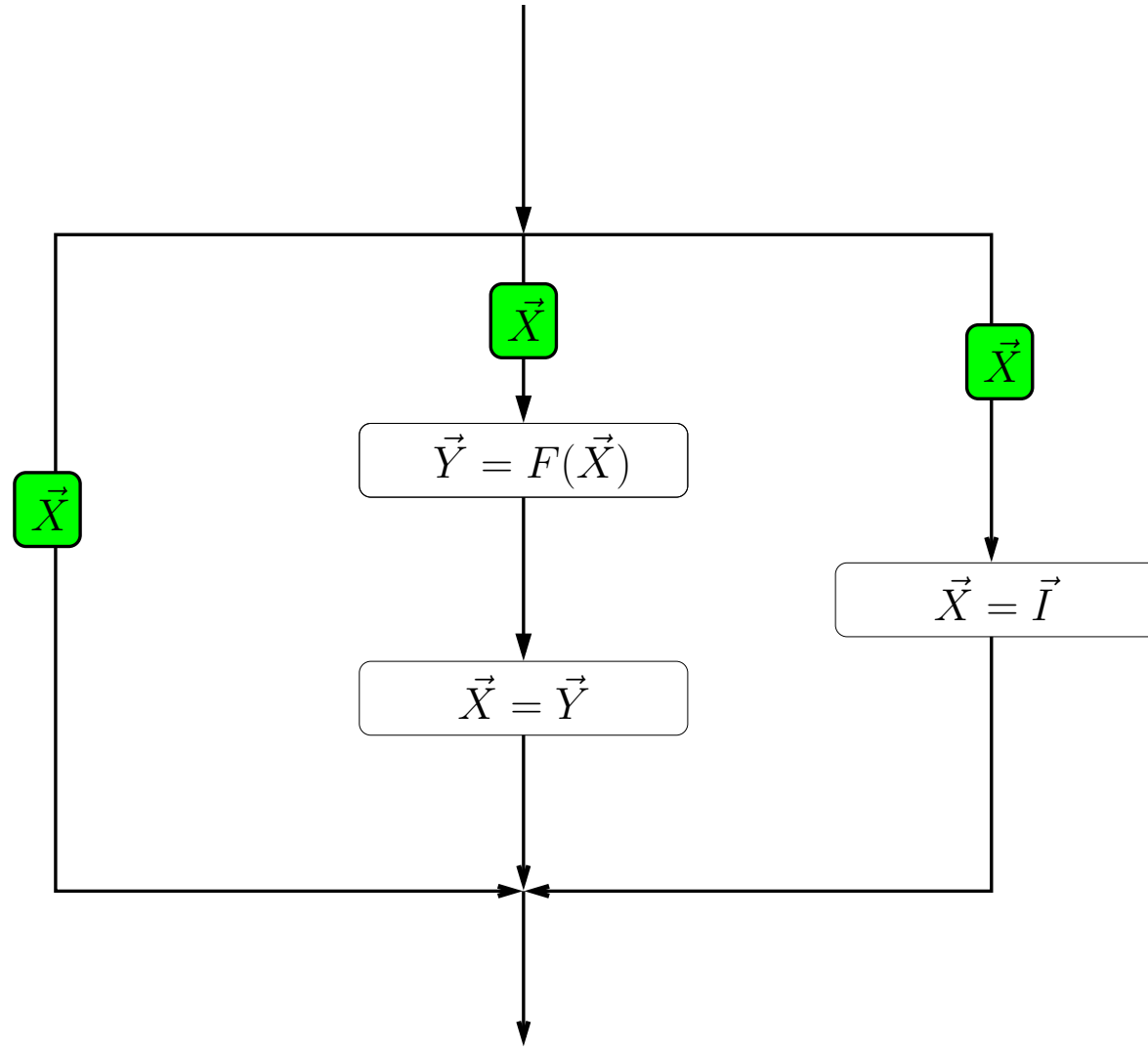
Merging computation paths



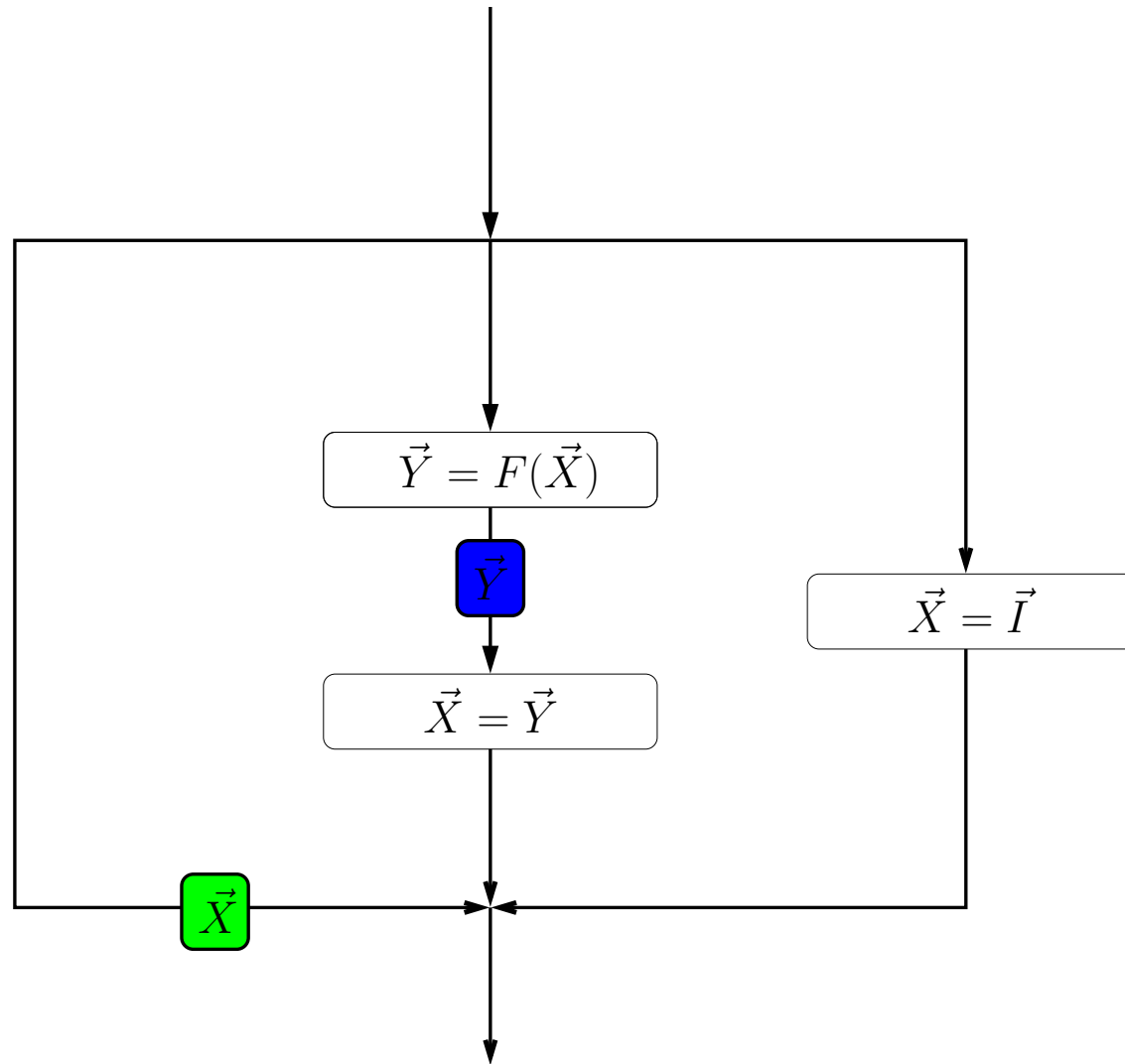
Merging computation paths



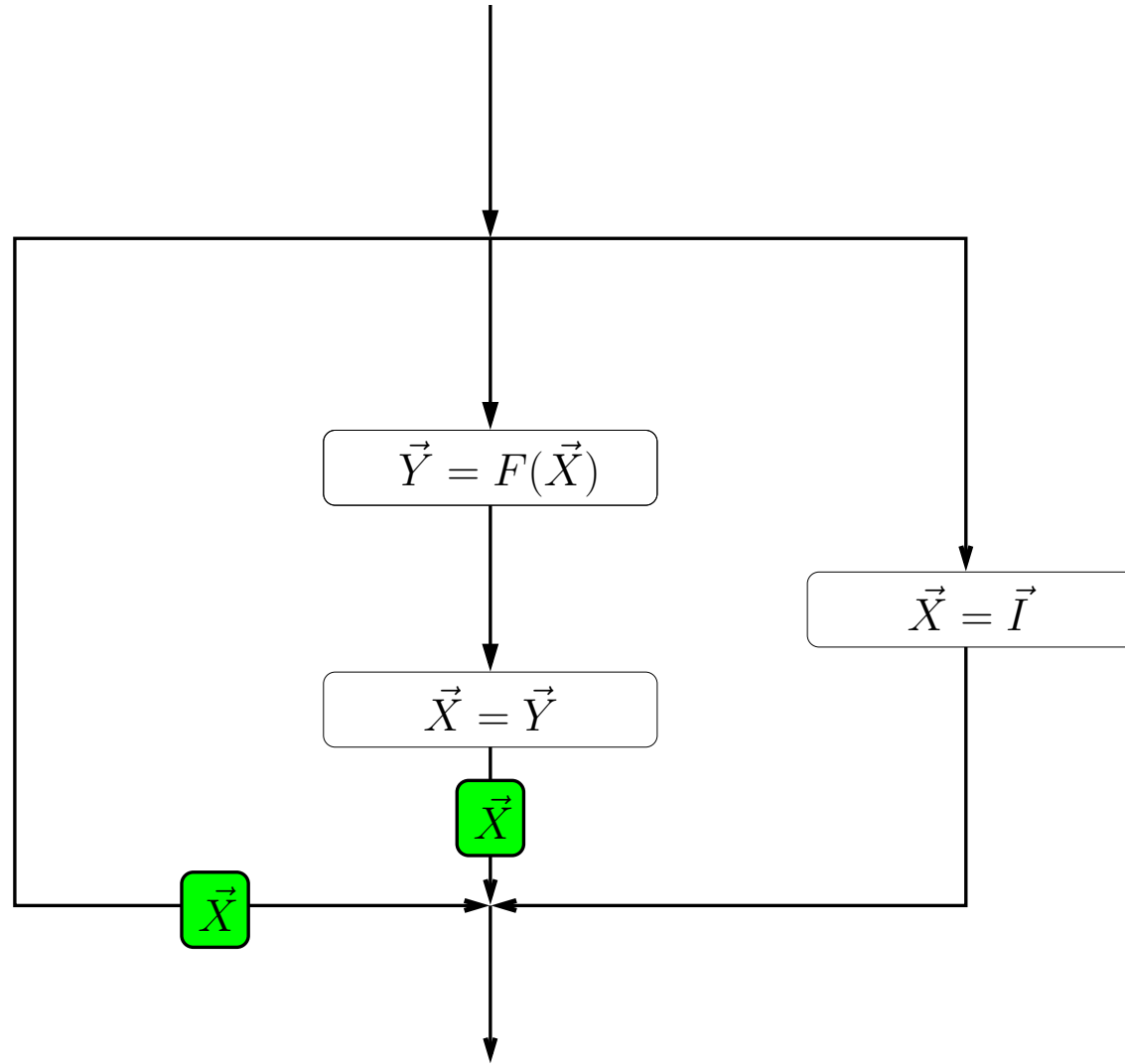
Merging computation paths



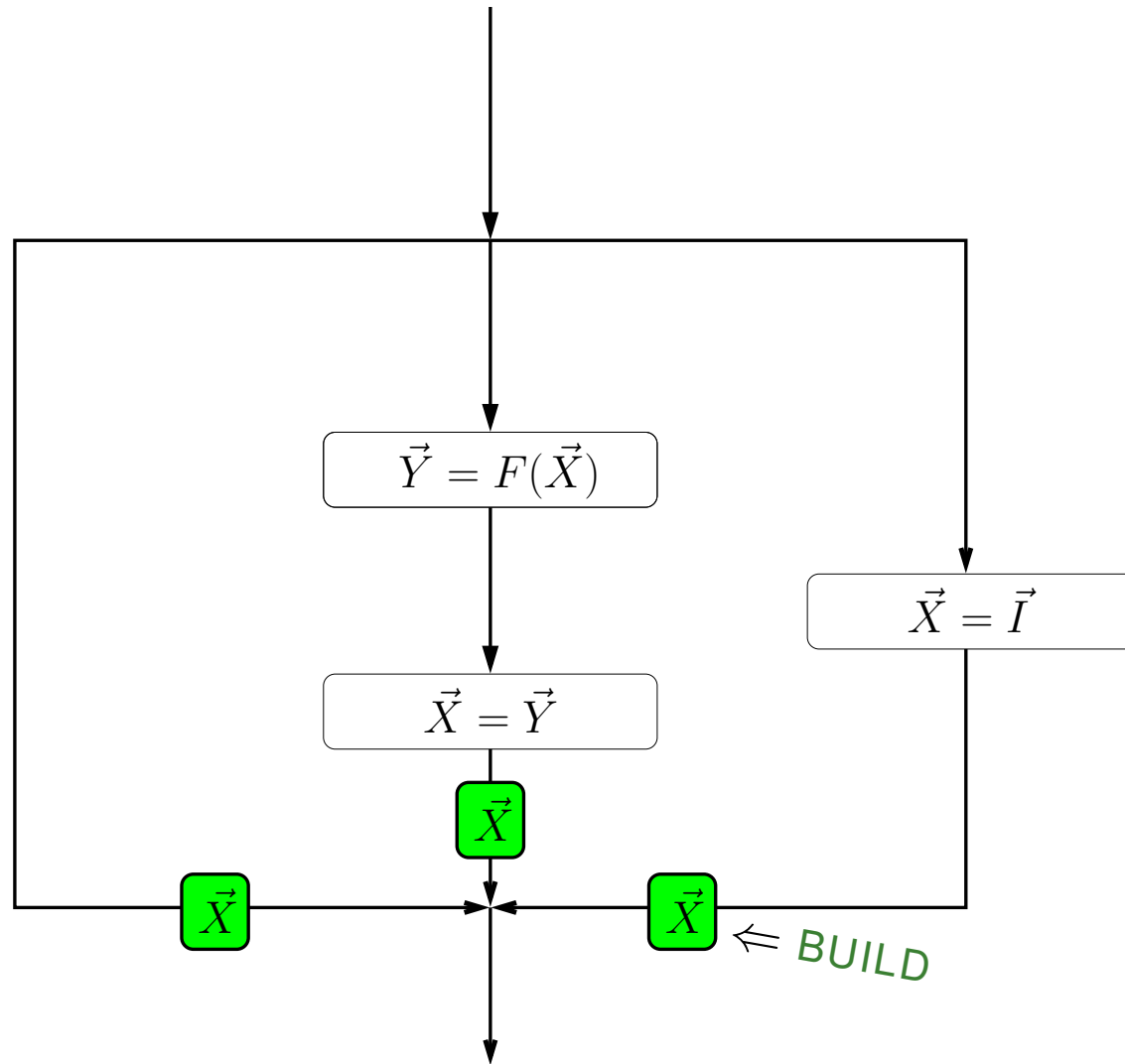
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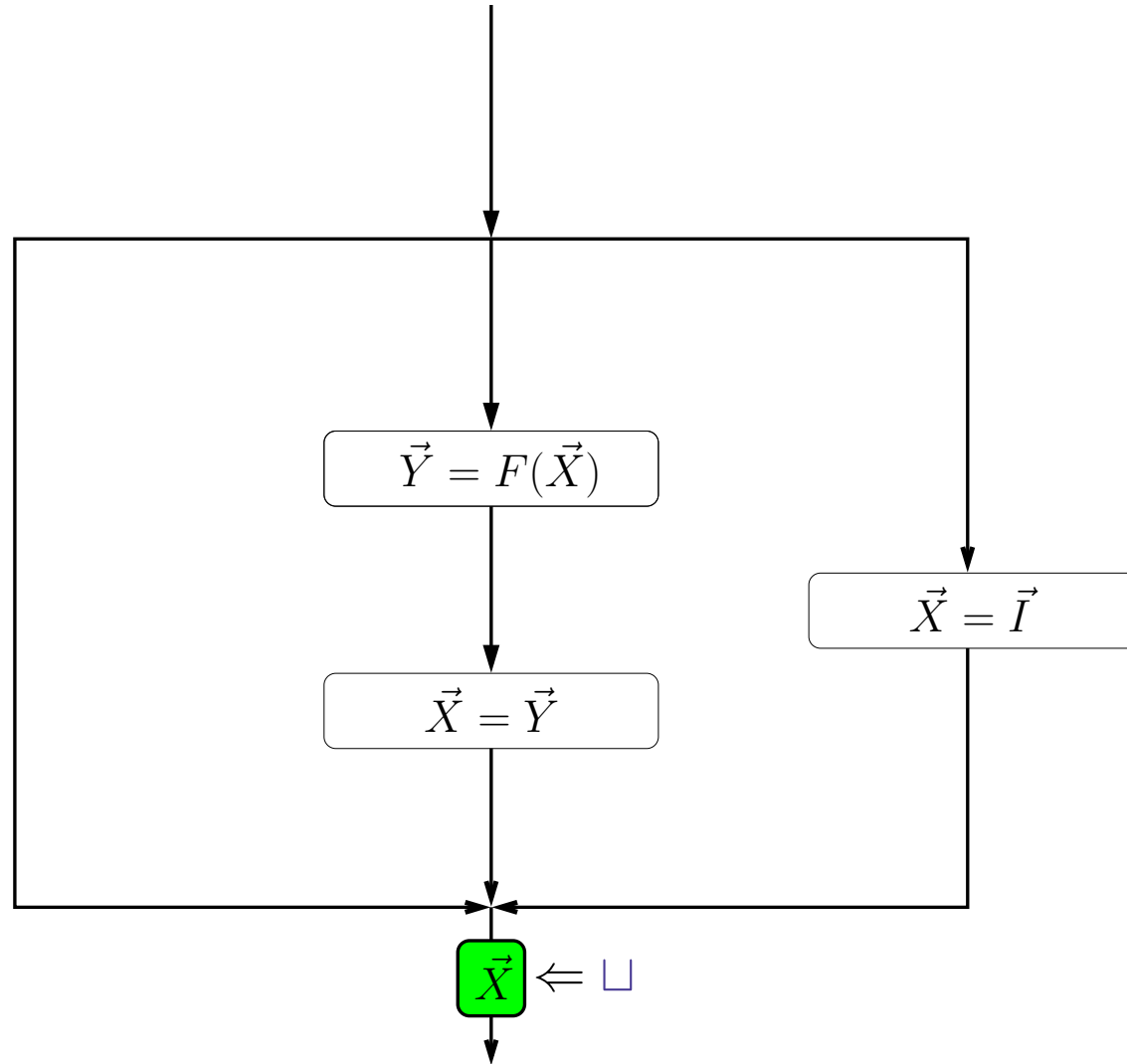
Merging computation paths



Merging computation paths



Merging computation paths



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Floating point domain

Let:

- \mathbb{F} be a finite subset of \mathbb{R} closed upon opposite,
- L is a finite subset of \mathbb{F} ;
- q, r two natural parameters for setting extrapolation strategy.

We define $\mathcal{F}_{q,r}$ as follows:

- $\mathcal{F}_{q,r} = \overline{\mathbb{F}} = \mathbb{F} \cup \{-\infty; +\infty\}$;
- $\gamma_{\overline{\mathbb{F}}} : \begin{cases} \overline{\mathbb{F}} & \mapsto \wp(\mathbb{R}) \\ a & \rightarrow \begin{cases} [-a; a] & \text{if } a \in \mathbb{F} \\ \mathbb{R} & \text{otherwise;} \end{cases} \end{cases}$
- $[_] : \begin{cases} \mathbb{R} \rightarrow \overline{\mathbb{F}} \\ r \rightarrow \min(\{f \in \overline{\mathbb{F}} \mid f \geq r\}); \end{cases}$
- $a \nabla_{\overline{\mathbb{F}}} b = \min(\{l \in L \cup \{a; +\infty\} \mid l \geq b\})$.

Extrapolation strategy

- Delayed widening:

$$(a_1, k_1) \nabla_{\mathcal{F}_{q,r}} (a_2, k_2) = \begin{cases} (a_1, k_1) & \text{if } a_1 \geq a_2 \\ (a_2, k_1 + 1) & \text{if } a_2 > a_1 \text{ and } k_1 < q \\ (a_1 \nabla_{\mathbb{F}} a_2, 0) & \text{otherwise;} \end{cases}$$

Constraints are only widened when they have been unstable (not necessarily successively) q times, since their last widening.

- Bounded narrowing:

$$(a_1, k_1) \Delta_{\mathcal{F}_{q,r}} (a_2, k_2) = \begin{cases} (a_1, k_1) & \text{if } a_1 \leq a_2 \text{ or } k_1 \leq (-r) \\ (a_2, \min(k_1, 0) - 1) & \text{if } a_2 < a_1 \text{ and } k_1 > (-r); \end{cases}$$

Constraints are only narrowed r times.

Approximating contracting functions

When analyzing filter, we iterate functions f such that:

- $f : I \times \mathbb{F} \rightarrow \mathbb{F}$
- $\forall i \in I$, the map $[x \rightarrow f(i, x)]$ is **contracting**;
- we can compute $f_l : I \rightarrow \mathbb{F}$ such that $\forall i \in I, f(i, f_l(i)) \leq f_l(i)$;

where I is a set of inputs.

Since $[x \rightarrow f(i, x)]$ is contracting, we have:

- $\forall i \in I, \forall x \geq f_l(i), f(i, x) \leq x$.

Our goal

We want to find a **iterating strategy** which ensures:

- **soundness** (even if f_l is unsound)
- **accuracy** (if f_l is sound):
 - do not jump directly at the limit f_l : (to analyze **not iterated filter, loop unrolling...**)
 - do **not jump higher than the limit** when the input is constant;
 - do **not jump higher than the limit** in most cases.
- **termination** (even if the input depend on the output).

Reduced product

We use an approximation of the **reduced product** of two domains:

Let q, r be two natural parameters.

1. the first domain iterates f in $\mathcal{F}_{0,r}$
 \implies widened at each step;
2. the second domain iterates $[(i, x) \rightarrow \max(f(i, x), f_l(i))]$ in $\mathcal{F}_{q,0}$
 \implies soundness does not depend on f_l
 \implies not widened at each step to wait until input are stables.

We use **the reduction**:

$$\rho : \begin{cases} \mathcal{F}_{0,r} \times \mathcal{F}_{q,0} & \mapsto \mathcal{F}_{0,r} \times \mathcal{F}_{q,0} \\ (x_0, m_0), (x_1, m_1) & \rightarrow (\min(x_0, x_1), m_0), (x_1, m_1) \end{cases}$$

after each computation step.

\implies The second domain is used to reduce the first one, when it is not accurate.

Unstable filters

In case the iterated function is not contracting, **filters** are very likely to **diverge**.
In case of linear filters, the **iterated function is linear**.
We may use the **arithmetic-geometric progression domain** [VMCAI'2005].
We require an external clock to relate the divergence to the value of the clock.

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Simplified second order filter

Theorem 5 (Including rounding errors)

Let $a, b, \varepsilon_a \geq 0, \varepsilon_b \geq 0, K \geq 0, m \geq 0, X, Y, Z$ be real numbers, such that:

1. $a^2 + 4b < 0,$
2. $X^2 - aXY - bY^2 \leq K,$
3. $aX + bY - (m + \varepsilon_a|X| + \varepsilon_b|Y|) \leq Z \leq aX + bY + (m + \varepsilon_a|X| + \varepsilon_b|Y|).$

We have

$$1. Z^2 - aZX - bX^2 \leq \left((\sqrt{-b} + \delta)\sqrt{K} + m \right)^2;$$

$$2. \begin{cases} \sqrt{-b} + \delta < 1 \\ K \geq \left(\frac{m}{1 - \sqrt{-b} - \delta} \right)^2 \end{cases} \implies Z^2 - aZX - bX^2 \leq K,$$

$$\text{where } \delta = 2 \frac{\varepsilon_b + \varepsilon_a \sqrt{-b}}{\sqrt{-(a^2 + 4b)}}.$$



Domain

- The domain relates the variables describing the **last two outputs** and the four **filter parameters** to the square root of **the ellipsis 'radius'**:

$\gamma_{\mathcal{B}_1}((X, Y, a, \varepsilon_a, b, \varepsilon_b), k)$ is given by the set of environments ρ that satisfy:

$$(\rho(X))^2 - a\rho(X)\rho(Y) - b(\rho(Y))^2 \leq k^2;$$

- in order to interpret assignment $Z = E$ under range constraints ρ^\sharp , we test whether E matches:

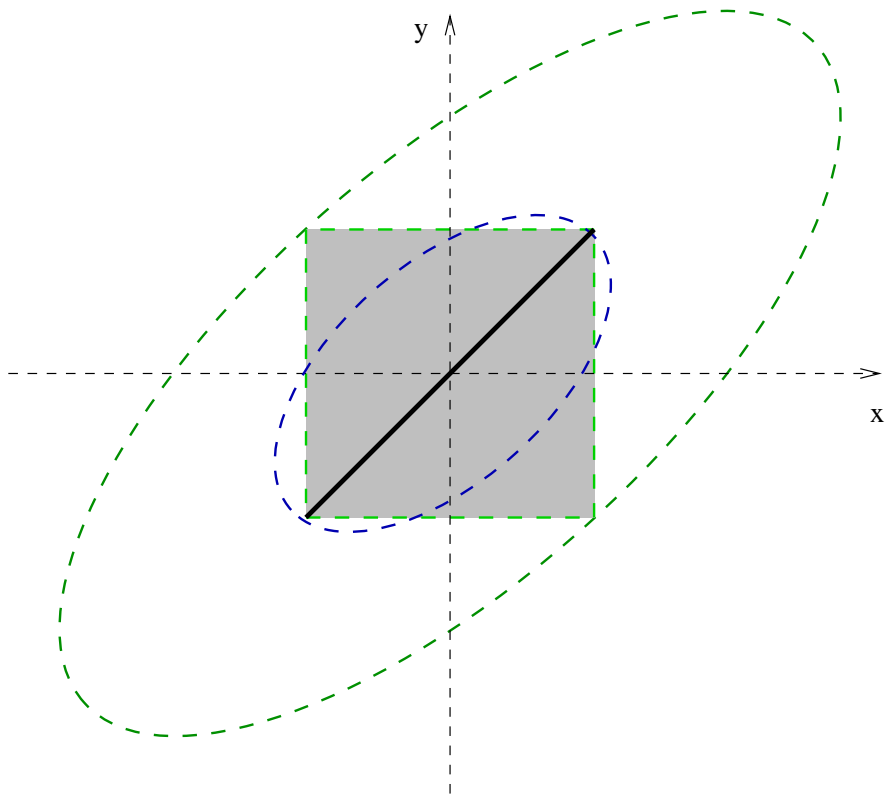
$$[a - \varepsilon_a; a + \varepsilon_a] \times X + [b - \varepsilon_b; b + \varepsilon_b] \times Y + E'$$

with $a^2 + 4b < 0$,

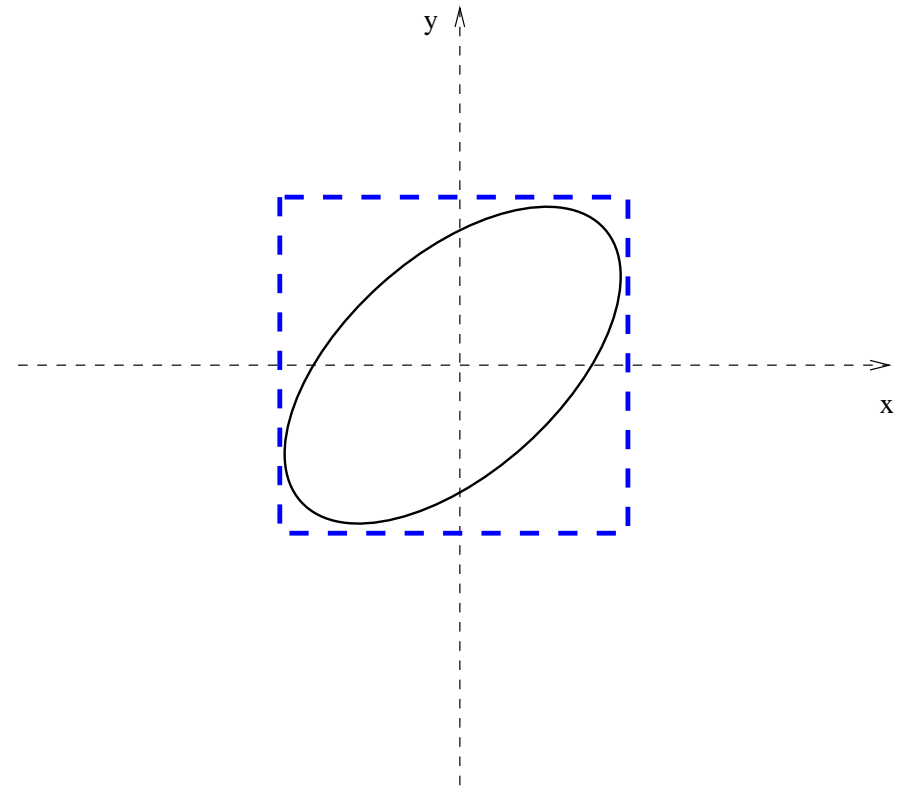
and capture:

- filter parameters: $(a, \varepsilon_a, b, \varepsilon_b)$;
- variables tied before (X, Y) and after the iteration (Z, X) ,
- an approximation of the current input: $\text{EVAL}^\sharp(E', \rho^\sharp)$.

Approximated reduced product



Initial conditions



Output refinement

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Higher order simplified filters

A simplified filter of class (k, l) is defined as a sequence:

$$S_{n+p} = a_1 S_n + \dots + a_p S_{n+p-1} + E_{n+p},$$

where the polynomial $P = X^p - a_p X^{p-1} - \dots - a_1 X^0$ has no multiple roots (in \mathbb{C}) and can be factored into the product of k second order irreducible polynomials $X^2 - \alpha_i X - \beta_i$ and l first order polynomials $X - \delta_j$.

Then, there exists sequences $(x_n^i)_{n \in \mathbb{N}}$ and $(y_n^j)_{n \in \mathbb{N}}$ such that:

$$\begin{cases} S_n = \left(\sum_{i=1}^k x_n^i \right) + \left(\sum_{j=1}^l y_n^j \right) \\ x_{n+2}^i = \alpha_i x_{n+1}^i + \beta_i x_n^i + F^i(E_{n+2}, E_{n+1}) \\ y_{n+1}^j = \delta_j y_n^j + G^j(E_{n+1}). \end{cases}$$

The initial outputs (x_0^i, x_1^i, y_0^j) and filter inputs F^i, G^j are given by solving symbolic linear systems, they only depend on the roots of P .

Higher order simplified filters

Whenever we meet an assignment $V_{n+p} = E_{n+p} + \sum_{k=1}^p I_k \times V_{n+k-1}$,

1. we consider the characteristic polynomial $P = X^p - \sum_{k=1}^p I_k \cdot X^{p-k}$,
2. we take a polynomial Q of the form $\prod_{i=1}^k (X^2 - A_i X - B_i) \prod_{j=1}^l (X - D_j)$ with $2k + l = p + 1$.
3. we expand Q into $X^p - \sum_{k=1}^p J_k \cdot X^{p-k}$.
4. we bound the expression $|\sum_{k=1}^p (I_k - J_k) \times V_{n+k-1}| \leq \mathbf{err}(V_n, \dots, V_{n+p-1})$;
5. we take the following assignment:

$$V_{n+p} = E_{n+p} + [-\mathbf{err}(V_n, \dots, V_{n+p-1}), +\mathbf{err}(V_n, \dots, V_{n+p-1})] + \sum_{k=1}^p J_k \times V_{n+k-1}$$

instead.

A sound factoring algorithm is not required !

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Other filters

We consider sequences of the following form:

$$\begin{cases} S_k = i_k, 0 \leq k < p \\ S_{n+p} = \overline{F}(S_n, \dots, S_{n+p-1}) \overline{+} \overline{G}(E_{n+p+1-q}, \dots, E_{n+p}) \end{cases}$$

Having bounds:

- on the **input** sequence (E_n) ,
- and on the **initial outputs** $(i_k)_{0 \leq k < p}$;

we want to **infer a bound on the output** sequence (S_n) .

Splitting S_n

We split the output sequence $S_n = R_n + \varepsilon_n$ into

- the contribution of the errors (ε_n) ;

$$\begin{cases} \varepsilon_k = 0, & 0 \leq k < p; \\ \varepsilon_{n+p} = F(\varepsilon_n, \dots, \varepsilon_{n+p-1}) + \mathbf{err}_{n+p} \end{cases}$$

- the ideal sequence (R_n) (in the real field);

$$\begin{cases} R_k = i_k, & 0 \leq k < p \\ R_{n+p} = F(R_n, \dots, R_{n+p-1}) + G(E_{n+p+1-q}, \dots, E_{n+p}) \end{cases}$$

Bounding R_n

To refine the output, we need to bound the sequence R_n :

1. We isolate the contribution of the N last inputs:

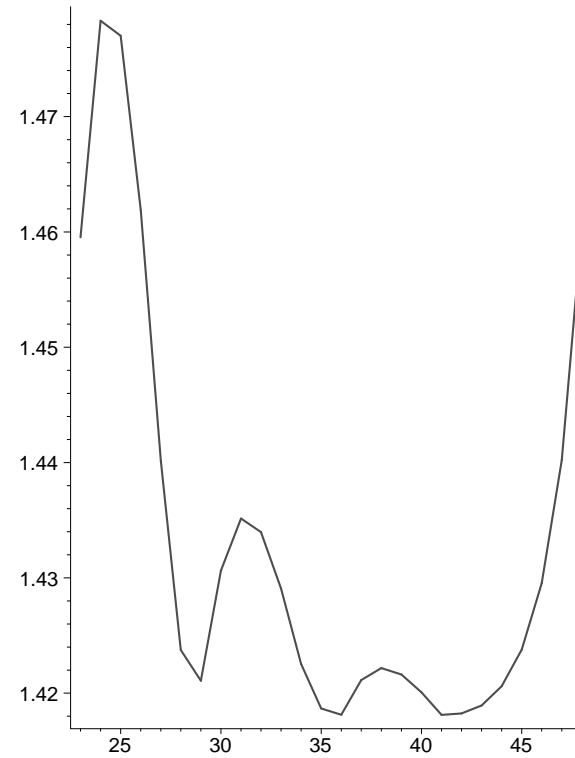
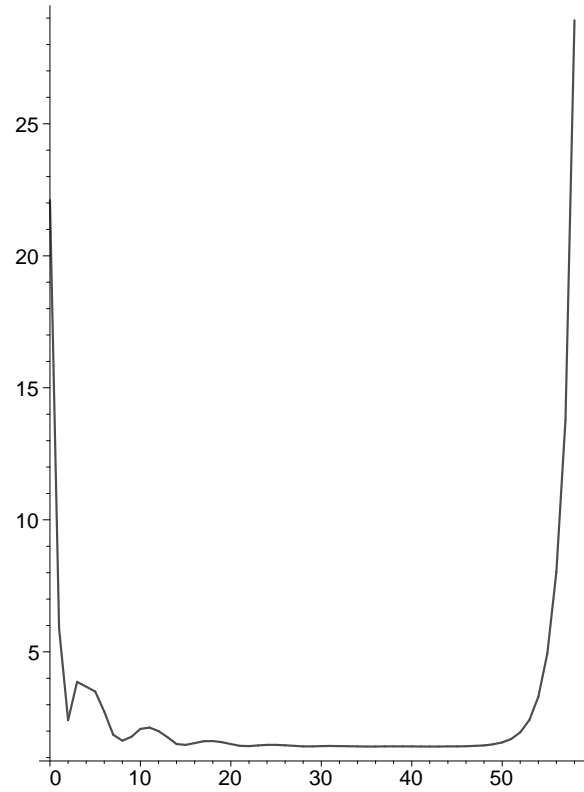
$$R_n = \text{last}_n^N(E_n, \dots, E_{n+1-N}) + \text{res}_n^N.$$

2. Since the filter is linear, we have, for $n > N + p$:

- $\text{last}_n^N(X_1, \dots, X_N) = \text{last}_{N+p}^N(X_1, \dots, X_N)$;
- res_n^N satisfies:

$$\text{res}_{n+p}^N = F(\text{res}_n^N, \dots, \text{res}_{n+p-1}^N) + G'_{[F,G]}(E_{n+p-N+1-q}, \dots, E_{n+p-N})$$

Abstract gain with respect to N



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Patterns

We use patterns to detect filter iterations:

$$P \triangleq (P \oplus P) \mid (P \ominus P) \mid (P \otimes P) \mid (\ominus P) \mid c \in \mathit{Var}_{cste} \mid V \in \mathit{Var}_{var}$$

Patterns are seen up to the following congruence relation:

$$\begin{aligned} (P_1 \odot P_2) &\equiv_P (P_2 \odot P_1) && \text{for } \odot \in \{\oplus, \otimes\} \\ ((P_1 \odot P_2) \odot P_3) &\equiv_P (P_1 \odot (P_2 \odot P_3)) && \text{for } \odot \in \{\oplus, \otimes\} \\ P_1 &\equiv_P (\ominus(\ominus P_1)) \\ (P_1 \ominus P_2) &\equiv_P (P_1 \oplus (\ominus P_2)) \\ \ominus(P_1 \odot P_2) &\equiv_P ((\ominus P_1) \odot (\ominus P_2)) && \text{for } \odot \in \{\oplus, \ominus\} \\ \ominus(P_1 \otimes P_2) &\equiv_P ((\ominus P_1) \otimes P_2) \\ \ominus(P_1 \otimes P_2) &\equiv_P (P_1 \otimes (\ominus P_2)) \end{aligned}$$

Expressions

We consider:

1. interval constraints:

$$\rho_I : \mathcal{V} \rightarrow \textit{Interval}$$

2. symbolic constraints [Miné: VMCAI'06]:

$$\rho_C : \mathcal{V} \rightarrow \textit{Expression} \cup \{\perp\}$$

Expressions in assignments are seen up the following compatible rewriting order:

$$\begin{aligned} \mathcal{C}[V] &\geq_E \mathcal{C}[\rho_I(V)] \\ \mathcal{C}[V] &\geq_E \mathcal{C}[\rho_C(V)] \text{ if } \rho_C(V) \neq \perp \end{aligned}$$

Pattern matching

Given $\rho_{cste} : \mathcal{V} \rightarrow Interval$ and $\rho_{var} : \mathcal{V} \rightarrow Expression \cup \{\top\}$,
we define the relation $\models_{\rho_{cste}, \rho_{var}}$ by induction as follows:

If: $E_1 \models_{\rho_{cste}, \rho_{var}} P_1$ and $E_2 \models_{\rho_{cste}, \rho_{var}} P_2$
then:

$$\begin{aligned} (E_1 \overline{+} E_2) &\models_{\rho_{cste}, \rho_{var}} (P_1 \oplus P_2) \\ (E_1 \overline{-} E_2) &\models_{\rho_{cste}, \rho_{var}} (P_1 \ominus P_2) \\ (E_1 \overline{\times} E_1) &\models_{\rho_{cste}, \rho_{var}} (P_1 \otimes P_2) \end{aligned}$$

$$\begin{array}{ll} E \models_{\rho_{cste}, \rho_{var}} c & \text{if } \rho_{cste}(c) = \tilde{\rho}_I(E) \\ E \models_{\rho_{cste}, \rho_{var}} (\ominus c) & \text{if } \rho_{cste}(c) = \tilde{\rho}_I(\overline{-}E) \\ X \models_{\rho_{cste}, \rho_{var}} V & \text{if } \rho_{var}(V) = X \end{array}$$

When $E \models_{\rho_{cste}, \rho_{var}} P$, we say that the expression E matches the pattern P
under the environments ρ_{cste} and ρ_{var} .

Abstract pattern matching

Given an expression E and a pattern P ,
find a set of tuples $(E', P', \rho_{cste}, \rho_{var})$ such that:

1. $E \geq_E E'$
2. $P \equiv_P P'$;
3. $E' \models_{\rho_{cste}, \rho_{var}} P'$.

We explore E and P in parallel, when necessary:

1. we reorder terms and factors in P ;
2. we introduce unary negations in P ;
3. we push negations toward the leaves of P ;
4. we replace variables in E with their symbolic constraints/ranges;

Memoization / Certificate

Exploration is costly (exponential in the size of P).

We use memoization to amortize this cost.

1. After each exploration, we memoize:

- successful patterns $((E', P', \rho_{cste}, \rho_{var}))$;
(it can be used as certificate for *a posteriori* checks)
- symbolic constraints that have been used;

2. At next iterations:

- when these symbolic constraints have changed,
we redo the exploration;
- otherwise we check whether the pattern is still valid.

We deal with rounding errors the usual way.

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Benchmarks

We analyze three programs in the same family on a **AMD Opteron 248, 8 Gb of RAM** (analyses use only **2 Gb of RAM**).

lines of C	70,000			216,000			379,000		
global variables	13,400			7,500			9,000		
iterations	72	41	37	161	75	53	151	187	74
time/iteration	52s	1mn18s	1mn16s	3mn07s	5mn08s	4mn40s	4mn35s	9mn25s	8mn17s
analysis time	1h02mn	53mn	47mn	8h23mn	6h25mn	4h08mn	11h34mn	30h26mn	10h14mn
false alarms	574	3	0	207	0	0	790	0	0

1. **without** filter domains;
2. with **simplified filter** domains;
3. with **expanded filter** domains.

Conclusion

- a highly **generic framework to analyze programs with digital filtering**:
a technical knowledge of used filters allows the design of the adequate abstract domain;
- the case of **linear filters is fully handled**:
we need to solve a symbolic linear system for each filter family;
we need an (not necessarily sound) polynomial reduction algorithm for each filter instance.
- filters are detected up to:
 - term recombination
 - and some laws of the real fields;

This framework has been used and was **necessary in the full certification** of the absence of run-time error **in industrial critical embedded software**.

<http://www.astree.ens.fr>