Static Analysis for Data Science

MPRI 2-6: Abstract Interpretation, Application to Verification and Static Analysis

Caterina Urban

February 7th, 2022

Year 2021-2022

Data Science is Everywhere

data is cheap and ubiquitous

web logs

mobile devices

sensors

transactions

data science is revolutionizing industries

retail
- personalized recommendations
- targeted marketing

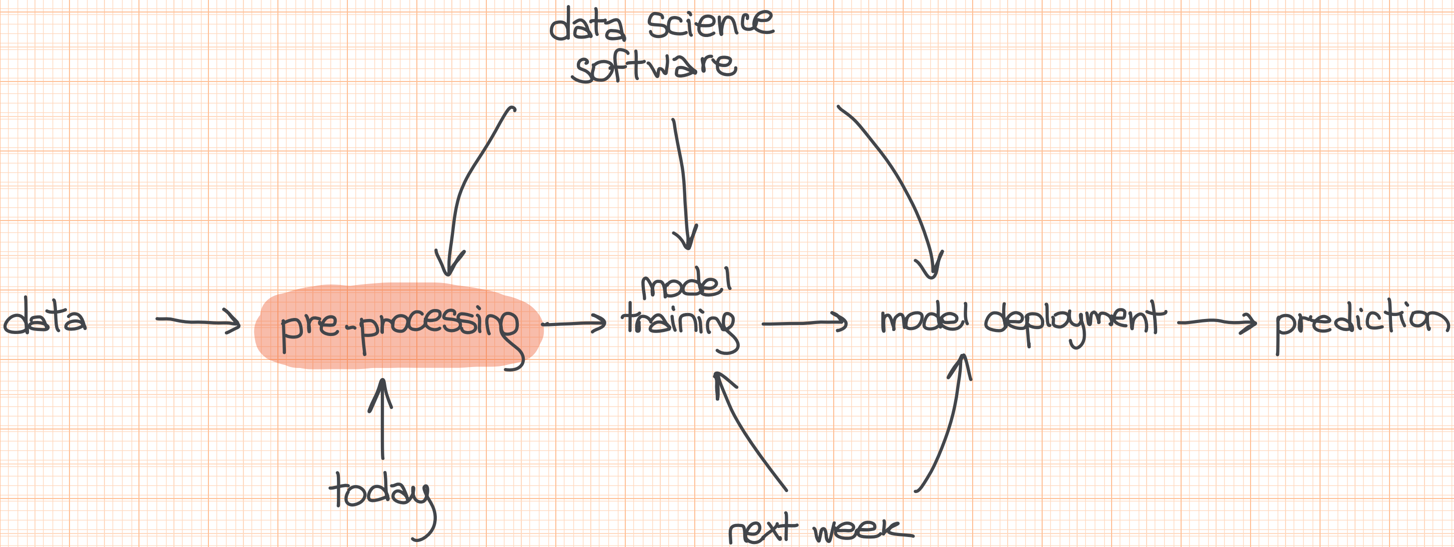
pharmaceutical
- predictive models
- patient selection

manufacturing
- equipment failure predictions
- internet of things

finance
- predictive models
- customized product offerings

energy
- exploration and discovery
- accident prevention

health care
- personalized treatments
- preventive care


data science software
data
pre-processing
model training
model deployment
prediction
today
next week

Ubiquitous Programming Errors

data science means **programming**

Excel

R

Python

Julia

programming means **programming errors**

programming errors that do not cause failures can have **serious consequences**

pharmaceutical

energy

finance

health care

Anomalously Unused Data

The Reinhart-Rogoff Paper

American Economic Review: Papers & Proceedings 100 (May 2010): 573–578
http://www.aeaweb.org/articles.php?doi=10.1257/aer.100.2.573

Growth in a Time of Debt

By Carmen M. Reinhart and Kenneth S. Rogoff[*]

In this paper, we exploit a new multi-country historical dataset on public (government) debt to search for a systemic relationship between high public debt levels, growth and inflation.[1] Our main result is that whereas the link between growth and debt seems relatively weak at "normal" debt levels, median growth rates for countries with public debt over roughly 90 percent of GDP are about one percent lower than otherwise; average (mean) growth rates are several percent lower. Surprisingly, the relationship between public debt and growth is remarkably similar across emerging markets and advanced economies. This is not the case for inflation. We find no systematic relationship between high debt levels and inflation for advanced economies as a group (albeit with individual country exceptions including the United States). By contrast, in emerging market countries, high public debt levels coincide with higher inflation.

Our topic would seem to be a timely one. Public debt has been soaring in the wake of the recent global financial maelstrom, especially in the epicenter countries. This should not be surprising, given the experience of earlier severe financial crises.[2] Outsized deficits and epic bank bailouts may be useful in fighting a downturn, but what is the long-run macroeconomic impact, especially against the backdrop of graying populations and rising social insurance costs? Are sharply elevated public debts ultimately a manageable policy challenge?

Our approach here is decidedly empirical, taking advantage of a broad new historical dataset on public debt (in particular, central government debt) first presented in Carmen M. Reinhart and Kenneth S. Rogoff (2008, 2009b). Prior to this dataset, it was exceedingly difficult to get more than two or three decades of public debt data even for many rich countries, and virtually impossible for most emerging markets. Our results incorporate data on 44 countries spanning about 200 years. Taken together, the data incorporate over 3,700 annual observations covering a wide range of political systems, institutions, exchange rate and monetary arrangements, and historic circumstances.

We also employ more recent data on external debt, including debt owed both by governments and by private entities. For emerging markets, we find that there exists a significantly more severe threshold for total gross external debt (public and private)—which is almost exclusively denominated in a foreign currency—than for total public debt (the domestically issued component of which is largely denominated in home currency). When gross external debt reaches 60 percent of GDP, annual growth declines by about two percent; for levels of external debt in excess of 90 percent of GDP, growth rates are roughly cut in half. We are not in a position to calculate separate total external debt thresholds (as opposed to public debt thresholds) for advanced countries. The available time-series is too recent, beginning only in 2000. We do note, however, that external debt levels in advanced countries now average nearly 200 percent of GDP, with external debt levels being particularly high across Europe.

The focus of this paper is on the longer term macroeconomic implications of much higher public and external debt. The final section, how-

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[*] Reinhart: Department of Economics, 4115 Tydings Hall, University of Maryland, College Park, MD 20742 (e-mail: creinhar@umd.edu); Rogoff: Economics Department, 216 Littauer Center, Harvard University, Cambridge MA 02138–3001 (e-mail: krogoff@harvard.edu). The authors would like to thank Olivier Jeanne and Vincent R. Reinhart for helpful comments.

[1] In this paper "public debt" refers to gross central government debt. "Domestic public debt" is government debt issued under domestic legal jurisdiction. Public debt does not include debts carrying a government guarantee. Total gross external debt includes the external debts of *all* branches of government as well as private debt that is issued by domestic private entities under a foreign jurisdiction.

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The Reinhart-Rog

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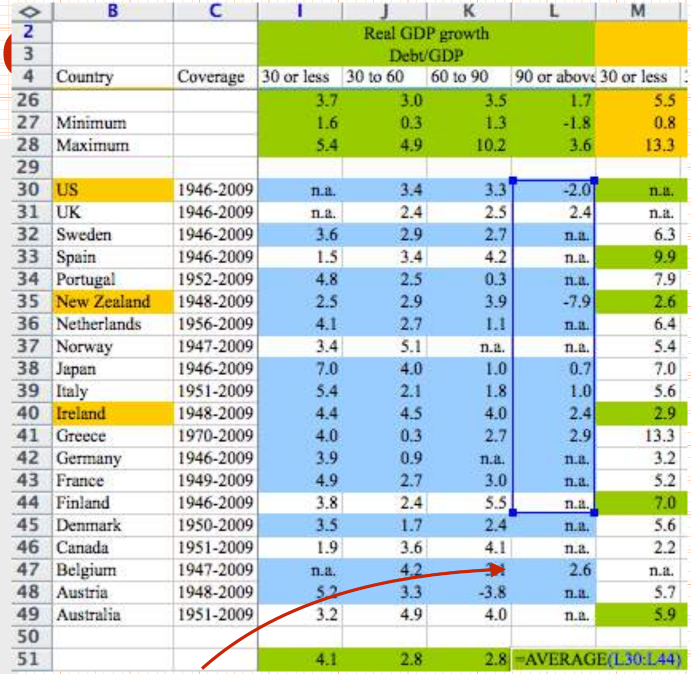
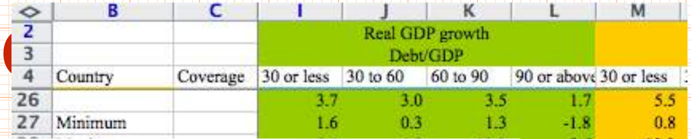


	B	C	I	J	K	L	M
2			Real GDP growth				
3			Debt/GDP				
4	Country	Coverage	30 or less	30 to 60	60 to 90	90 or above	30 or less
26			3.7	3.0	3.5	1.7	5.5
27	Minimum		1.6	0.3	1.3	-1.8	0.8
28	Maximum		5.4	4.9	10.2	3.6	13.3
29							
30	US	1946-2009	n.a.	3.4	3.3	-2.0	n.a.
31	UK	1946-2009	n.a.	2.4	2.5	2.4	n.a.
32	Sweden	1946-2009	3.6	2.9	2.7	n.a.	6.3
33	Spain	1946-2009	1.5	3.4	4.2	n.a.	9.9
34	Portugal	1952-2009	4.8	2.5	0.3	n.a.	7.9
35	New Zealand	1948-2009	2.5	2.9	3.9	-7.9	2.6
36	Netherlands	1956-2009	4.1	2.7	1.1	n.a.	6.4
37	Norway	1947-2009	3.4	5.1	n.a.	n.a.	5.4
38	Japan	1946-2009	7.0	4.0	1.0	0.7	7.0
39	Italy	1951-2009	5.4	2.1	1.8	1.0	5.6
40	Ireland	1948-2009	4.4	4.5	4.0	2.4	2.9
41	Greece	1970-2009	4.0	0.3	2.7	2.9	13.3
42	Germany	1946-2009	3.9	0.9	n.a.	n.a.	3.2
43	France	1949-2009	4.9	2.7	3.0	n.a.	5.2
44	Finland	1946-2009	3.8	2.4	5.5	n.a.	7.0
45	Denmark	1950-2009	3.5	1.7	2.4	n.a.	5.6
46	Canada	1951-2009	1.9	3.6	4.1	n.a.	2.2
47	Belgium	1947-2009	n.a.	4.2	2.1	2.6	n.a.
48	Austria	1948-2009	5.2	3.3	-3.8	n.a.	5.7
49	Australia	1951-2009	3.2	4.9	4.0	n.a.	5.9
50							
51			4.1	2.8	2.8	=AVERAGE(L30:L44)	

data excluded from the analysis

The Reinhart-Rog



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						-2.0	n.a.
						2.4	n.a.
						n.a.	6.3
						n.a.	9.9
						n.a.	7.9

FAQ: Reinhart, Rogoff, and the Excel Error That Changed History

By Peter Coy | April 18, 2013

The Excel Depression

By PAUL KRUGMAN
Published: April 18, 2013 | 470 Comments

In this age of information, math errors can lead to disaster. NASA's Mars Orbiter crashed because engineers forgot to convert to metric measurements; JPMorgan Chase's "London Whale" venture went bad in part because modelers divided by a sum instead of an average. So, did an Excel coding error destroy the economies of the Western world?

Enlarge This Image

The story so far: At the beginning of 2010, two Harvard economists, Carmen Reinhart and Kenneth Rogoff, circulated a paper, "Growth in a Time of Debt," that purported to identify a critical "threshold," a tipping point, for government indebtedness. Once debt exceeds 90 percent of gross domestic product, they claimed, economic growth drops off sharply.

Ms. Reinhart and Mr. Rogoff had credibility thanks to a widely admired earlier book on the history of financial

England Covid-19 Cases Error

SCIENCE | US & WORLD | TECH

Excel spreadsheet error blamed for UK's 16,000 missing coronavirus cases

The case went missing after the spreadsheet hit its filesize limit

By James Vincent | Oct 5, 2020, 9:41am EDT

NEWS

Check for updates

The BMJ

Cite this as: *BMJ* 2020;371:m3891

http://dx.doi.org/10.1136/bmj.m3891

Published: 06 October 2020

Covid-19: Only half of 16 000 patients missed from England's official figures have been contacted

Elisabeth Mahase

Details of nearly 16 000 cases of covid-19 were not transferred to England's NHS Test and Trace service and were missed from official figures because of an error in the process for updating the data.

England's health and social care secretary, Matt Hancock, told the House of Commons on Monday 5 October that after the error was discovered on Friday 2 October "6500 hours of extra contact tracing" had been carried out over the weekend. But as at Monday morning only half (51%) of the people had been reached by contact tracers.

In response, Labour's shadow health secretary, Jonathan Ashworth, said, "Thousands of people are blissfully unaware they have been exposed to covid,

data and furthermore have issued guidance on validation and risk management for these products if they are to be used in such a safety critical manner."

The error came as the Labour Party's leader, Keir Starmer, said that the prime minister had "lost control" of covid-19, with no clear strategy for beating it. Speaking to the *Observer*, Starmer set out his five point plan for covid-19, which starts with publishing the criteria for local restrictions, as the German government did. Secondly, he said public health messaging should be improved by adding a feature to the NHS covid-19 app so people can search their postcode and find out their local restrictions.

Starmer has also said he would fix the contact tracing system by investing in NHS and university laboratories to expand testing and at the same time put local public health teams in charge of contact

Example

```
english = bool(input())
math = bool(input())
science = bool(input())
bonus = bool(input())

passing = True
if not english:
    english = False
if not math:
    passing = False or bonus
if not math:
    passing = False or bonus

print(passing)
```

INPUT VARIABLES

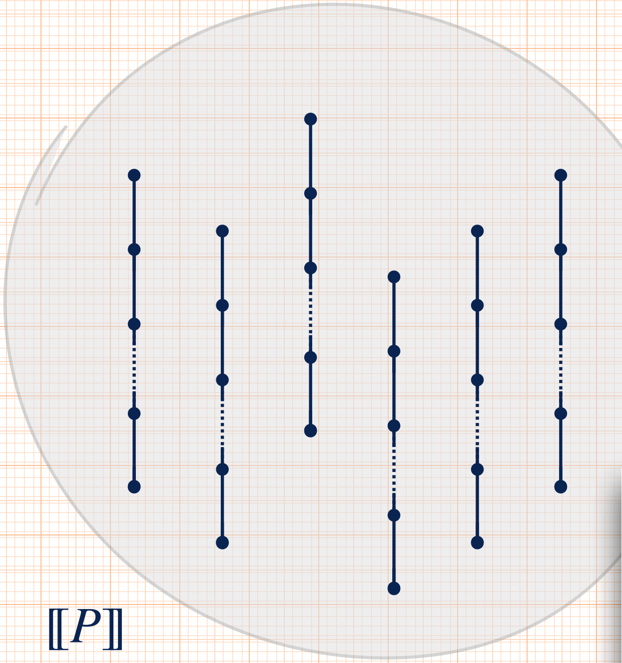
ERROR: english SHOULD BE passing

ERROR: math SHOULD BE science

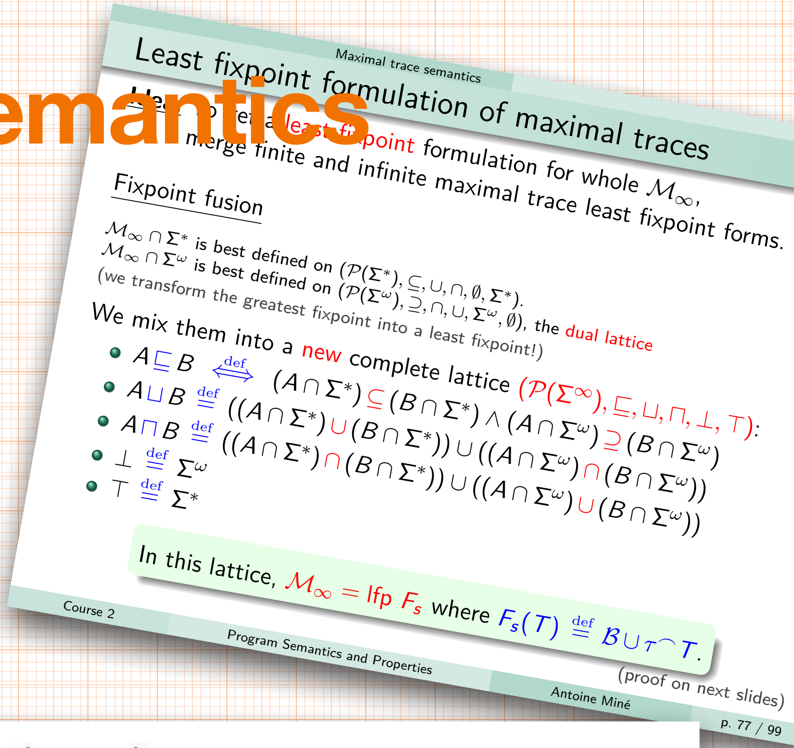
OUTPUT VARIABLES

the input variables english and science are unused

Maximal Trace Semantics



$[\![P]\!]$



Maximal trace semantics

Least fixpoint formulation of maximal traces

Idea: to get a least fixpoint formulation for whole $\mathcal{M}_\infty$, merge finite and infinite maximal trace least fixpoint forms.

Fixpoint fusion

$\mathcal{M}_\infty \cap \Sigma^*$ is best defined on $(\mathcal{P}(\Sigma^*), \subseteq, \cup, \cap, \emptyset, \Sigma^*)$.
$\mathcal{M}_\infty \cap \Sigma^\omega$ is best defined on $(\mathcal{P}(\Sigma^\omega), \supseteq, \cap, \cup, \Sigma^\omega, \emptyset)$, the dual lattice
(we transform the greatest fixpoint into a least fixpoint!)

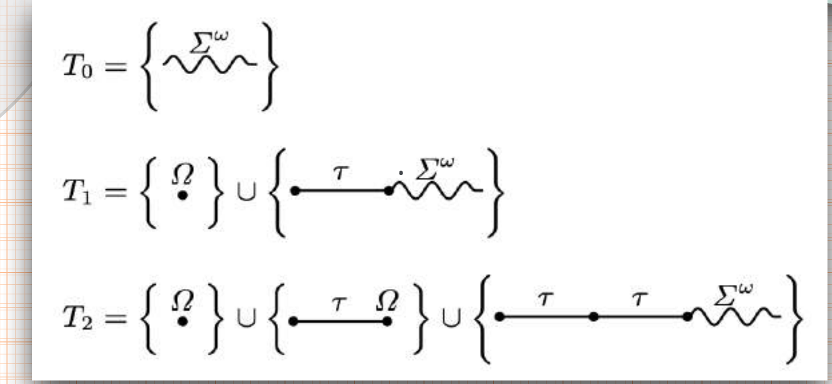
We mix them into a new complete lattice $(\mathcal{P}(\Sigma^\infty), \sqsubseteq, \sqcup, \sqcap, \bot, \top)$:

- $A \sqsubseteq B \overset{\text{def}}{\iff} (A \cap \Sigma^*) \subseteq (B \cap \Sigma^*) \wedge (A \cap \Sigma^\omega) \supseteq (B \cap \Sigma^\omega)$
- $A \sqcup B \overset{\text{def}}{=} ((A \cap \Sigma^*) \cup (B \cap \Sigma^*)) \cup ((A \cap \Sigma^\omega) \cap (B \cap \Sigma^\omega))$
- $A \sqcap B \overset{\text{def}}{=} ((A \cap \Sigma^*) \cap (B \cap \Sigma^*)) \cup ((A \cap \Sigma^\omega) \cup (B \cap \Sigma^\omega))$
- $\bot \overset{\text{def}}{=} \Sigma^\omega$
- $\top \overset{\text{def}}{=} \Sigma^*$

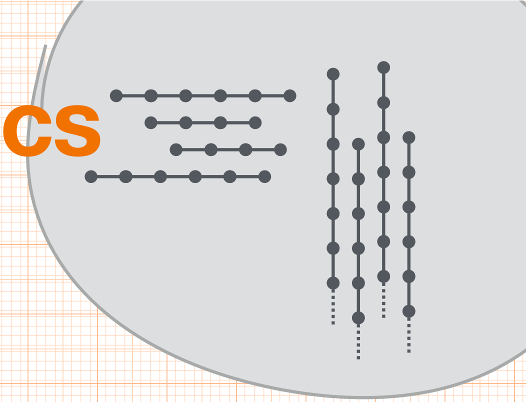
In this lattice, $\mathcal{M}_\infty = \text{lfp } F_s$ where $F_s(T) \overset{\text{def}}{=} \mathcal{B} \cup \tau^\frown T$.

(proof on next slides)

Course 2 | Program Semantics and Properties | Antoine Miné | p. 77 / 99



Maximal Trace Semantics



$[\![P]\!]$

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if not math:
    passing = False or bonus
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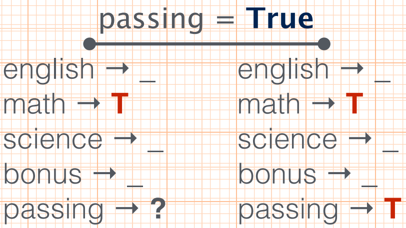
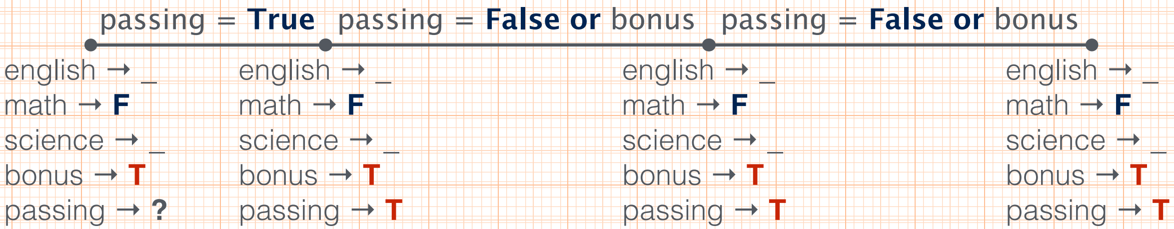
```



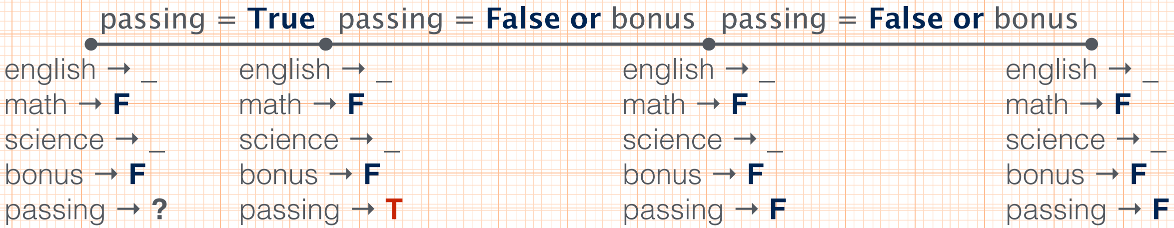
passing = **True**

english → _	english → _
math → **T**	math → **T**
science → _	science → _
bonus → _	bonus → _
passing → **?**	passing → **T**



passing = **True** passing = **False or** bonus passing = **False or** bonus

english → _	english → _	english → _	english → _
math → **F**	math → **F**	math → **F**	math → **F**
science → _	science → _	science → _	science → _
bonus → **T**	bonus → **T**	bonus → **T**	bonus → **T**
passing → **?**	passing → **T**	passing → **T**	passing → **T**



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bonus → **F**	bonus → **F**	bonus → **F**	bonus → **F**
passing → **?**	passing → **T**	passing → **F**	passing → **F**

Input Data (Non-)Usage

set of all input variables of program P

$$\mathcal{N}_J \overset{\text{def}}{=} \{[\![P]\!] \in \mathcal{P}(\Sigma^{+\infty}) \mid \forall i \in J \subseteq I_P \colon \text{UNUSED}_i([\![P]\!])\}$$

$\mathcal{N}_J$ is the set of all programs P (or, rather, their semantics $[\![P]\!]$) that **do not use** the value of the input variables in $J \subseteq I_P$

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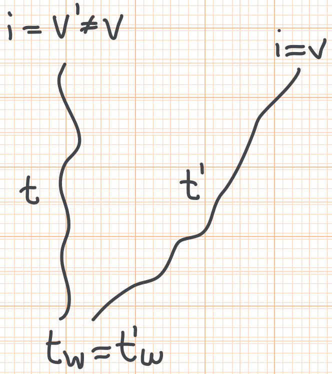
set of all possible variable values

$$\begin{aligned}\text{UNUSED}_i([\![M]\!]) \overset{\text{def}}{=} \; & \forall t \in [\![P]\!], v \in \mathscr{V} \colon t_0(i) \neq v \Rightarrow \exists t' \in [\![P]\!] \colon \\ & (\forall 0 \leq j \leq |I_P| \colon j \neq i \Rightarrow t_0(j) = t'_0(j)) \\ & \wedge t'_0(i) = v \\ & \wedge t_\omega = t'_\omega\end{aligned}$$



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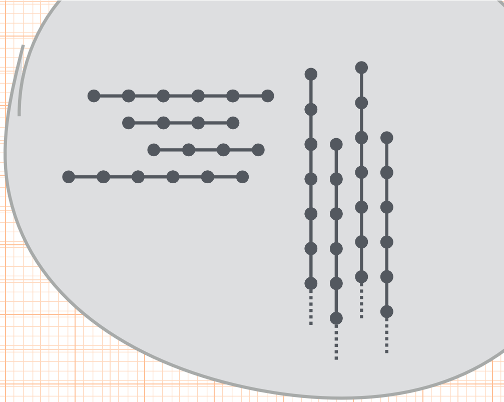
Intuitively: **any possible program outcome** is possible **from any value of the input** variable i

Input Data (Non-)Usage



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science → _	science → _
bonus → _	bonus → _
passing → **?**	passing → **T**

passing = **True** passing = **False or** bonus passing = **False or** bonus

english → _	english → _	english → _	english → _
math → **F**	math → **F**	math → **F**	math → **F**
science → _	science → _	science → _	science → _
bonus → **T**	bonus → **T**	bonus → **T**	bonus → **T**
passing → **?**	passing → **T**	passing → **T**	passing → **T**

only the input value 'math → F' yields the outcome 'passing → F'

passing = **True** passing = **False or** bonus passing = **False or** bonus

english → _	english → _	english → _	english → _
math → **F**	math → **F**	math → **F**	math → **F**
science → _	science → _	science → _	science → _
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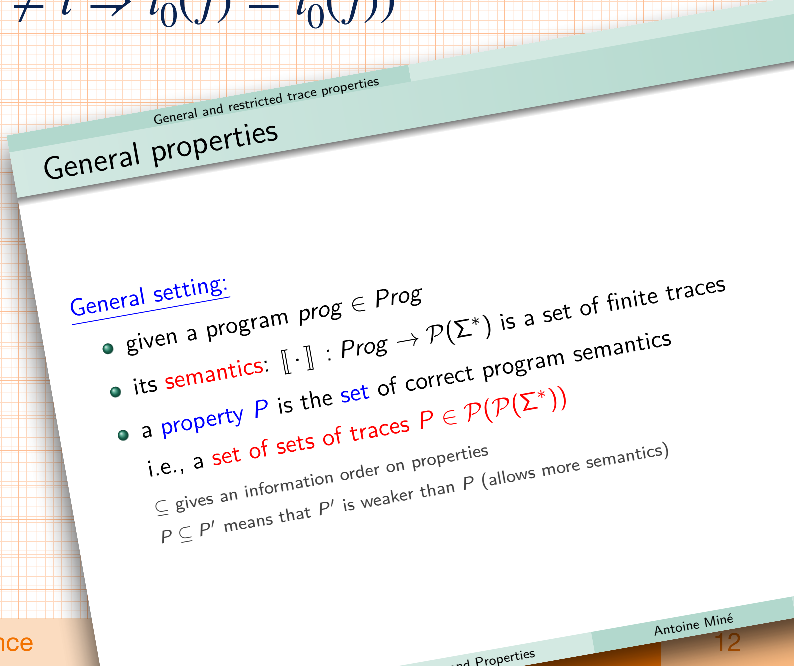


General and restricted trace properties

General properties

General setting:

- given a program $prog \in Prog$
- its semantics: $[\![\cdot]\!] : Prog \to \mathcal{P}(\Sigma^*)$ is a set of finite traces
- a property P is the set of correct program semantics

  i.e., a set of sets of traces $P \in \mathcal{P}(\mathcal{P}(\Sigma^*))$

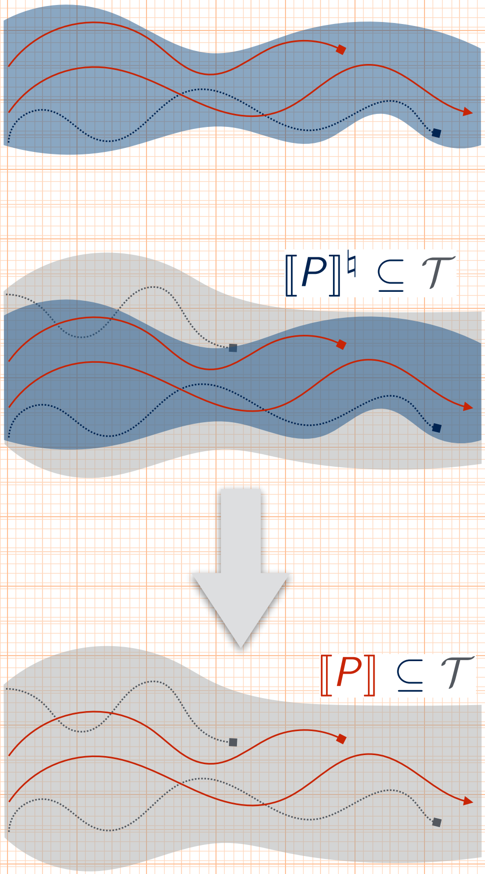
$\subseteq$ gives an information order on properties

$P \subseteq P'$ means that P' is weaker than P (allows more semantics)

nd Properties

Antoine Miné

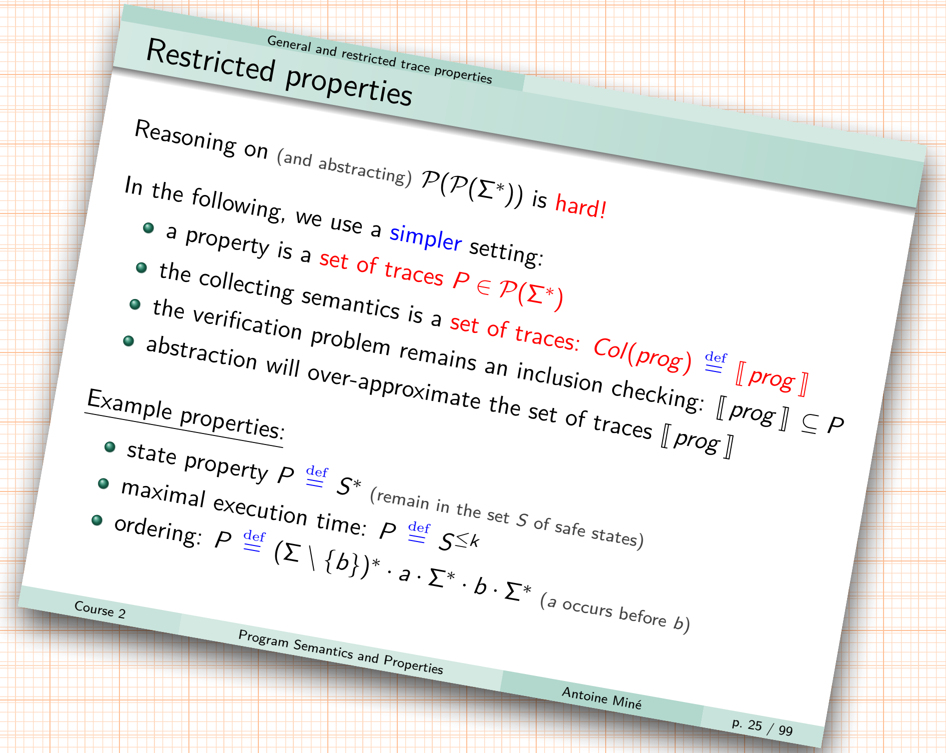
Trace Properties



$[\![P]\!] \subseteq [\![P]\!]^\natural$

$[\![P]\!]^\natural \subseteq \mathcal{T}$

$[\![P]\!] \subseteq \mathcal{T}$



General and restricted trace properties

Restricted properties

Reasoning on (and abstracting) $\mathcal{P}(\mathcal{P}(\Sigma^*))$ is hard!

In the following, we use a simpler setting:

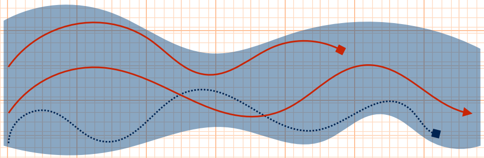
- a property is a set of traces $P \in \mathcal{P}(\Sigma^*)$
- the collecting semantics is a set of traces: $Col(prog) \stackrel{\text{def}}{=} [\![\, prog \,]\!]$
- the verification problem remains an inclusion checking: $[\![\, prog \,]\!] \subseteq P$
- abstraction will over-approximate the set of traces $[\![\, prog \,]\!]$

Example properties:

- state property $P \stackrel{\text{def}}{=} S^*$ (remain in the set S of safe states)
- maximal execution time: $P \stackrel{\text{def}}{=} S^{\leq k}$
- ordering: $P \stackrel{\text{def}}{=} (\Sigma \setminus \{b\})^* \cdot a \cdot \Sigma^* \cdot b \cdot \Sigma^*$ (a occurs before b)

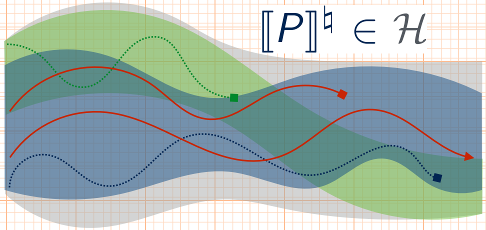
Course 2 Program Semantics and Properties Antoine Miné p. 25 / 99

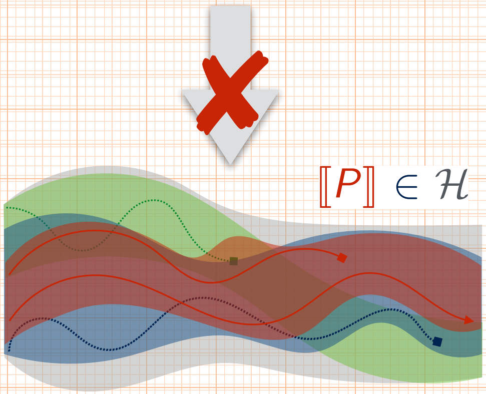
Program Properties



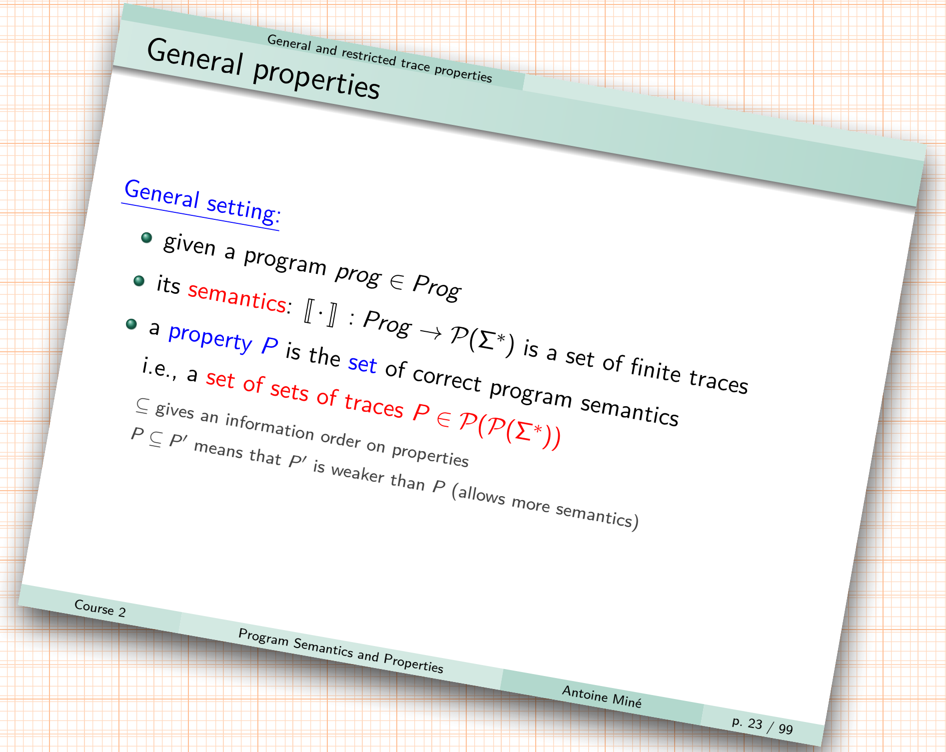
$[\![P]\!] \subseteq [\![P]\!]^\natural$

$[\![P]\!]^\natural \in \mathcal{H}$





$[\![P]\!] \in \mathcal{H}$



General and restricted trace properties

General properties

General setting:

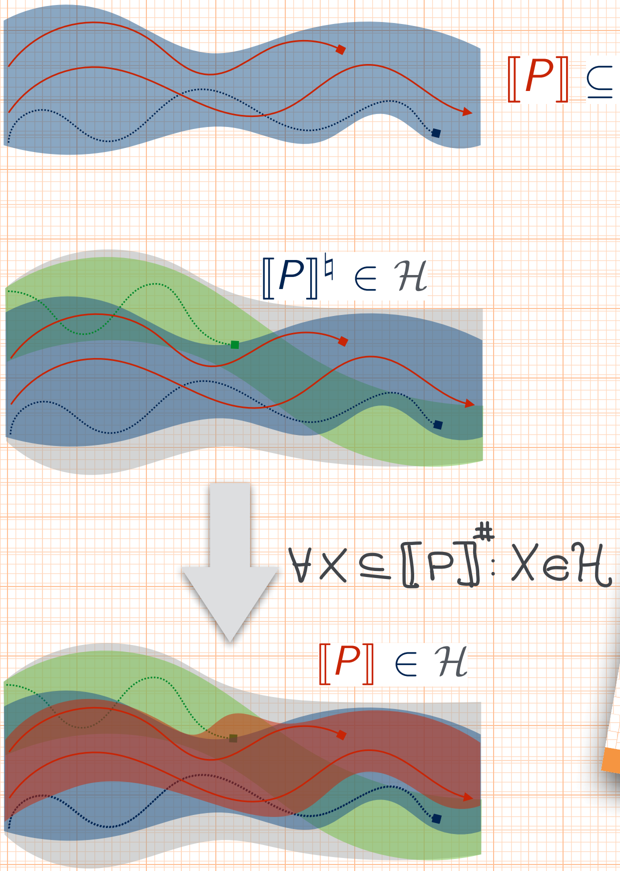
- given a program $prog \in Prog$
- its semantics: $[\![\cdot]\!] : Prog \to \mathcal{P}(\Sigma^*)$ is a set of finite traces
- a property P is the set of correct program semantics
  i.e., a set of sets of traces $P \in \mathcal{P}(\mathcal{P}(\Sigma^*))$

$\subseteq$ gives an information order on properties

$P \subseteq P'$ means that P' is weaker than P (allows more semantics)

Course 2 | Program Semantics and Properties | Antoine Miné | p. 23 / 99

Subset-Closed Properties

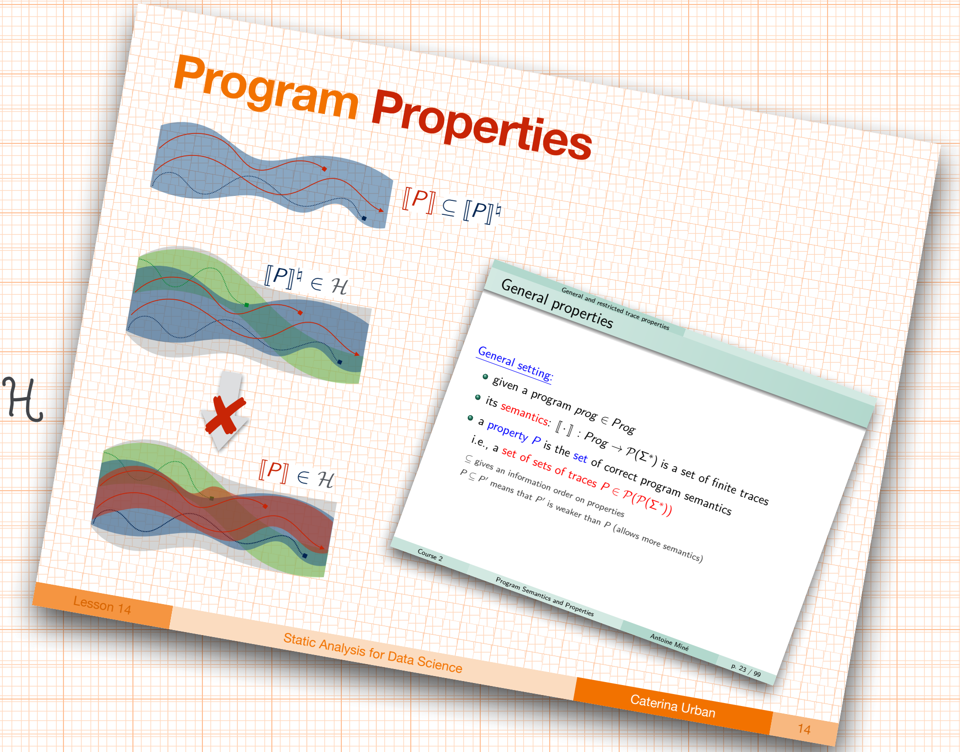


$$[\![P]\!] \subseteq [\![P]\!]^\natural$$

$$[\![P]\!]^\natural \in \mathcal{H}$$

$$\forall X \subseteq [\![P]\!]^\#: X \in \mathcal{H}$$

$$[\![P]\!] \in \mathcal{H}$$



Program Properties

$$[\![P]\!] \subseteq [\![P]\!]^\natural$$

$$[\![P]\!]^\natural \in \mathcal{H}$$

$$[\![P]\!] \in \mathcal{H}$$

General properties

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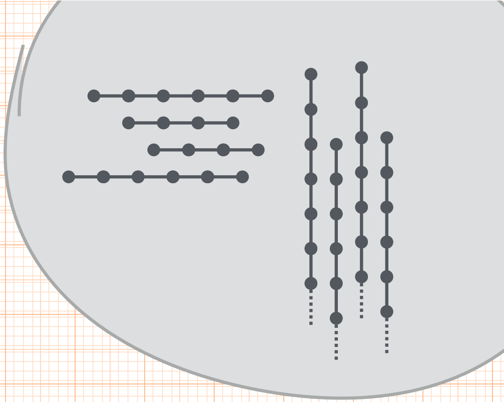
Course 2 | Program Semantics and Properties | Antoine Miné | p. 23 / 99

Lesson 14 | Static Analysis for Data Science | Caterina Urban | 14

Input Data (Non-)Usage



$[\![P]\!]$

```
passing = True
if not english:
    english = False
if not math:
    passing = False or bonus
if not math:
    passing = False or bonus
```

passing = True

english → _
math → **T**
science → _
bonus → _
passing → **?**

english → _
math → **T**
science → _
bonus → _
passing → **T**

passing = True | passing = False or bonus | passing = False or bonus

english → _
math → **F**
science → _
bonus → **T**
passing → **?**

english → _
math → **F**
science → _
bonus → **T**
passing → **T**

english → _
math → **F**
science → _
bonus → **T**
passing → **T**

english → _
math → **F**
science → _
bonus → **T**
passing → **T**

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math → **F**
science → F
bonus → **F**
passing → **?**

english → _
math → **F**
science → F
bonus → **F**
passing → **T**

english → _
math → **F**
science → F
bonus → **F**
passing → **F**

english → _
math → **F**
science → F
bonus → **F**
passing → **F**

NOT a subset-closed property!

Input Data (Non-)Usage

$$\mathcal{N}_J \overset{\text{def}}{=} \{[\![P]\!] \in \mathscr{P}(\Sigma^{+\infty}) \mid \forall i \in J \subseteq I_P \colon \text{UNUSED}_i([\![P]\!])\}$$

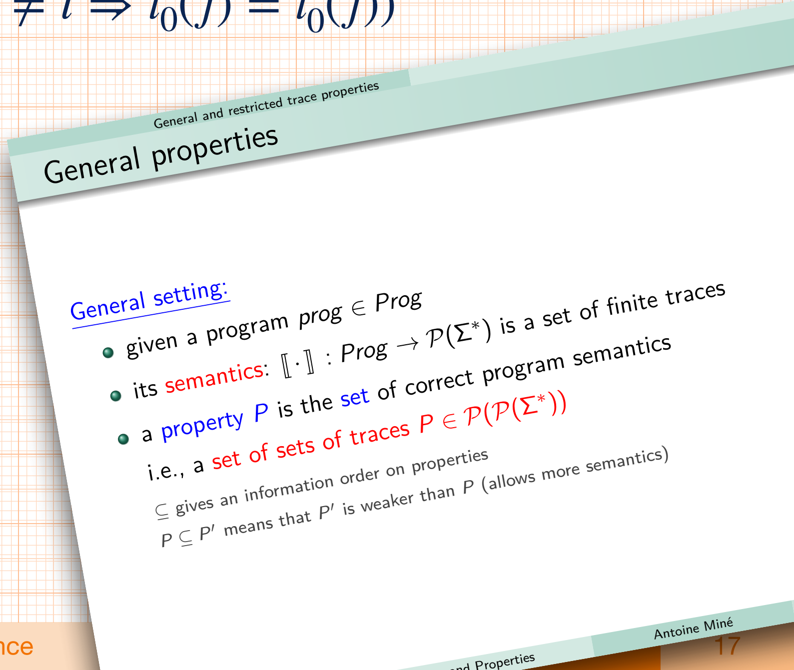
$\mathcal{N}_J$ is the set of all programs P (or, rather, their semantics $[\![P]\!]$) that **do not use** the value of the input variables in $J \subseteq I_P$

$$\begin{aligned}\text{UNUSED}_i([\![M]\!]) \overset{\text{def}}{=}\ & \forall t \in [\![P]\!], v \in \mathscr{V} \colon t_0(i) \neq v \Rightarrow \exists t' \in [\![P]\!] \colon \\ & (\forall 0 \leq j \leq |I_P| \colon j \neq i \Rightarrow t_0(j) = t'_0(j)) \\ & \wedge t'_0(i) = v \\ & \wedge t_\omega = t'_\omega\end{aligned}$$

Intuitively: **any possible program outcome** is possible **from any value of the input** variable i

Theorem

$$P \models \mathcal{N}_J \Leftrightarrow \{[\![P]\!]\} \subseteq \mathcal{N}_J$$



General and restricted trace properties

General properties

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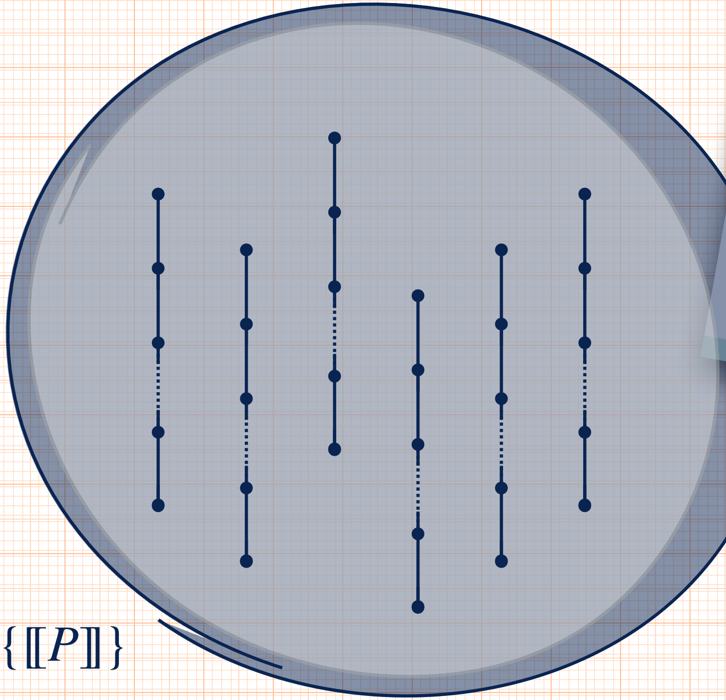
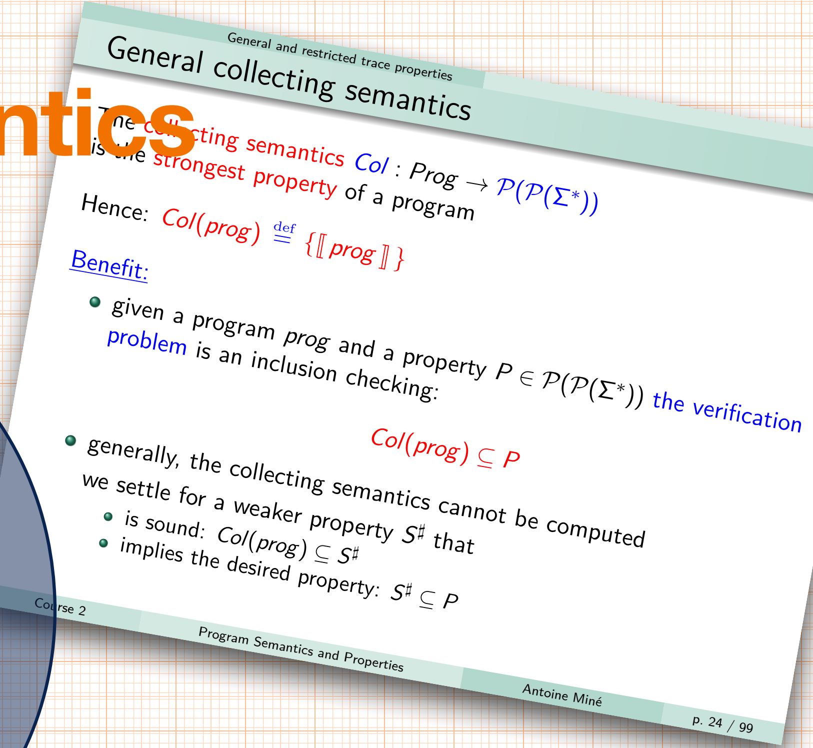
nd Properties

Antoine Miné

Collecting Semantics





General and restricted trace properties

General collecting semantics

The collecting semantics $Col : Prog \to \mathcal{P}(\mathcal{P}(\Sigma^*))$
is the strongest property of a program

Hence: $Col(prog) \stackrel{\text{def}}{=} \{[\![\, prog \,]\!]\}$

Benefit:

- given a program *prog* and a property $P \in \mathcal{P}(\mathcal{P}(\Sigma^*))$ the verification problem is an inclusion checking:

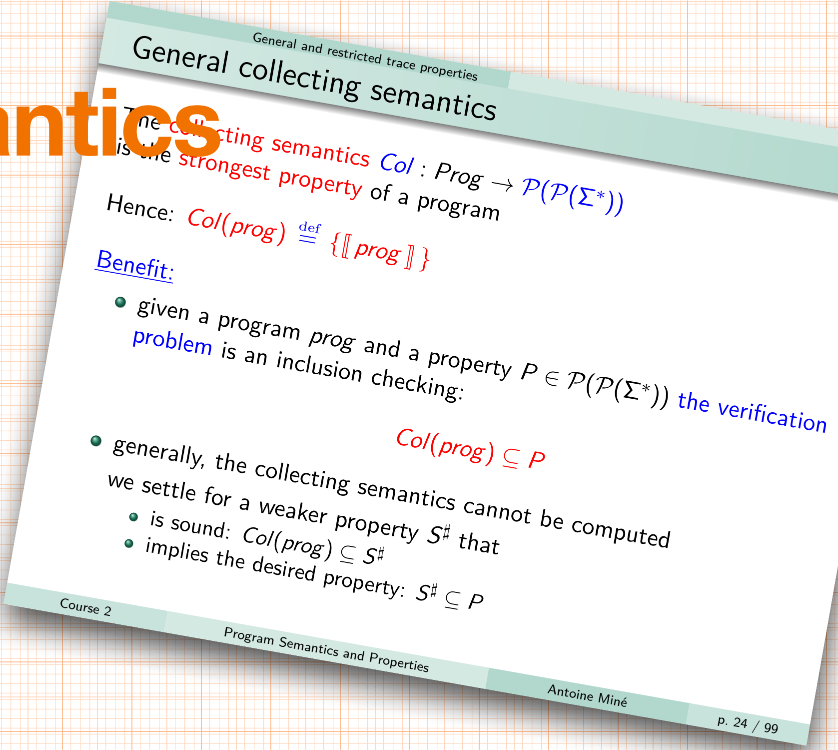
$$Col(prog) \subseteq P$$

- generally, the collecting semantics cannot be computed
  we settle for a weaker property $S^\sharp$ that
  - is sound: $Col(prog) \subseteq S^\sharp$
  - implies the desired property: $S^\sharp \subseteq P$

Course 2 | Program Semantics and Properties | Antoine Miné | p. 24 / 99

Collecting Semantics

Intuition



General and restricted trace properties

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  - implies the desired property: $S^\sharp \subseteq P$

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Property (*by extension*): set of elements that have that property

Property “being Patrick Cousot”

Property “being program P”

$$\{\llbracket P \rrbracket\}$$

(Another) **Hierarchy** of Semantics

$$\begin{tikzcd}
\llbracket P \rrbracket_{\rightsquigarrow} & \text{dependency semantics} \\
\llbracket P \rrbracket_{\bullet} \arrow[u, "\alpha_{\rightsquigarrow}"'] & \text{outcome semantics} \\
\{\llbracket P \rrbracket\} \arrow[u, "\alpha_{\bullet}"'] & \text{collecting semantics}
\end{tikzcd}$$

(Another) Hierarchy of Semantics

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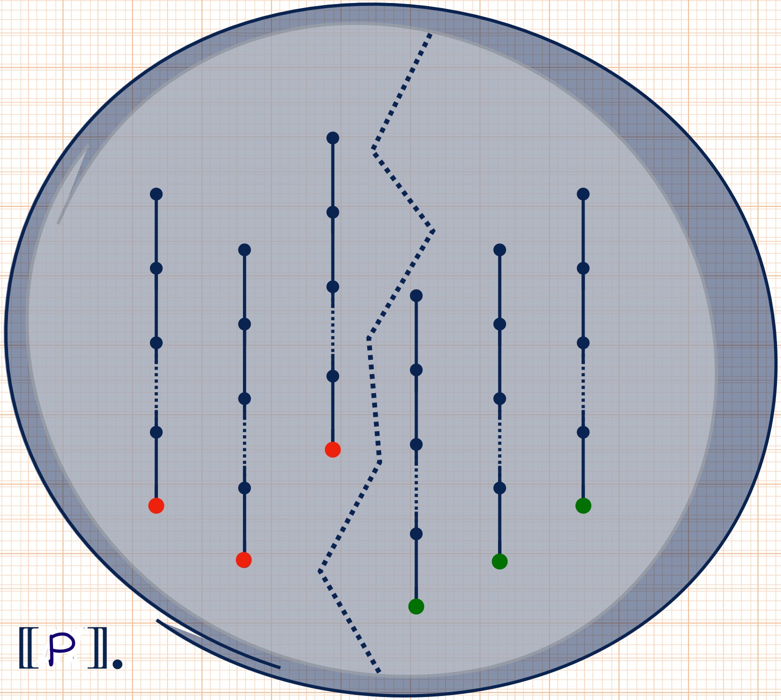
Outcome Semantics



partitioning a set of traces that satisfies input data (non-)usage **with respect to the program outcome** yields sets of traces that also satisfy input data (non-)usage

Outcome Semantics

finite sequences of states with values $v_1, \dots, v_k$ for output variables $o_1, \dots, o_k$

$$\mathbb{O} \overset{\text{def}}{=} \{\Sigma^+_{o_1=v_1,\dots,o_k=v_k} \mid v_1, \dots, v_k \in \mathscr{V}\} \cup \{\Sigma^\omega\}$$

outcomes

Lemma

$$P \models \mathscr{N}_J \Leftrightarrow \{[\![P]\!] \cap O \mid O \in \mathbb{O}\} \subseteq \mathscr{N}_J$$

input data (non-) usage can be decided independently for each possible outcome

Outcome Semantics

$$\mathbb{O} \overset{\text{def}}{=} \{\Sigma^{+}_{o_1=v_1,\ldots,o_k=v_k} \mid v_1,\ldots,v_k \in \mathcal{V}\} \cup \{\Sigma^{\omega}\}$$ **outcomes**

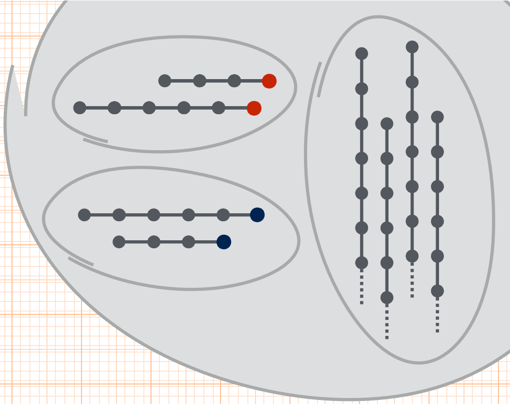
Lemma

$$P \models \mathcal{N}_J \Leftrightarrow \{[\![P]\!] \cap O \mid O \in \mathbb{O}\} \subseteq \mathcal{N}_J$$

$$\begin{tikzcd}
\langle \mathscr{P}(\mathscr{P}(\Sigma^{+\infty})), \subseteq \rangle \arrow[r, bend right, "\alpha_\bullet"'] & \langle \mathscr{P}(\mathscr{P}(\Sigma^{+\infty})), \subseteq_\bullet \rangle \arrow[l, bend right, "\gamma_\bullet"']
\end{tikzcd}$$

$$\alpha_\bullet(S) \overset{\text{def}}{=} \{T \cap O \mid T \in S \wedge O \in \mathbb{O}\}$$ **outcome abstraction**

Outcome Semantics



⟦P⟧•

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```

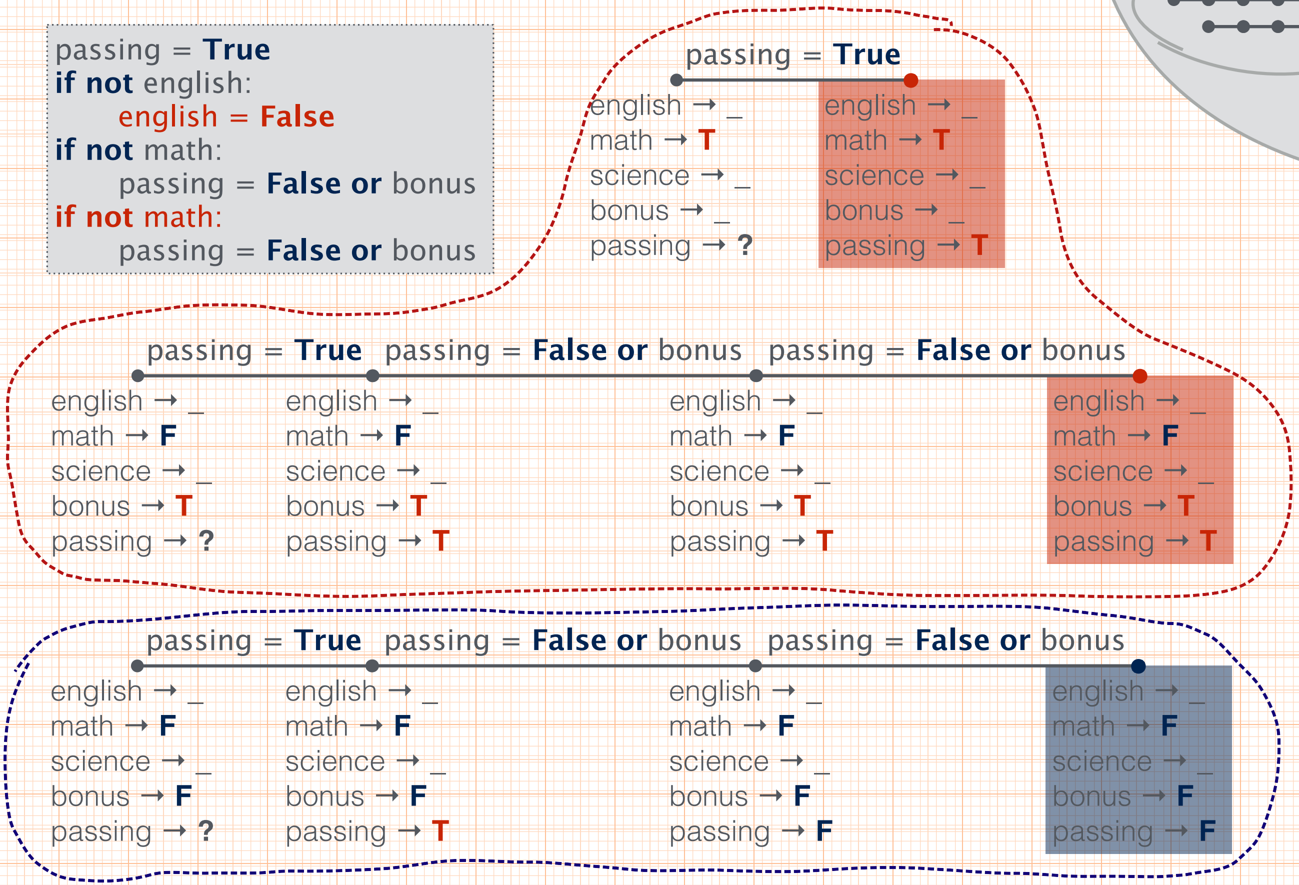

passing = True

english → _
math → T
science → _
bonus → _
passing → ?

english → _
math → T
science → _
bonus → _
passing → T

passing = True passing = False or bonus passing = False or bonus

english → _
math → F
science → _
bonus → T
passing → ?

english → _
math → F
science → _
bonus → T
passing → T

english → _
math → F
science → _
bonus → T
passing → T

english → _
math → F
science → _
bonus → T
passing → T

passing = True passing = False or bonus passing = False or bonus

english → _
math → F
science → _
bonus → F
passing → ?

english → _
math → F
science → _
bonus → F
passing → T

english → _
math → F
science → _
bonus → F
passing → F

english → _
math → F
science → _
bonus → F
passing → F

∅

unfeasible
non-termination
outcome

Outcome Semantics

$\mathbb{O} \overset{\text{def}}{=} \{\Sigma^{+}_{o_1=v_1,\ldots,o_k=v_k} \mid v_1,\ldots,v_k \in \mathscr{V}\} \cup \{\Sigma^{\omega}\}$ **outcomes**

Lemma

$P \models \mathscr{N}_J \Leftrightarrow \{[\![P]\!] \cap O \mid O \in \mathbb{O}\} \subseteq \mathscr{N}_J$

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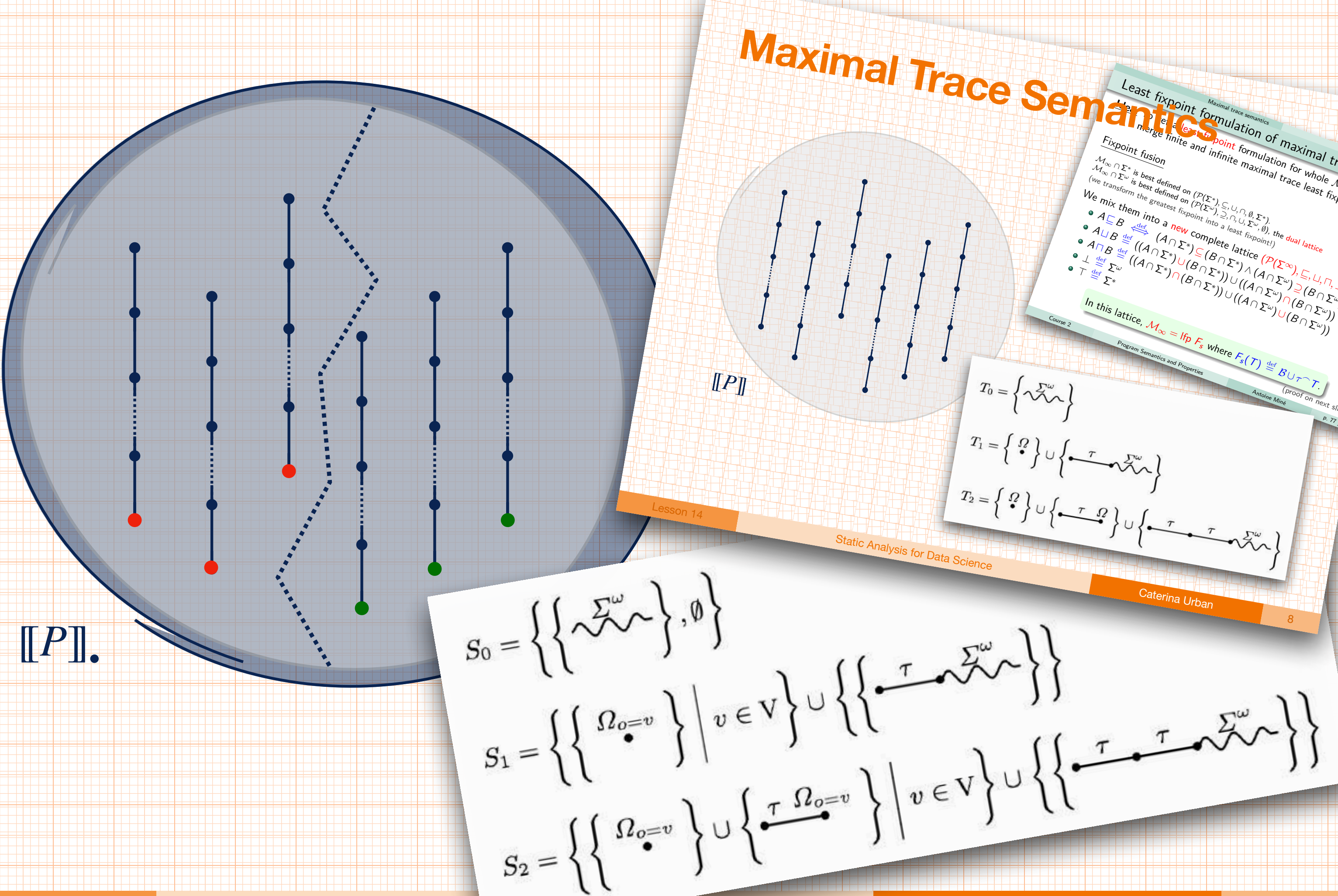
\end{tikzcd}$$

$\alpha_\bullet(S) \overset{\text{def}}{=} \{T \cap O \mid T \in S \wedge O \in \mathbb{O}\}$ **outcome abstraction**

$$[\![P]\!]_\bullet \overset{\text{def}}{=} \alpha_\bullet(\{[\![P]\!]\}) = \{[\![P]\!] \cap O \mid O \in \mathbb{O}\}$$

Outcome Semantics



Maximal Trace Semantics

Maximal trace semantics

Least fixpoint formulation of maximal tr

least fixpoint formulation for whole

merge finite and infinite maximal trace least fixp

Fixpoint fusion

$\mathcal{M}_\infty \cap \Sigma^*$ is best defined on $(\mathcal{P}(\Sigma^*), \subseteq, \cup, \cap, \emptyset, \Sigma^*)$.

$\mathcal{M}_\infty \cap \Sigma^\omega$ is best defined on $(\mathcal{P}(\Sigma^\omega), \supseteq, \cap, \cup, \Sigma^\omega, \emptyset)$, the dual lattice

(we transform the greatest fixpoint into a least fixpoint!)

We mix them into a new complete lattice $(\mathcal{P}(\Sigma^\infty), \sqsubseteq, \sqcup, \sqcap,$

- $A \sqsubseteq B \overset{\text{def}}{\iff} (A \cap \Sigma^*) \subseteq (B \cap \Sigma^*) \wedge (A \cap \Sigma^\omega) \supseteq (B \cap \Sigma^\omega$
- $A \sqcup B \overset{\text{def}}{=} ((A \cap \Sigma^*) \cup (B \cap \Sigma^*)) \cup ((A \cap \Sigma^\omega) \cap (B \cap \Sigma^\omega))$
- $A \sqcap B \overset{\text{def}}{=} ((A \cap \Sigma^*) \cap (B \cap \Sigma^*)) \cup ((A \cap \Sigma^\omega) \cup (B \cap \Sigma^\omega))$
- $\bot \overset{\text{def}}{=} \Sigma^\omega$
- $\top \overset{\text{def}}{=} \Sigma^*$

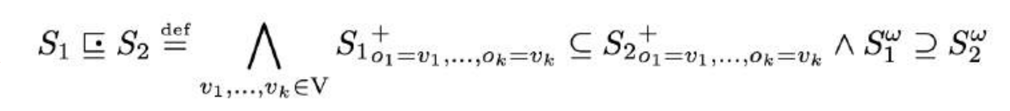
In this lattice, $\mathcal{M}_\infty = \text{lfp } F_s$ where $F_s(T) \overset{\text{def}}{=} \mathcal{B} \cup \tau^\frown T$.

(proof on next sl

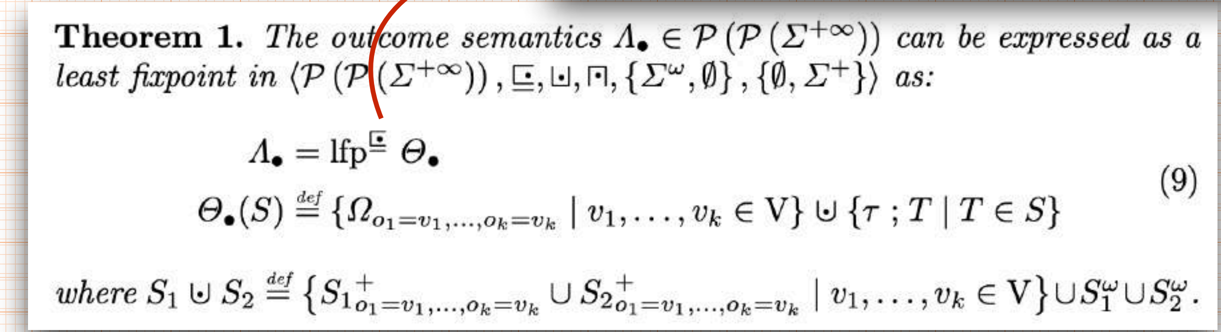
Course 2 | Program Semantics and Properties | Antoine Miné | p. 77

Lesson 14 | Static Analysis for Data Science | Caterina Urban | 8

Outcome Semantics



$$S_1 \mathrel{\dot{\sqsubseteq}} S_2 \overset{\text{def}}{=} \bigwedge_{v_1,\dots,v_k \in \mathrm{V}} {S_1}^{+}_{o_1=v_1,\dots,o_k=v_k} \subseteq {S_2}^{+}_{o_1=v_1,\dots,o_k=v_k} \wedge S_1^{\omega} \supseteq S_2^{\omega}$$

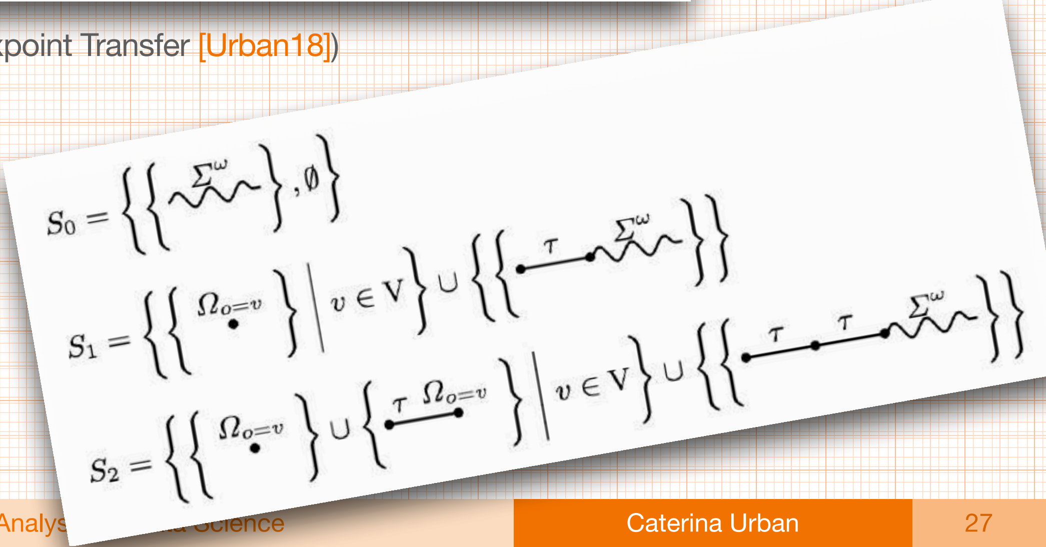


Theorem 1. *The outcome semantics* $\Lambda_\bullet \in \mathcal{P}\left(\mathcal{P}\left(\Sigma^{+\infty}\right)\right)$ *can be expressed as a least fixpoint in* $\langle \mathcal{P}\left(\mathcal{P}\left(\Sigma^{+\infty}\right)\right), \dot{\sqsubseteq}, \dot{\sqcup}, \dot{\sqcap}, \{\Sigma^{\omega}, \emptyset\}, \{\emptyset, \Sigma^{+}\}\rangle$ *as:*

$$\begin{aligned} \Lambda_\bullet &= \mathrm{lfp}^{\dot{\sqsubseteq}}\ \Theta_\bullet \\ \Theta_\bullet(S) &\overset{\mathit{def}}{=} \{\Omega_{o_1=v_1,\dots,o_k=v_k} \mid v_1,\dots,v_k \in \mathrm{V}\} \mathbin{\dot{\cup}} \{\tau \,;\, T \mid T \in S\} \end{aligned} \tag{9}$$

where $S_1 \mathbin{\dot{\cup}} S_2 \overset{\mathit{def}}{=} \{{S_1}^{+}_{o_1=v_1,\dots,o_k=v_k} \cup {S_2}^{+}_{o_1=v_1,\dots,o_k=v_k} \mid v_1,\dots,v_k \in \mathrm{V}\} \cup S_1^{\omega} \cup S_2^{\omega}$.

(proof by Kleenian Fixpoint Transfer [Urban18])



Input Data (Non-)Usage

$$\mathcal{N}_J \overset{\text{def}}{=} \{[\![P]\!] \in \mathscr{P}(\Sigma^{+\infty}) \mid \forall i \in J \subseteq I_P \colon \text{UNUSED}_i([\![P]\!])\}$$

$\mathcal{N}_J$ is the set of all programs P (or, rather, their semantics $[\![P]\!]$) that **do not use** the value of the input variables in $J \subseteq I_P$

$$t_0(i) \neq v \Rightarrow \exists t' \in [\![P]\!] :$$
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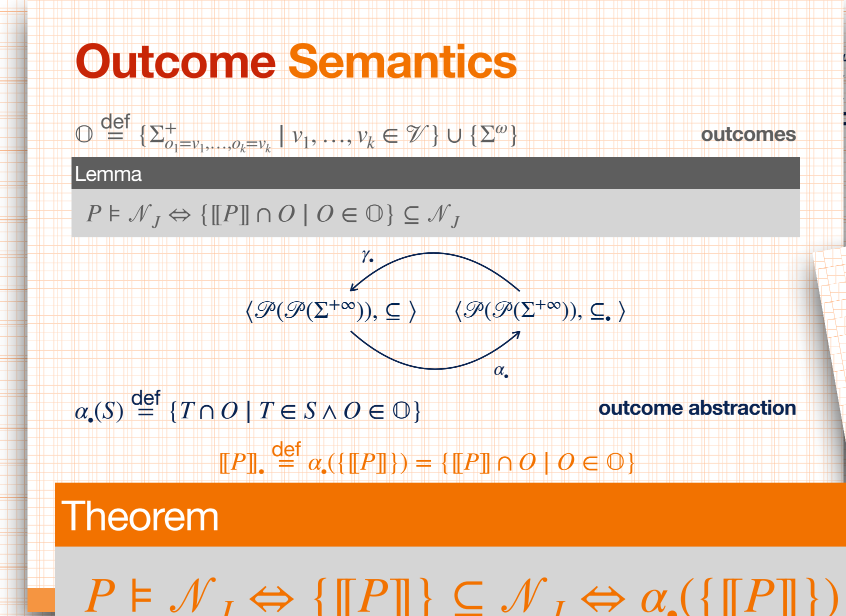

Outcome Semantics

$\mathbb{O} \overset{\text{def}}{=} \{\Sigma^+_{o_1=v_1,\ldots,o_k=v_k} \mid v_1, \ldots, v_k \in \mathscr{V}\} \cup \{\Sigma^\omega\}$ **outcomes**

Lemma

$P \models \mathcal{N}_J \Leftrightarrow \{[\![P]\!] \cap O \mid O \in \mathbb{O}\} \subseteq \mathcal{N}_J$

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$\alpha_\bullet(S) \overset{\text{def}}{=} \{T \cap O \mid T \in S \wedge O \in \mathbb{O}\}$ **outcome abstraction**

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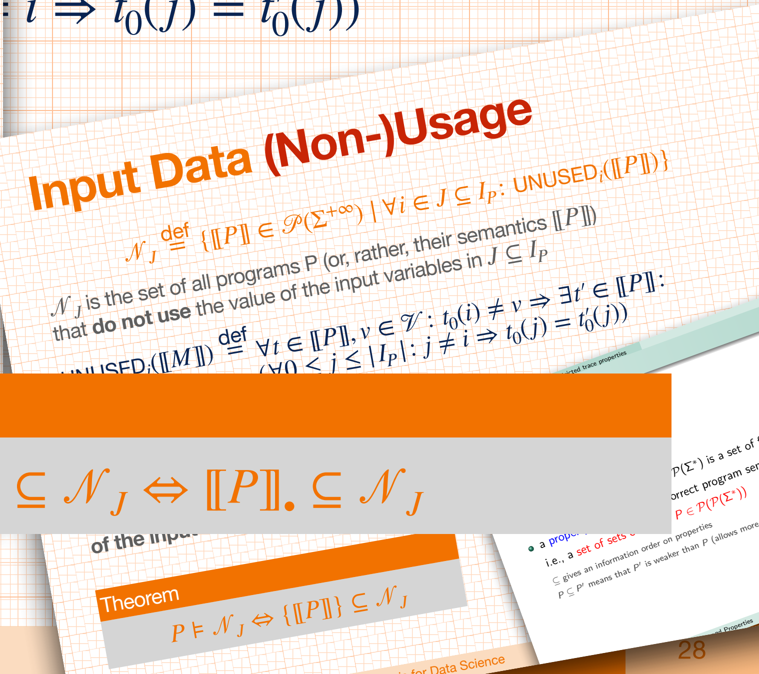

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$(\forall 0 < j \leq |I_P| : j \neq i \Rightarrow t_0(j) = t'_0(j))$

of the inpu

Theorem

$P \models \mathcal{N}_J \Leftrightarrow \{[\![P]\!]\} \subseteq \mathcal{N}_J$

for Data Science

icted trace properties

$\mathcal{P}(\Sigma^*)$ is a set of

orrect program se

$P \in \mathcal{P}(\mathcal{P}(\Sigma^*))$

- a prope
  i.e., a set of sets
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d Properties

Theorem

$$P \models \mathcal{N}_J \Leftrightarrow \{[\![P]\!]\} \subseteq \mathcal{N}_J \Leftrightarrow \alpha_\bullet(\{[\![P]\!]\}) \subseteq \mathcal{N}_J \Leftrightarrow [\![P]\!]_\bullet \subseteq \mathcal{N}_J$$

(Another) Hierarchy of Semantics

$$\begin{tikzcd}
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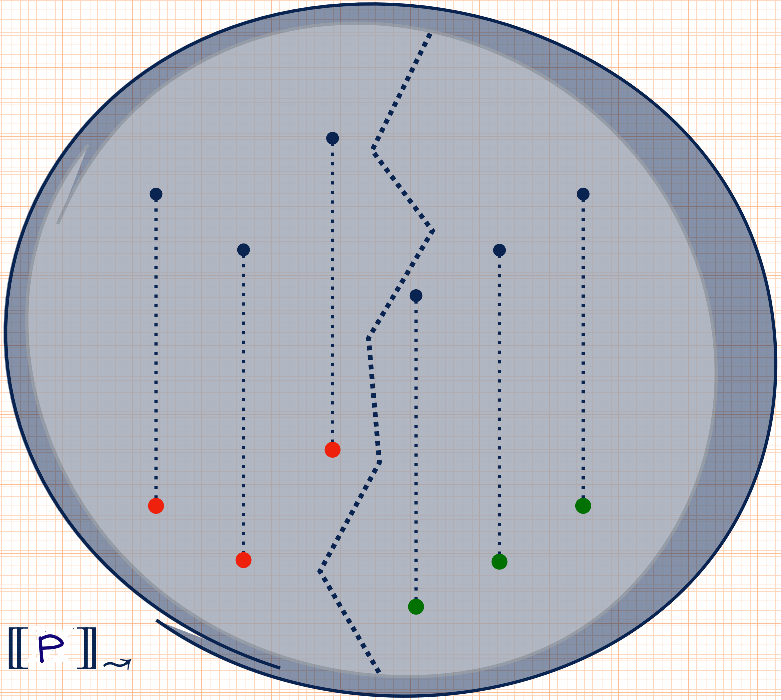
\end{tikzcd}$$

Dependency Semantics



💡 to reason about input data (non-)usage **we do not need to consider all intermediate computations** between the initial and final states of a trace (if any)

Dependency Semantics

$\Sigma_\perp \overset{\text{def}}{=} \Sigma \cup \{\perp\}$

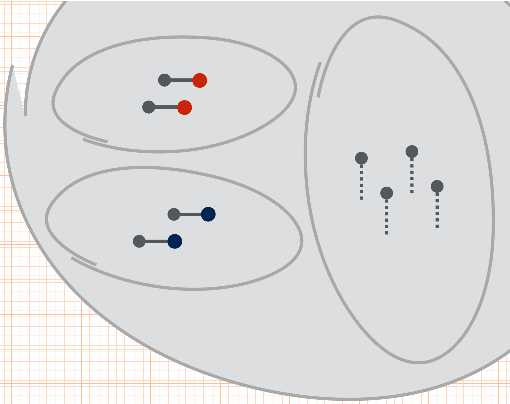
$$\begin{tikzcd}
\langle \mathscr{P}(\mathscr{P}(\Sigma^{+\infty})), \subseteq_{\bullet} \rangle \arrow[r, bend right, "\alpha_{\rightsquigarrow}"'] & \langle \mathscr{P}(\mathscr{P}(\Sigma \times \Sigma_\perp)), \subseteq_{\rightsquigarrow} \rangle \arrow[l, bend right, "\gamma_{\rightsquigarrow}"']
\end{tikzcd}$$

final state, if any,
or $\perp$ if
the trace is infinite

$$\alpha_{\rightsquigarrow}(S) \overset{\text{def}}{=} \{\{\langle t_0, t_\omega \rangle \in \Sigma \times \Sigma_\perp \mid t \in T\} \mid T \in S\}$$

$$\gamma_{\rightsquigarrow}(S) \overset{\text{def}}{=} \{T \in \mathscr{P}(\Sigma^{+\infty}) \mid \{\langle t_0, t_\omega \rangle \in \Sigma \times \Sigma_\perp \mid t \in T\} \in S\}$$

Dependency Semantics



$[\![P]\!]_{\rightsquigarrow}$

```
passing = True
if not english:
        english = False
if not math:
        passing = False or bonus
if not math:
        passing = False or bonus
```

english → _
math → T
science → _
bonus → _
passing → ?

english → _
math → T
science → _
bonus → _
passing → T

english → _
math → F
science → _
bonus → T
passing → ?

english → _
math → F
science → _
bonus → T
passing → T

english → _
math → F
science → _
bonus → F
passing → ?

english → _
math → F
science → _
bonus → F
passing → F

Dependency Semantics

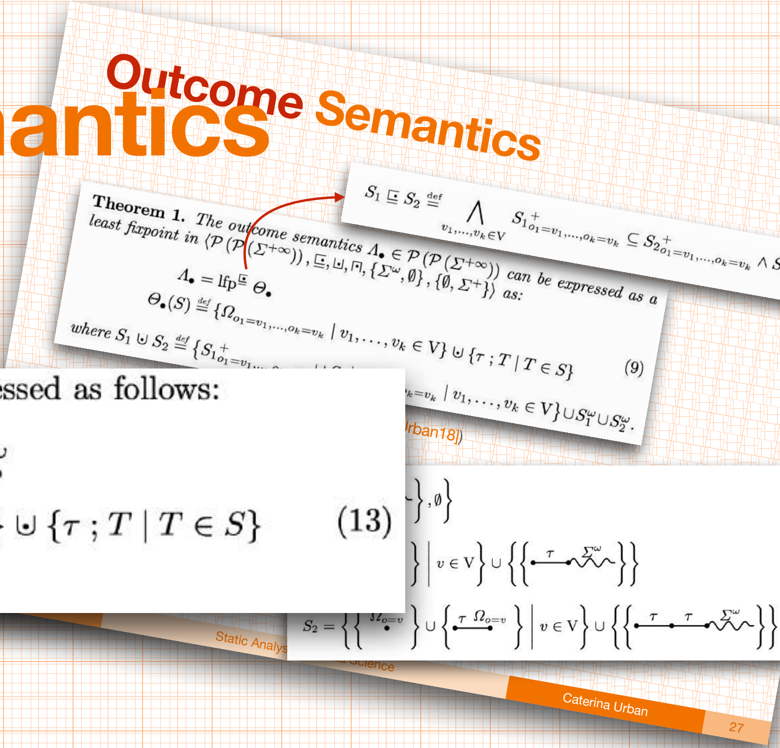
$$\begin{tikzcd}
\langle \mathscr{P}(\mathscr{P}(\Sigma^{+\infty})), \subseteq_{\bullet} \rangle \arrow[r, bend right, "\alpha_{\rightsquigarrow}"'] & \langle \mathscr{P}(\mathscr{P}(\Sigma \times \Sigma_{\perp})), \subseteq_{\rightsquigarrow} \rangle \arrow[l, bend right, "\gamma_{\rightsquigarrow}"']
\end{tikzcd}$$

$$\alpha_{\rightsquigarrow}(S) \overset{\text{def}}{=} \{\{\langle t_0, t_\omega \rangle \in \Sigma \times \Sigma_\perp \mid t \in T\} \mid T \in S\}$$

$$\gamma_{\rightsquigarrow}(S) \overset{\text{def}}{=} \{T \in \mathscr{P}(\Sigma^{+\infty}) \mid \{\langle t_0, t_\omega \rangle \in \Sigma \times \Sigma_\perp \mid t \in T\} \in S\}$$

$$[\![P]\!]_{\rightsquigarrow} \overset{\text{def}}{=} \alpha_{\rightsquigarrow}([\![P]\!]_{\bullet}) = \{\{\langle t_0, t_\omega \rangle \in \Sigma \times \Sigma \mid t \in [\![P]\!] \cap O\} \mid O \in \mathbb{O}\}$$

Dependency Semantics



Outcome Semantics

Theorem 1. *The outcome semantics* $\Lambda_\bullet \in \mathcal{P}(\mathcal{P}(\Sigma^{+\infty}))$ *can be expressed as a least fixpoint in* $\langle \mathcal{P}(\mathcal{P}(\Sigma^{+\infty})), \dot\sqsubseteq, \dot\sqcup, \dot\sqcap, \{\Sigma^\omega, \emptyset\}, \{\emptyset, \Sigma^+\}\rangle$ *as:*

$$\Lambda_\bullet = \text{lfp}^{\dot\sqsubseteq}\, \Theta_\bullet$$

$$\Theta_\bullet(S) \overset{\text{def}}{=} \{\Omega_{o_1=v_1,\dots,o_k=v_k} \mid v_1,\dots,v_k \in \text{V}\} \dot\cup \{\tau\,;T \mid T \in S\} \quad (9)$$

where $S_1 \dot\cup S_2 \overset{\text{def}}{=} \{S_{1\,o_1=v_1,\dots}^{+}$

$=v_k \mid v_1,\dots,v_k \in \text{V}\} \cup S_1^\omega \cup S_2^\omega.$

$$S_1 \dot\sqsubseteq S_2 \overset{\text{def}}{=} \bigwedge_{v_1,\dots,v_k \in \text{V}} S_{1\,o_1=v_1,\dots,o_k=v_k}^{+} \subseteq S_{2\,o_1=v_1,\dots,o_k=v_k}^{+} \wedge S$$

rban18])

Static Analys Science

Caterina Urban

27

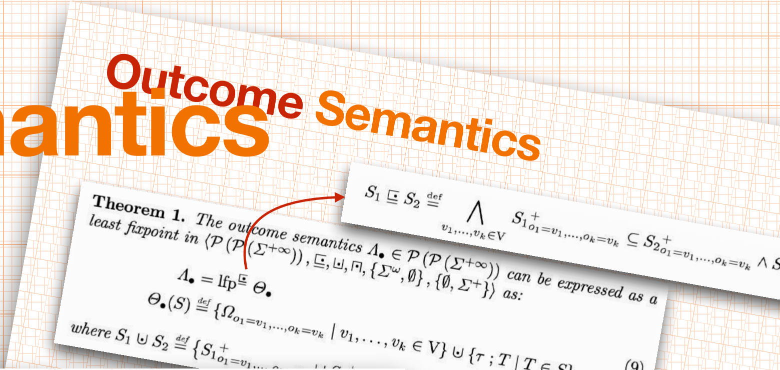
the outcome semantics $\Lambda_\bullet$ can be equivalently expressed as follows:

$$\Lambda_\bullet = \Lambda_\bullet^+ \cup \Lambda_\bullet^\omega = \text{lfp}_{\emptyset}^{\dot\sqsubseteq}\, \Theta_\bullet^+ \cup \text{lfp}_{\{\Sigma^\omega\}}^{\dot\sqsubseteq}\, \Theta_\bullet^\omega$$

$$\Theta_\bullet^+(S) \overset{\text{def}}{=} \{\Omega_{o_1=v_1,\dots,o_k=v_k} \mid v_1,\dots,v_k \in \text{V}\} \dot\cup \{\tau\,;T \mid T \in S\} \quad (13)$$

$$\Theta_\bullet^\omega(S) \overset{\text{def}}{=} \{\tau\,;T \mid T \in S\}$$

Dependency Semantics



Outcome Semantics

$$S_1 \sqsubseteq S_2 \stackrel{\text{def}}{=} \bigwedge_{v_1,\dots,v_k \in \mathrm{V}} S_{1\,o_1=v_1,\dots,o_k=v_k}^{+} \subseteq S_{2\,o_1=v_1,\dots,o_k=v_k}^{+} \wedge S$$

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where $S_1 \sqcup S_2 \stackrel{\text{def}}{=} \{S_{1\,o_1=v_1}^{+}$

$\mid v_1,\dots,v_k \in \mathrm{V}\} \cup S_1^\omega \cup S_2^\omega.$

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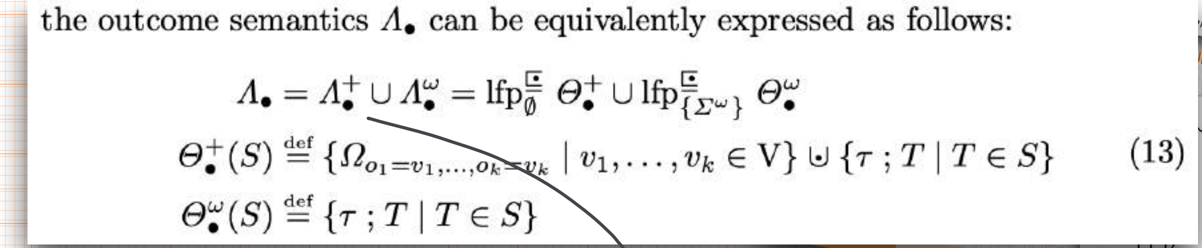

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$$\Lambda_\bullet = \Lambda_\bullet^{+} \cup \Lambda_\bullet^{\omega} = \mathrm{lfp}_{\emptyset}^{\sqsubseteq}\, \Theta_\bullet^{+} \cup \mathrm{lfp}_{\{\Sigma^\omega\}}^{\sqsubseteq}\, \Theta_\bullet^{\omega}$$

$$\Theta_\bullet^{+}(S) \stackrel{\text{def}}{=} \{\Omega_{o_1=v_1,\dots,o_k=v_k} \mid v_1,\dots,v_k \in \mathrm{V}\} \sqcup \{\tau \,;\, T \mid T \in S\} \quad (13)$$

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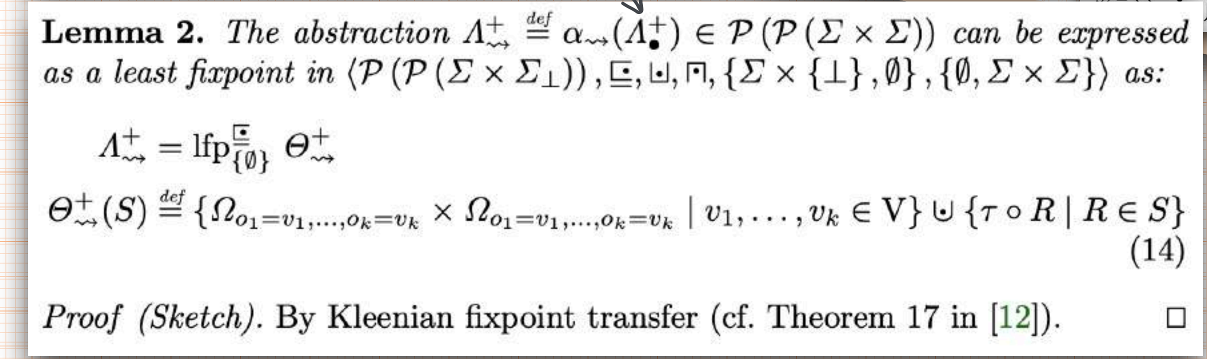


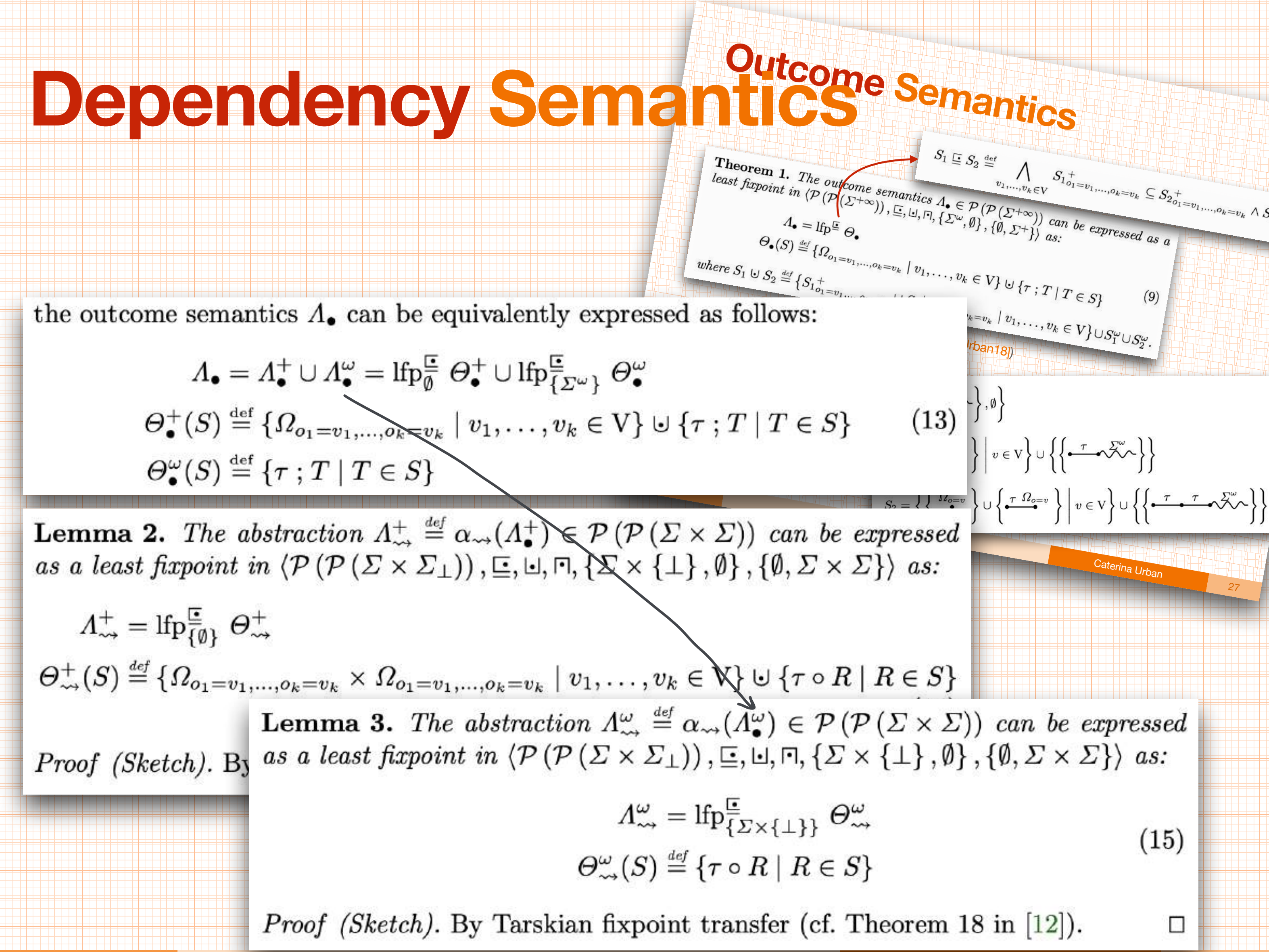
Lemma 2. *The abstraction $\Lambda_{\rightsquigarrow}^{+} \stackrel{\text{def}}{=} \alpha_{\rightsquigarrow}(\Lambda_\bullet^{+}) \in \mathcal{P}(\mathcal{P}(\Sigma \times \Sigma))$ can be expressed as a least fixpoint in $\langle \mathcal{P}(\mathcal{P}(\Sigma \times \Sigma_\bot)), \sqsubseteq, \sqcup, \sqcap, \{\Sigma \times \{\bot\}, \emptyset\}, \{\emptyset, \Sigma \times \Sigma\}\rangle$ as:*

$$\Lambda_{\rightsquigarrow}^{+} = \mathrm{lfp}_{\{\emptyset\}}^{\sqsubseteq}\, \Theta_{\rightsquigarrow}^{+}$$

$$\Theta_{\rightsquigarrow}^{+}(S) \stackrel{\text{def}}{=} \{\Omega_{o_1=v_1,\dots,o_k=v_k} \times \Omega_{o_1=v_1,\dots,o_k=v_k} \mid v_1,\dots,v_k \in \mathrm{V}\} \sqcup \{\tau \circ R \mid R \in S\} \quad (14)$$

Proof (Sketch). By Kleenian fixpoint transfer (cf. Theorem 17 in [12]). ☐

Dependency Semantics



Outcome Semantics

$$S_1 \mathrel{\dot\sqsubseteq} S_2 \overset{\text{def}}{=} \bigwedge_{v_1,\dots,v_k \in \mathrm{V}} {S_1}^+_{o_1=v_1,\dots,o_k=v_k} \subseteq {S_2}^+_{o_1=v_1,\dots,o_k=v_k} \wedge S$$

Theorem 1. *The outcome semantics $\Lambda_\bullet \in \mathcal{P}(\mathcal{P}(\Sigma^{+\infty}))$ can be expressed as a least fixpoint in $\langle \mathcal{P}(\mathcal{P}(\Sigma^{+\infty})), \dot\sqsubseteq, \dot\sqcup, \dot\sqcap, \{\Sigma^\omega, \emptyset\}, \{\emptyset, \Sigma^+\}\rangle$ as:*

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$$\Theta_\bullet(S) \overset{\text{def}}{=} \{\Omega_{o_1=v_1,\dots,o_k=v_k} \mid v_1,\dots,v_k \in \mathrm{V}\} \mathbin{\dot\cup} \{\tau ; T \mid T \in S\} \tag{9}$$

where $S_1 \mathbin{\dot\cup} S_2 \overset{\text{def}}{=} \{{S_1}^+_{o_1=v_1,\dots}$ … $_{k=v_k} \mid v_1,\dots,v_k \in \mathrm{V}\} \cup S_1^\omega \cup S_2^\omega$.

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the outcome semantics $\Lambda_\bullet$ can be equivalently expressed as follows:

$$\Lambda_\bullet = \Lambda_\bullet^+ \cup \Lambda_\bullet^\omega = \mathrm{lfp}^{\dot\sqsubseteq}_{\emptyset}\, \Theta_\bullet^+ \cup \mathrm{lfp}^{\dot\sqsubseteq}_{\{\Sigma^\omega\}}\, \Theta_\bullet^\omega$$

$$\Theta_\bullet^+(S) \overset{\text{def}}{=} \{\Omega_{o_1=v_1,\dots,o_k=v_k} \mid v_1,\dots,v_k \in \mathrm{V}\} \mathbin{\dot\cup} \{\tau ; T \mid T \in S\} \tag{13}$$

$$\Theta_\bullet^\omega(S) \overset{\text{def}}{=} \{\tau ; T \mid T \in S\}$$

Lemma 2. *The abstraction $\Lambda_\rightsquigarrow^+ \overset{\text{def}}{=} \alpha_\rightsquigarrow(\Lambda_\bullet^+) \in \mathcal{P}(\mathcal{P}(\Sigma \times \Sigma))$ can be expressed as a least fixpoint in $\langle \mathcal{P}(\mathcal{P}(\Sigma \times \Sigma_\perp)), \dot\sqsubseteq, \dot\sqcup, \dot\sqcap, \{\Sigma \times \{\perp\}, \emptyset\}, \{\emptyset, \Sigma \times \Sigma\}\rangle$ as:*

$$\Lambda_\rightsquigarrow^+ = \mathrm{lfp}^{\dot\sqsubseteq}_{\{\emptyset\}}\, \Theta_\rightsquigarrow^+$$

$$\Theta_\rightsquigarrow^+(S) \overset{\text{def}}{=} \{\Omega_{o_1=v_1,\dots,o_k=v_k} \times \Omega_{o_1=v_1,\dots,o_k=v_k} \mid v_1,\dots,v_k \in \mathrm{V}\} \mathbin{\dot\cup} \{\tau \circ R \mid R \in S\}$$

Proof (Sketch). By

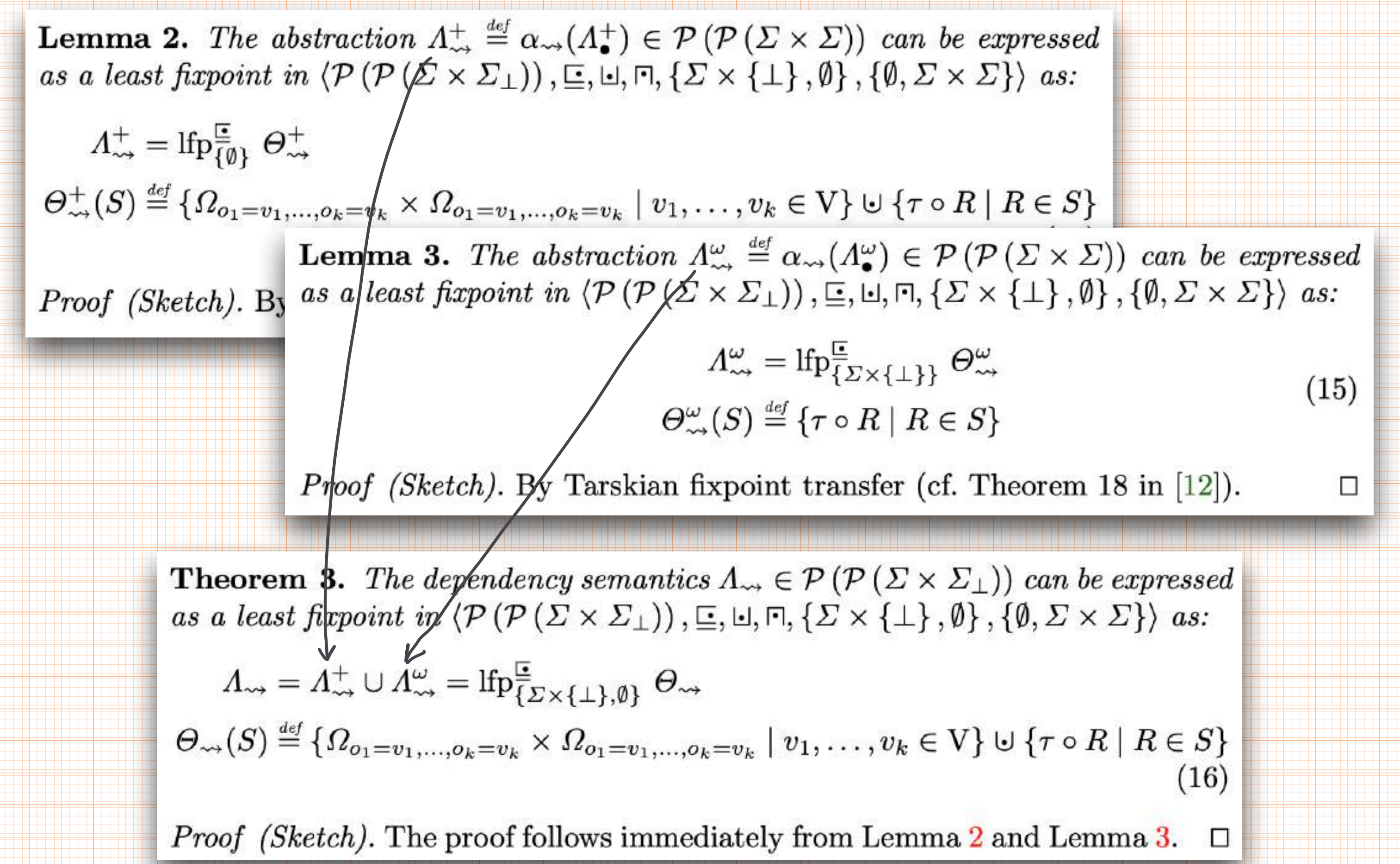
Lemma 3. *The abstraction $\Lambda_\rightsquigarrow^\omega \overset{\text{def}}{=} \alpha_\rightsquigarrow(\Lambda_\bullet^\omega) \in \mathcal{P}(\mathcal{P}(\Sigma \times \Sigma))$ can be expressed as a least fixpoint in $\langle \mathcal{P}(\mathcal{P}(\Sigma \times \Sigma_\perp)), \dot\sqsubseteq, \dot\sqcup, \dot\sqcap, \{\Sigma \times \{\perp\}, \emptyset\}, \{\emptyset, \Sigma \times \Sigma\}\rangle$ as:*

$$\Lambda_\rightsquigarrow^\omega = \mathrm{lfp}^{\dot\sqsubseteq}_{\{\Sigma\times\{\perp\}\}}\, \Theta_\rightsquigarrow^\omega \tag{15}$$

$$\Theta_\rightsquigarrow^\omega(S) \overset{\text{def}}{=} \{\tau \circ R \mid R \in S\}$$

Proof (Sketch). By Tarskian fixpoint transfer (cf. Theorem 18 in [12]). ☐

Dependency Semantics



Lemma 2. *The abstraction* $\Lambda^{+}_{\leadsto} \overset{def}{=} \alpha_{\leadsto}(\Lambda^{+}_{\bullet}) \in \mathcal{P}(\mathcal{P}(\Sigma \times \Sigma))$ *can be expressed as a least fixpoint in* $\langle \mathcal{P}(\mathcal{P}(\Sigma \times \Sigma_{\bot})), \dot{\sqsubseteq}, \dot{\sqcup}, \dot{\sqcap}, \{\Sigma \times \{\bot\}, \emptyset\}, \{\emptyset, \Sigma \times \Sigma\}\rangle$ *as:*

$$\Lambda^{+}_{\leadsto} = \mathrm{lfp}^{\dot{\sqsubseteq}}_{\{\emptyset\}}\, \Theta^{+}_{\leadsto}$$

$$\Theta^{+}_{\leadsto}(S) \overset{def}{=} \{\Omega_{o_1=v_1,\dots,o_k=v_k} \times \Omega_{o_1=v_1,\dots,o_k=v_k} \mid v_1, \dots, v_k \in \mathrm{V}\} \dot{\sqcup} \{\tau \circ R \mid R \in S\}$$

Proof (Sketch). By

Lemma 3. *The abstraction* $\Lambda^{\omega}_{\leadsto} \overset{def}{=} \alpha_{\leadsto}(\Lambda^{\omega}_{\bullet}) \in \mathcal{P}(\mathcal{P}(\Sigma \times \Sigma))$ *can be expressed as a least fixpoint in* $\langle \mathcal{P}(\mathcal{P}(\Sigma \times \Sigma_{\bot})), \dot{\sqsubseteq}, \dot{\sqcup}, \dot{\sqcap}, \{\Sigma \times \{\bot\}, \emptyset\}, \{\emptyset, \Sigma \times \Sigma\}\rangle$ *as:*

$$\begin{aligned} \Lambda^{\omega}_{\leadsto} &= \mathrm{lfp}^{\dot{\sqsubseteq}}_{\{\Sigma \times \{\bot\}\}}\, \Theta^{\omega}_{\leadsto} \\ \Theta^{\omega}_{\leadsto}(S) &\overset{def}{=} \{\tau \circ R \mid R \in S\} \end{aligned} \tag{15}$$

Proof (Sketch). By Tarskian fixpoint transfer (cf. Theorem 18 in [12]). ☐

Theorem 3. *The dependency semantics* $\Lambda_{\leadsto} \in \mathcal{P}(\mathcal{P}(\Sigma \times \Sigma_{\bot}))$ *can be expressed as a least fixpoint in* $\langle \mathcal{P}(\mathcal{P}(\Sigma \times \Sigma_{\bot})), \dot{\sqsubseteq}, \dot{\sqcup}, \dot{\sqcap}, \{\Sigma \times \{\bot\}, \emptyset\}, \{\emptyset, \Sigma \times \Sigma\}\rangle$ *as:*

$$\Lambda_{\leadsto} = \Lambda^{+}_{\leadsto} \cup \Lambda^{\omega}_{\leadsto} = \mathrm{lfp}^{\dot{\sqsubseteq}}_{\{\Sigma \times \{\bot\}, \emptyset\}}\, \Theta_{\leadsto}$$

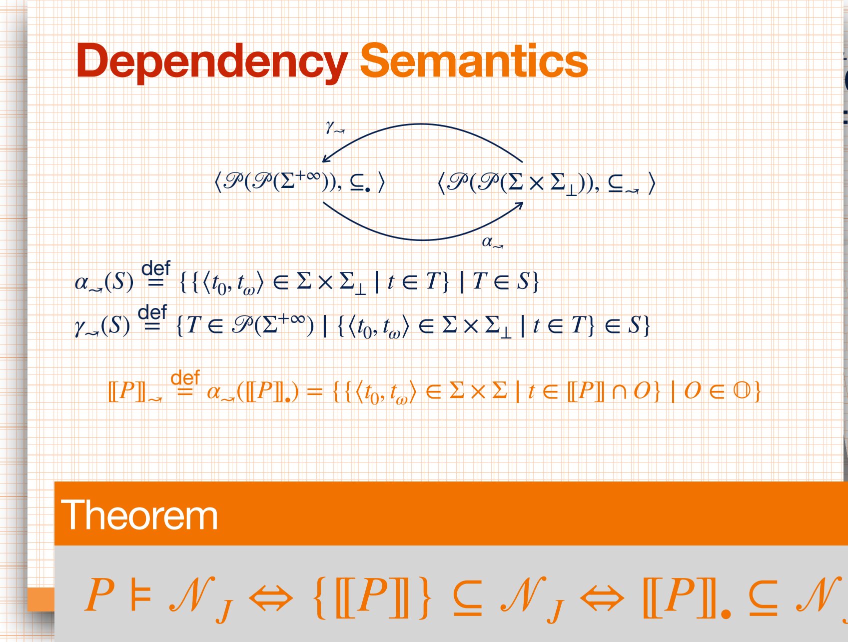
$$\Theta_{\leadsto}(S) \overset{def}{=} \{\Omega_{o_1=v_1,\dots,o_k=v_k} \times \Omega_{o_1=v_1,\dots,o_k=v_k} \mid v_1, \dots, v_k \in \mathrm{V}\} \dot{\sqcup} \{\tau \circ R \mid R \in S\} \tag{16}$$

Proof (Sketch). The proof follows immediately from Lemma 2 and Lemma 3. ☐

Input Data (Non-)Usage

$$\mathcal{N}_J \overset{\text{def}}{=} \{[\![P]\!] \in \mathscr{P}(\Sigma^{+\infty}) \mid \forall i \in J \subseteq I_P \colon \text{UNUSED}_i([\![P]\!])\}$$

$\mathcal{N}_J$ is the set of all programs P (or, rather, their semantics $[\![P]\!]$) that **do not use** the value of the input variables in $J \subseteq I_P$



Dependency Semantics

$$\begin{tikzcd}\langle \mathscr{P}(\mathscr{P}(\Sigma^{+\infty})), \subseteq_\bullet \rangle \arrow[r, bend right, "\alpha_\rightsquigarrow"'] & \langle \mathscr{P}(\mathscr{P}(\Sigma \times \Sigma_\bot)), \subseteq_\rightsquigarrow \rangle \arrow[l, bend right, "\gamma_\rightsquigarrow"']\end{tikzcd}$$

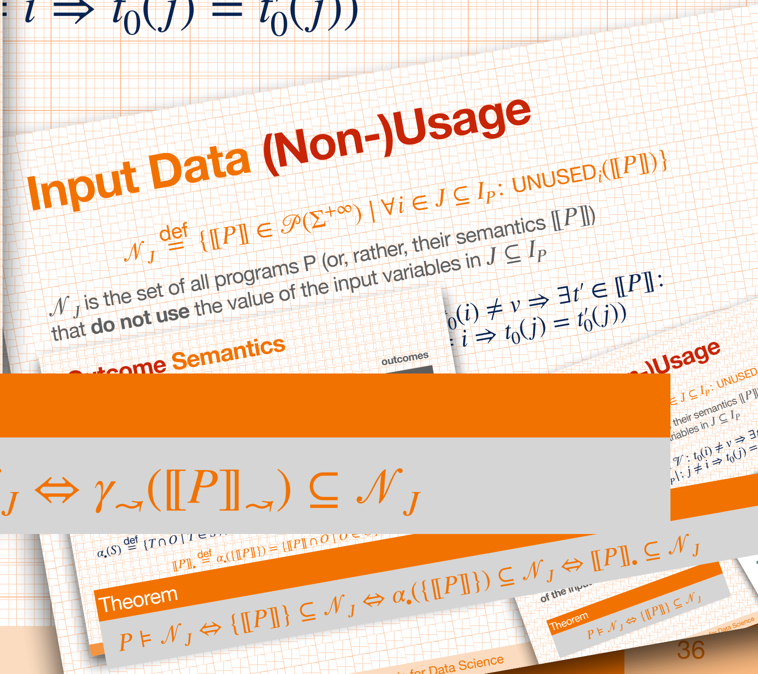
$$\alpha_\rightsquigarrow(S) \overset{\text{def}}{=} \{\{\langle t_0, t_\omega \rangle \in \Sigma \times \Sigma_\bot \mid t \in T\} \mid T \in S\}$$

$$\gamma_\rightsquigarrow(S) \overset{\text{def}}{=} \{T \in \mathscr{P}(\Sigma^{+\infty}) \mid \{\langle t_0, t_\omega \rangle \in \Sigma \times \Sigma_\bot \mid t \in T\} \in S\}$$

$$[\![P]\!]_\rightsquigarrow \overset{\text{def}}{=} \alpha_\rightsquigarrow([\![P]\!]_\bullet) = \{\{\langle t_0, t_\omega \rangle \in \Sigma \times \Sigma \mid t \in [\![P]\!] \cap O\} \mid O \in \mathbb{O}\}$$

$${}_0(i) \neq v \Rightarrow \exists t' \in [\![P]\!]:$$

$$i \Rightarrow t_0(j) = t'_0(j))$$



Input Data (Non-)Usage

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$\mathcal{N}_J$ is the set of all programs P (or, rather, their semantics $[\![P]\!]$) that **do not use** the value of the input variables in $J \subseteq I_P$

tcome Semantics

Theorem

$P \models \mathcal{N}_J \Leftrightarrow \{[\![P]\!]\} \subseteq \mathcal{N}_J \Leftrightarrow \alpha_\bullet(\{[\![P]\!]\}) \subseteq \mathcal{N}_J \Leftrightarrow [\![P]\!]_\bullet \subseteq \mathcal{N}_J$

Theorem

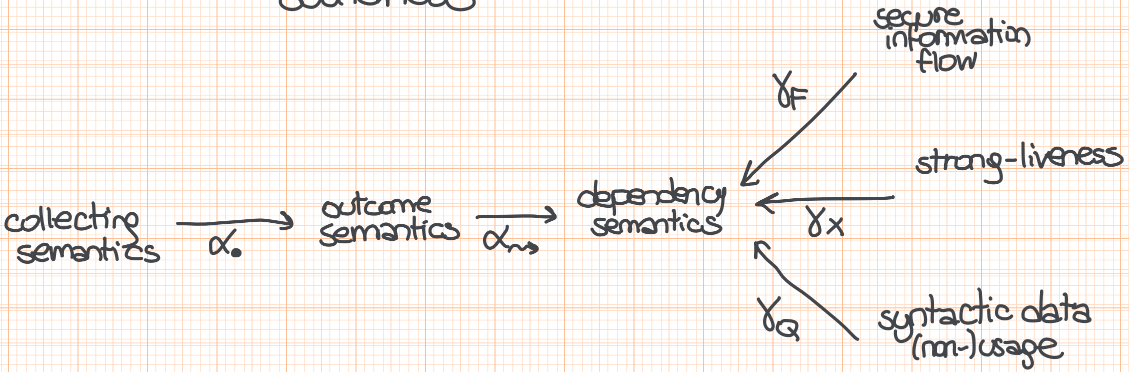
$$P \models \mathcal{N}_J \Leftrightarrow \{[\![P]\!]\} \subseteq \mathcal{N}_J \Leftrightarrow [\![P]\!]_\bullet \subseteq \mathcal{N}_J \Leftrightarrow \gamma_\rightsquigarrow([\![P]\!]_\rightsquigarrow) \subseteq \mathcal{N}_J$$

Input Data (Non-)Usage Abstractions

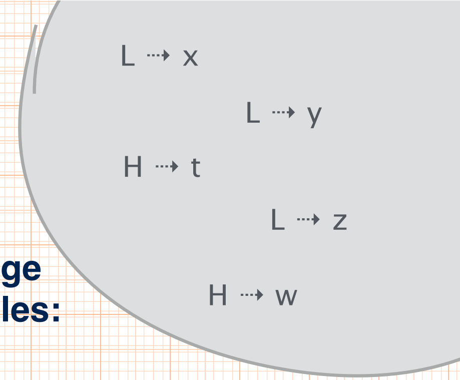
Over-Approximation of the Used Input Data

⇒ **Under-Approximation** of the Unused Input Data

$$P \models \mathcal{N}_{J^\natural \subseteq J} \underset{\substack{\uparrow \\ \text{soundness}}}{\Leftarrow} \gamma_{\rightsquigarrow}(\gamma_A(\llbracket P \rrbracket_A)) \subseteq \mathcal{N}_{J^\natural \subseteq J}$$



Secure Information Flow



possibilistic non-interference coincides with input data (non-)usage when the set J of unused input variables contains *all* input variables:

- **input variables are high-security variables**
- **output variables are low-security variables**

$[\![P]\!]_{\mathrm{F}}$

$$e ::= v \mid x \mid \texttt{not}\ e \mid e\ \texttt{and}\ e \mid e\ \texttt{or}\ e \qquad \text{(expressions)}$$
$$s ::= \texttt{skip} \mid x = e \mid \texttt{if}\ e\texttt{:}\ s\ \texttt{else:}\ s \mid \texttt{while}\ e\texttt{:}\ s \mid s\ s \qquad \text{(statements)}$$

explicit usage flows

implicit usage flows

$$\Theta_{\mathrm{F}}[\![\texttt{skip}]\!](S) \overset{\text{def}}{=} S$$

$$\Theta_{\mathrm{F}}[\![x = e]\!](S) \overset{\text{def}}{=} \{L \rightsquigarrow y \in S \mid y \neq x\} \cup \{L \rightsquigarrow x \mid \mathcal{V}_{\mathrm{F}}[\![e]\!]S\}$$

$$\Theta_{\mathrm{F}}[\![\texttt{if}\ e\texttt{:}\ s_1\ \texttt{else:}\ s_2]\!](S) \overset{\text{def}}{=} \begin{cases} \Theta_{\mathrm{F}}[\![s_1]\!](S) \sqcup_{\mathrm{F}} \Theta_{\mathrm{F}}[\![s_2]\!](S) & \text{if } \mathcal{V}_{\mathrm{F}}[\![e]\!]S \\ \{L \rightsquigarrow x \in S \mid x \notin \mathrm{W}(s_1) \cup \mathrm{W}(s_2)\} & \text{otherwise} \end{cases}$$

$$\Theta_{\mathrm{F}}[\![\texttt{while}\ e\texttt{:}\ s]\!](S) \overset{\text{def}}{=} \mathrm{lfp}_{S}^{\sqsubseteq_{\mathrm{F}}}\ \Theta_{\mathrm{F}}[\![\texttt{if}\ e\texttt{:}\ s\ \texttt{else:}\ \texttt{skip}]\!]$$

$$\Theta_{\mathrm{F}}[\![s_1\ s_2]\!](S) \overset{\text{def}}{=} \Theta_{\mathrm{F}}[\![s_2]\!] \circ \Theta_{\mathrm{F}}[\![s_1]\!](S)$$

S guarantees a unique value fore independently of values of input variables

$\mathcal{V}_{\mathrm{F}}[\![x]\!]S \iff L \rightsquigarrow x \in S$

set of variables modified by s_i

$\mathcal{L} \overset{\text{def}}{=} \{L, H\}$: set of security levels

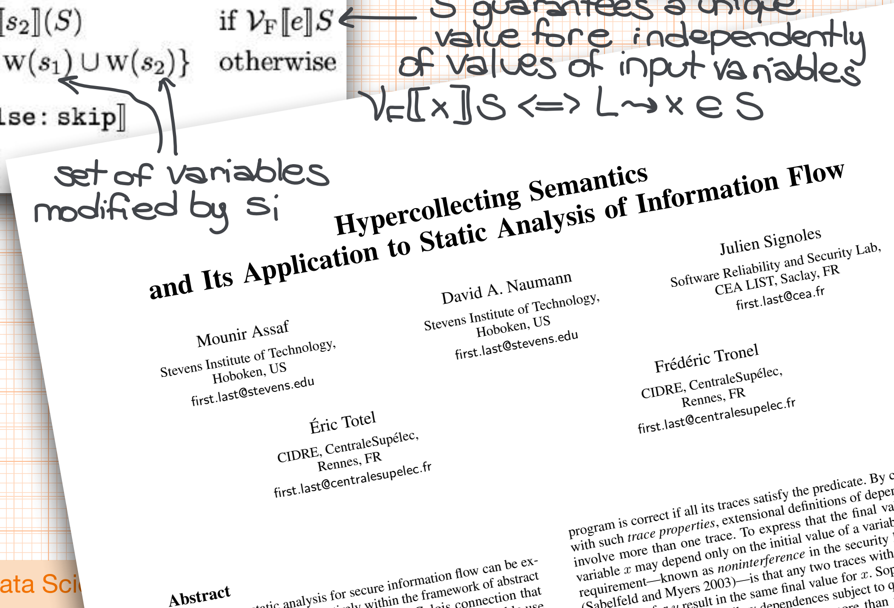
$L \rightsquigarrow x$: dependency constraint

$\mathrm{F} \overset{\text{def}}{=} \{L \rightsquigarrow x \mid x \in X\}$

$\langle \mathcal{P}(\mathrm{F}), \sqsubseteq_{\mathrm{F}}, \sqcup_{\mathrm{F}} \rangle$: abstract domain

$S_1 \sqsubseteq_{\mathrm{F}} S_2 \overset{\text{def}}{=} S_1 \supseteq S_2$

$S_1 \sqcup_{\mathrm{F}} S_2 \overset{\text{def}}{=} S_1 \cap S_2$



Hypercollecting Semantics and Its Application to Static Analysis of Information Flow

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Abstract

tatic analysis for secure information flow can be ex-
ly within the framework of abstract
alois connection that

program is correct if all its traces satisfy the predicate. By c
with such *trace properties*, extensional definitions of depe
involve more than one trace. To express that the final va
variable x may depend only on the initial value of a variab
requirement—known as *noninterference* in the security
(Sabelfeld and Myers 2003)—is that any two traces with
result in the same final value for x. Sop

Secure Information Flow

L ⇢ x

L ⇢ y

H ⇢ t

L ⇢ z

H ⇢ w

possibilistic non-interference coincides with input data (non-)usage when the set J of unused input variables contains *all* input variables:

- **input variables are high-security variables**
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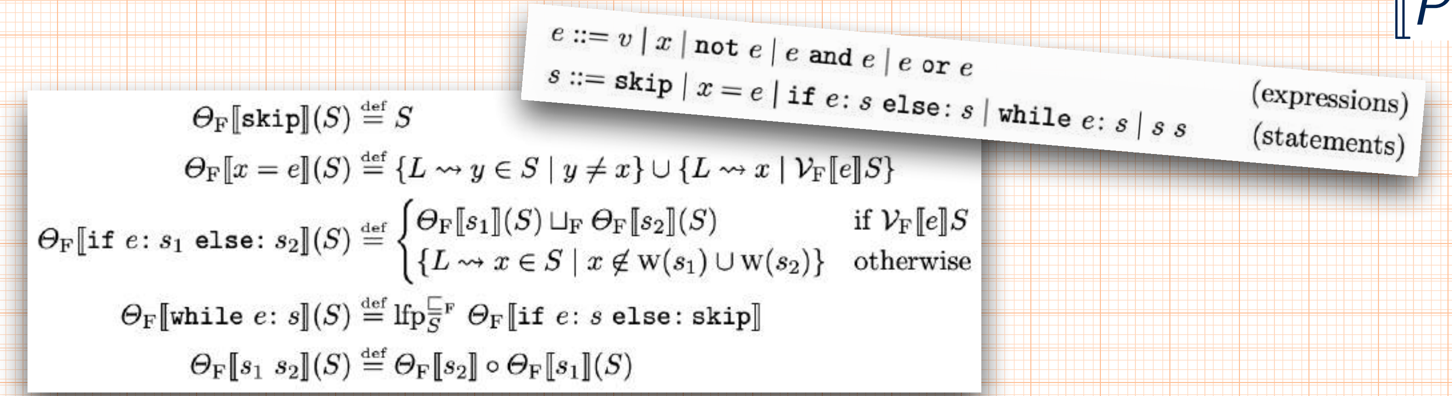
$[\![P]\!]_{\mathrm{F}}$



$$e ::= v \mid x \mid \texttt{not}\ e \mid e\ \texttt{and}\ e \mid e\ \texttt{or}\ e \qquad \text{(expressions)}$$

$$s ::= \texttt{skip} \mid x = e \mid \texttt{if}\ e\texttt{:}\ s\ \texttt{else:}\ s \mid \texttt{while}\ e\texttt{:}\ s \mid s\ s \qquad \text{(statements)}$$

$$\Theta_{\mathrm{F}}[\![\texttt{skip}]\!](S) \stackrel{\text{def}}{=} S$$

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$$\Theta_{\mathrm{F}}[\![\texttt{if}\ e\texttt{:}\ s_1\ \texttt{else:}\ s_2]\!](S) \stackrel{\text{def}}{=} \begin{cases} \Theta_{\mathrm{F}}[\![s_1]\!](S) \sqcup_{\mathrm{F}} \Theta_{\mathrm{F}}[\![s_2]\!](S) & \text{if } \mathcal{V}_{\mathrm{F}}[\![e]\!]S \\ \{L \rightsquigarrow x \in S \mid x \notin \mathrm{W}(s_1) \cup \mathrm{W}(s_2)\} & \text{otherwise} \end{cases}$$

$$\Theta_{\mathrm{F}}[\![\texttt{while}\ e\texttt{:}\ s]\!](S) \stackrel{\text{def}}{=} \mathrm{lfp}_S^{\sqsubseteq_{\mathrm{F}}}\ \Theta_{\mathrm{F}}[\![\texttt{if}\ e\texttt{:}\ s\ \texttt{else:}\ \texttt{skip}]\!]$$

$$\Theta_{\mathrm{F}}[\![s_1\ s_2]\!](S) \stackrel{\text{def}}{=} \Theta_{\mathrm{F}}[\![s_2]\!] \circ \Theta_{\mathrm{F}}[\![s_1]\!](S)$$

```
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    english = False
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    passing = False or bonus
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```
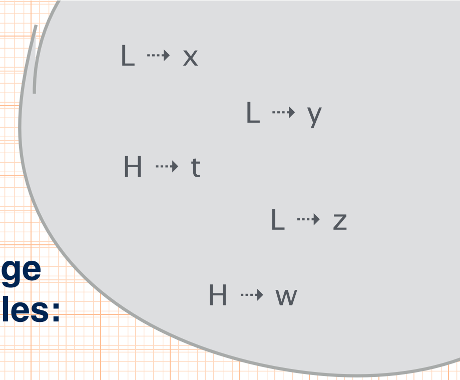

Secure Information Flow


L ⇢ x

L ⇢ y

H ⇢ t

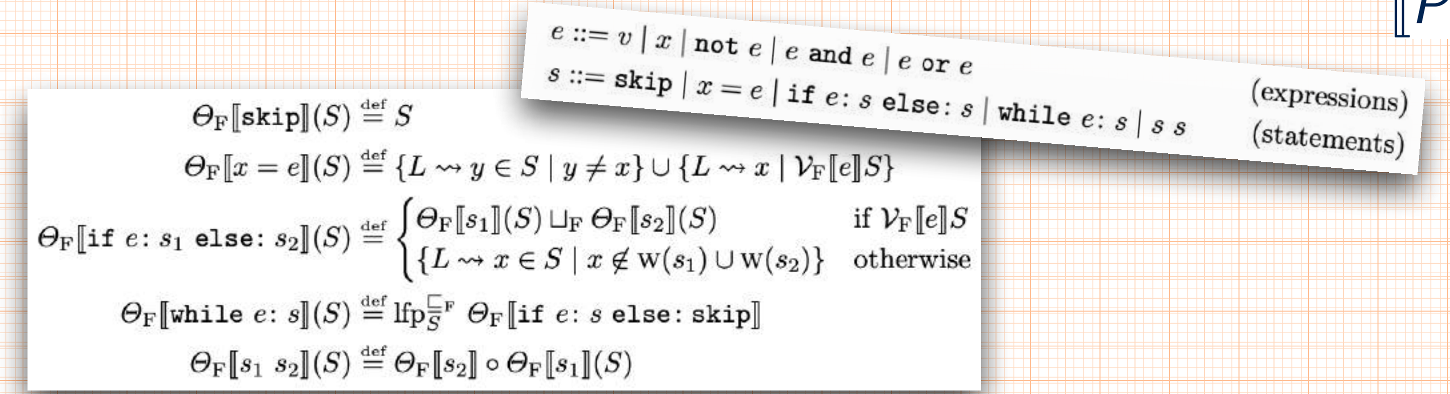
L ⇢ z

H ⇢ w

possibilistic non-interference coincides with input data (non-)usage when the set J of unused input variables contains *all* input variables:

- **input variables are high-security variables**
- **output variables are low-security variables**

$[\![P]\!]_{\mathrm{F}}$


$$e ::= v \mid x \mid \texttt{not}\ e \mid e\ \texttt{and}\ e \mid e\ \texttt{or}\ e \qquad \text{(expressions)}$$

$$s ::= \texttt{skip} \mid x = e \mid \texttt{if}\ e\texttt{:}\ s\ \texttt{else:}\ s \mid \texttt{while}\ e\texttt{:}\ s \mid s\ s \qquad \text{(statements)}$$

$$\Theta_{\mathrm{F}}[\![\texttt{skip}]\!](S) \stackrel{\text{def}}{=} S$$

$$\Theta_{\mathrm{F}}[\![x = e]\!](S) \stackrel{\text{def}}{=} \{L \rightsquigarrow y \in S \mid y \neq x\} \cup \{L \rightsquigarrow x \mid \mathcal{V}_{\mathrm{F}}[\![e]\!]S\}$$

$$\Theta_{\mathrm{F}}[\![\texttt{if}\ e\texttt{:}\ s_1\ \texttt{else:}\ s_2]\!](S) \stackrel{\text{def}}{=} \begin{cases} \Theta_{\mathrm{F}}[\![s_1]\!](S) \sqcup_{\mathrm{F}} \Theta_{\mathrm{F}}[\![s_2]\!](S) & \text{if } \mathcal{V}_{\mathrm{F}}[\![e]\!]S \\ \{L \rightsquigarrow x \in S \mid x \notin \mathrm{W}(s_1) \cup \mathrm{W}(s_2)\} & \text{otherwise} \end{cases}$$

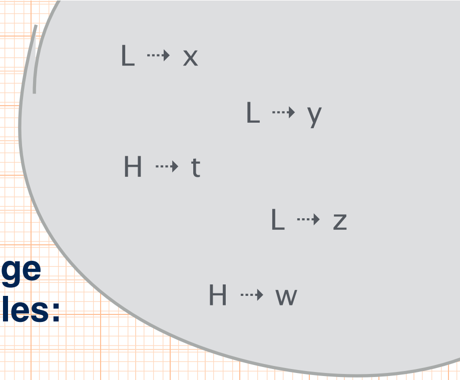
$$\Theta_{\mathrm{F}}[\![\texttt{while}\ e\texttt{:}\ s]\!](S) \stackrel{\text{def}}{=} \mathrm{lfp}_{S}^{\sqsubseteq_{\mathrm{F}}}\ \Theta_{\mathrm{F}}[\![\texttt{if}\ e\texttt{:}\ s\ \texttt{else:}\ \texttt{skip}]\!]$$

$$\Theta_{\mathrm{F}}[\![s_1\ \ s_2]\!](S) \stackrel{\text{def}}{=} \Theta_{\mathrm{F}}[\![s_2]\!] \circ \Theta_{\mathrm{F}}[\![s_1]\!](S)$$

```
passing = True
if not english:
    english = False
if not math:
    passing = False or bonus
if not math:
    passing = False or bonus
```

◀······ L ⇢ passing,(H ⇢ english, math, science, bonus)

Secure Information Flow



$L \dashrightarrow x$

$L \dashrightarrow y$

$H \dashrightarrow t$

$L \dashrightarrow z$

$H \dashrightarrow w$

possibilistic non-interference coincides with input data (non-)usage when the set J of unused input variables contains *all* input variables:

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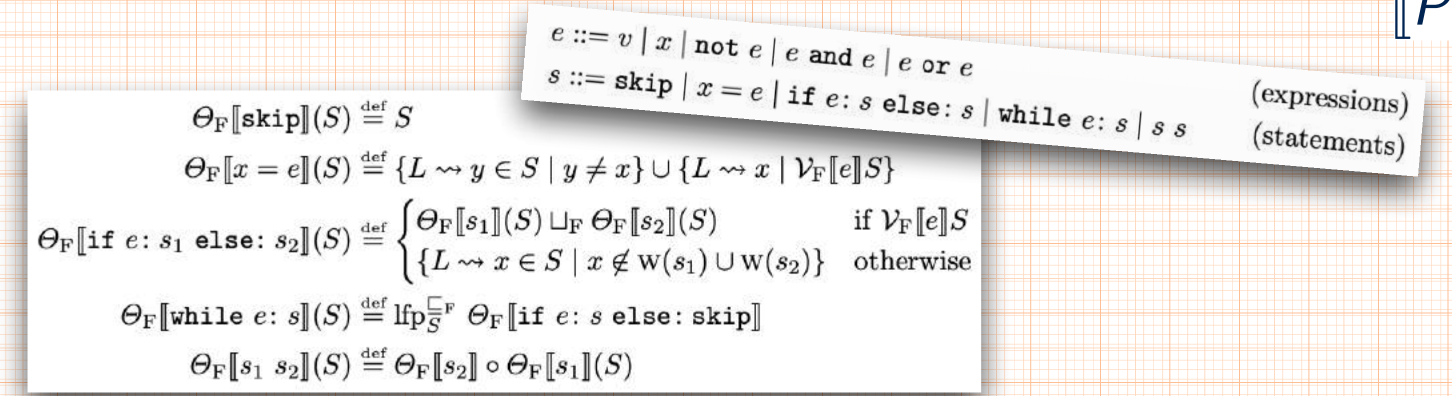
$$[\![P]\!]_{\mathrm{F}}$$



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$$\Theta_{\mathrm{F}}[\![s_1\ s_2]\!](S) \overset{\text{def}}{=} \Theta_{\mathrm{F}}[\![s_2]\!] \circ \Theta_{\mathrm{F}}[\![s_1]\!](S)$$

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passing = True
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```

$L \dashrightarrow \text{passing}\left(H \dashrightarrow \text{english, math, science, bonus}\right)$

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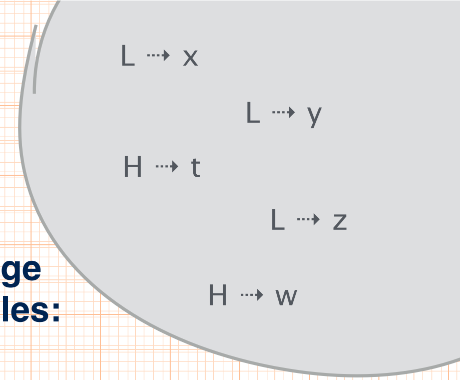
Secure Information Flow



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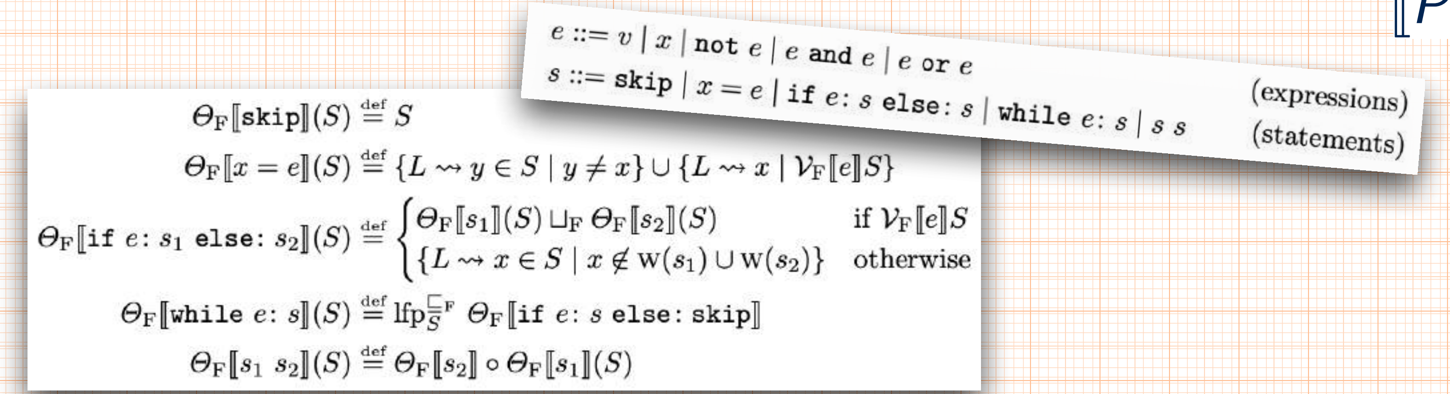
$[\![P]\!]_{\mathrm{F}}$



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L ⇢ passing,(H ⇢ english, math, science, bonus)

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Secure Information Flow

L ⇢ x

L ⇢ y

H ⇢ t

L ⇢ z

H ⇢ w

$[\![P]\!]_{\mathrm{F}}$

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$$
\begin{aligned}
e &::= v \mid x \mid \mathtt{not}\ e \mid e\ \mathtt{and}\ e \mid e\ \mathtt{or}\ e && \text{(expressions)}\\
s &::= \mathtt{skip} \mid x = e \mid \mathtt{if}\ e\colon s\ \mathtt{else}\colon s \mid \mathtt{while}\ e\colon s \mid s\ s && \text{(statements)}
\end{aligned}
$$

$$\Theta_{\mathrm{F}}[\![\mathtt{skip}]\!](S) \stackrel{\text{def}}{=} S$$

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L ⇢ passing,(H ⇢ english, math, science, bonus)

(H ⇢ english, math, science, bonus, passing)

depends on input variables

implicit usage flow

Secure Information Flow

L ⇢ x

L ⇢ y

H ⇢ t

L ⇢ z

H ⇢ w

possibilistic non-interference coincides with input data (non-)usage when the set J of unused input variables contains *all* input variables:

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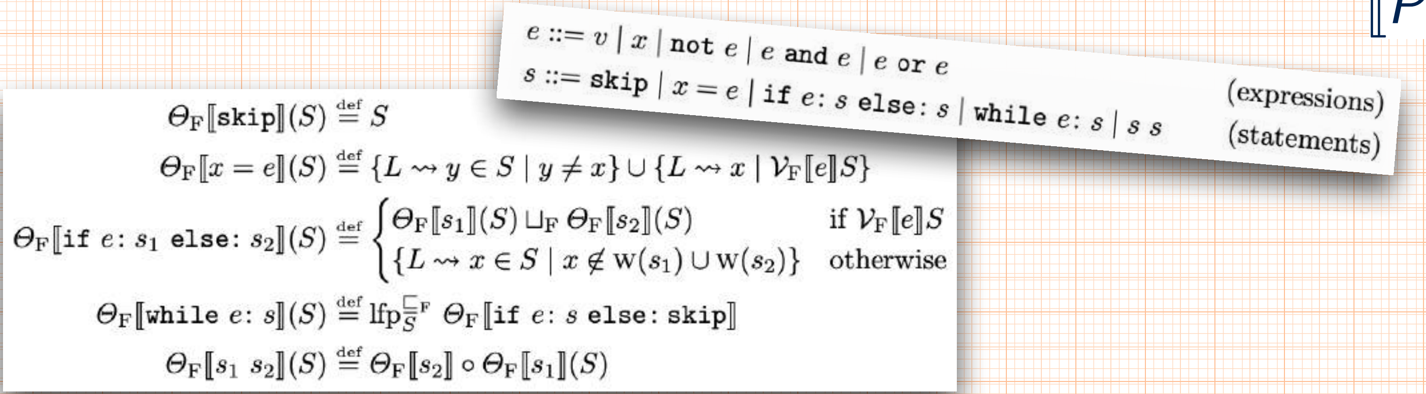
$[\![P]\!]_{\mathrm{F}}$



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\begin{aligned}
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\end{aligned}
$$

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L ⇢ passing,(H ⇢ english, math, science, bonus)

(H ⇢ english, math, science, bonus, passing)

(H ⇢ english, math, science, bonus, passing)

Secure Information Flow

L ⇢ x

L ⇢ y

H ⇢ t

L ⇢ z

H ⇢ w

$[\![P]\!]_{\mathrm{F}}$

possibilistic non-interference coincides with input data (non-)usage when the set J of unused input variables contains *all* input variables:

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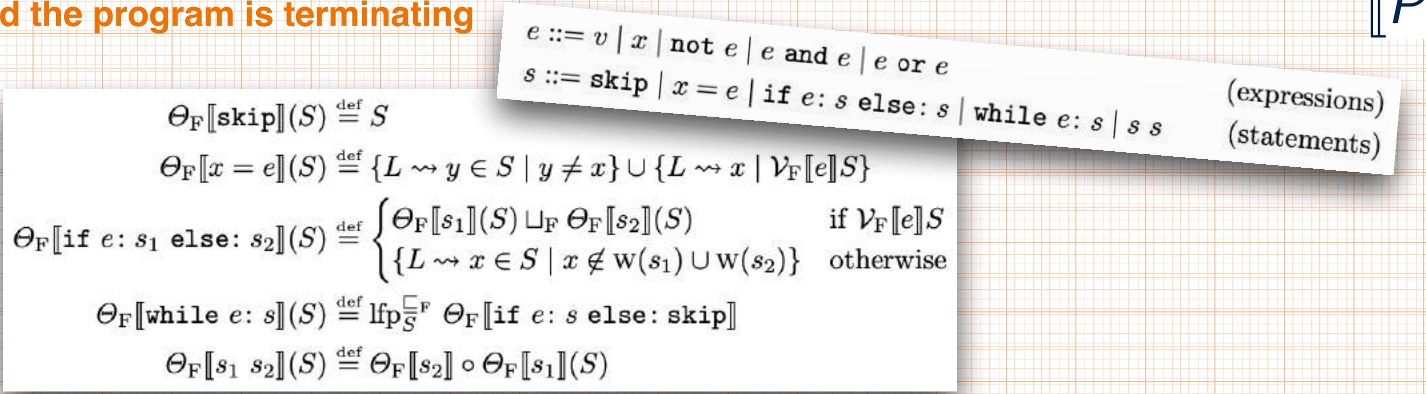
and the program is terminating



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```
passing = True
while not english:
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```

L ⇢ passing, (H ⇢ english, math, science, bonus)

L ⇢ passing, (H ⇢ english, math, science, bonus)

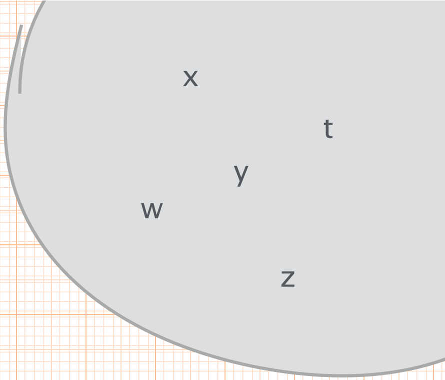
L ⇢ passing, (H ⇢ english, math, science, bonus)

Theorem

$$\gamma_F(S) \stackrel{\text{def}}{=} \{R \in \mathcal{P}(\Sigma \times \Sigma) \mid \alpha_F(R) \sqsubseteq_F S\}$$

$$P \models \mathcal{N}^*_{I_P} \Leftarrow \gamma_{\rightsquigarrow}(\gamma_F([\![P]\!]_F)) \subseteq \mathcal{N}^*_{I_P}$$

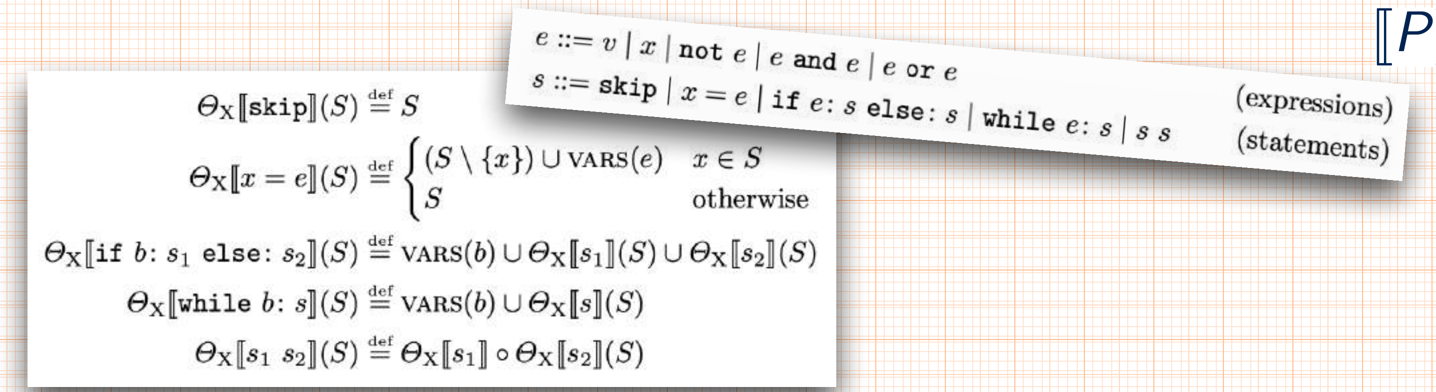
Strong-Liveness



$[\![P]\!]_{\mathrm{X}}$

a variable is strongly live if

- **it is used in an assignment to another strongly live variable**
- **it is used in a statement other than an assignment**



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$$\Theta_{\mathrm{X}}[\![\texttt{if}\ b\texttt{:}\ s_1\ \texttt{else:}\ s_2]\!](S) \stackrel{\text{def}}{=} \textsc{vars}(b) \cup \Theta_{\mathrm{X}}[\![s_1]\!](S) \cup \Theta_{\mathrm{X}}[\![s_2]\!](S)$$

$$\Theta_{\mathrm{X}}[\![\texttt{while}\ b\texttt{:}\ s]\!](S) \stackrel{\text{def}}{=} \textsc{vars}(b) \cup \Theta_{\mathrm{X}}[\![s]\!](S)$$

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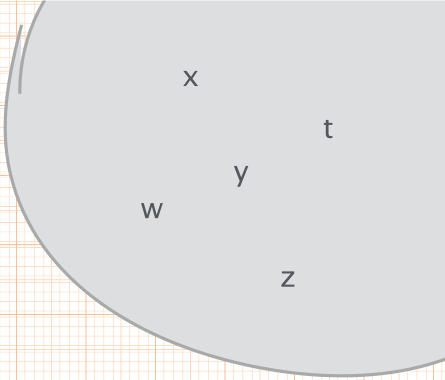
$\langle \mathcal{P}(\mathrm{X}), \subseteq, \cup, \cap, \emptyset, \mathrm{X} \rangle$: abstract domain

Strong-Liveness



$\llbracket P \rrbracket_{\mathrm{X}}$

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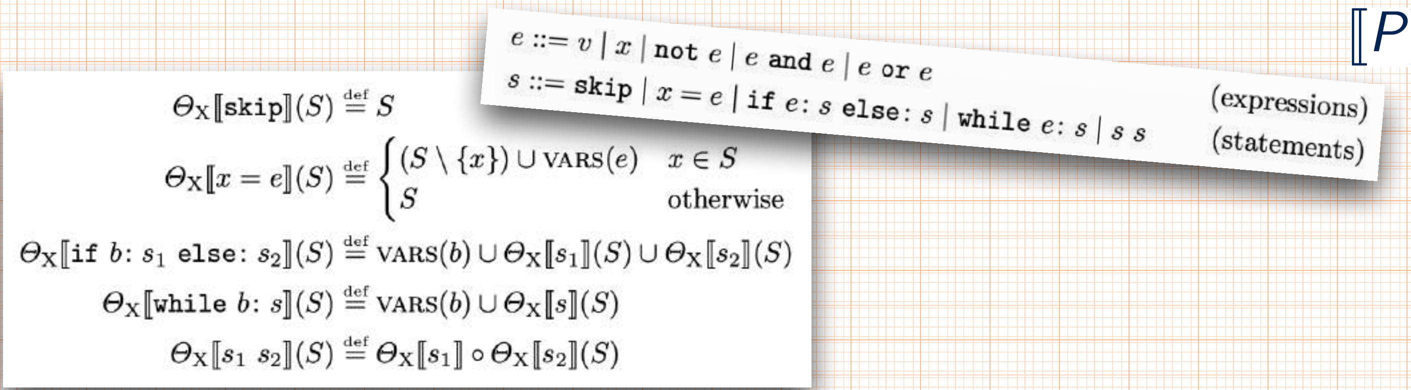

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$$\Theta_{\mathrm{X}}\llbracket \texttt{while}\ b\texttt{:}\ s \rrbracket(S) \stackrel{\text{def}}{=} \text{VARS}(b) \cup \Theta_{\mathrm{X}}\llbracket s \rrbracket(S)$$

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```
passing = True
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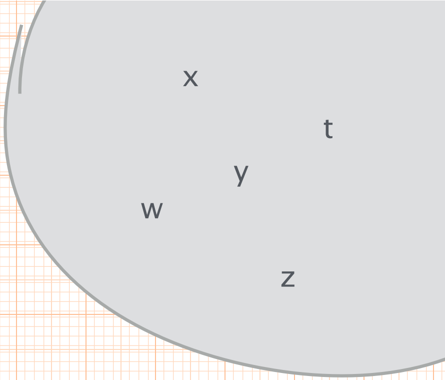
```

{ passing }

the initial set of strongly live variables is the set of output variables

Strong-Liveness



$[\![P]\!]_{\mathrm{X}}$

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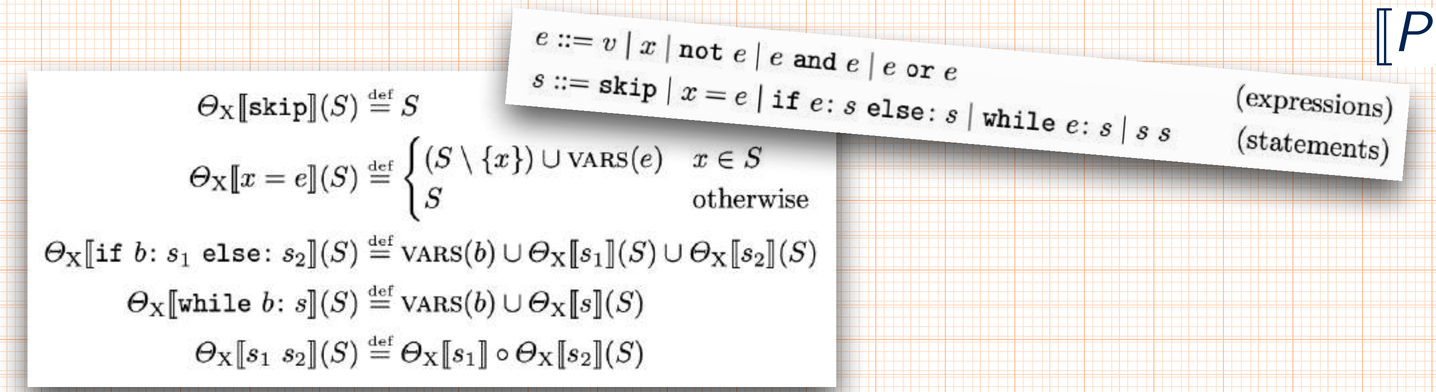

$e ::= v \mid x \mid \texttt{not}\ e \mid e\ \texttt{and}\ e \mid e\ \texttt{or}\ e$ (expressions)

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if not math:
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```

{ bonus, math, english } → the input variable science is definetly not used by the program

{ bonus, math, english }

{ bonus, math }

{ bonus, math }

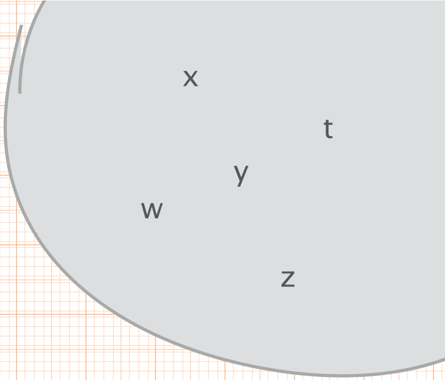
{ bonus, math }

{ bonus, math }

{ bonus }

{ passing }

Strong-Liveness



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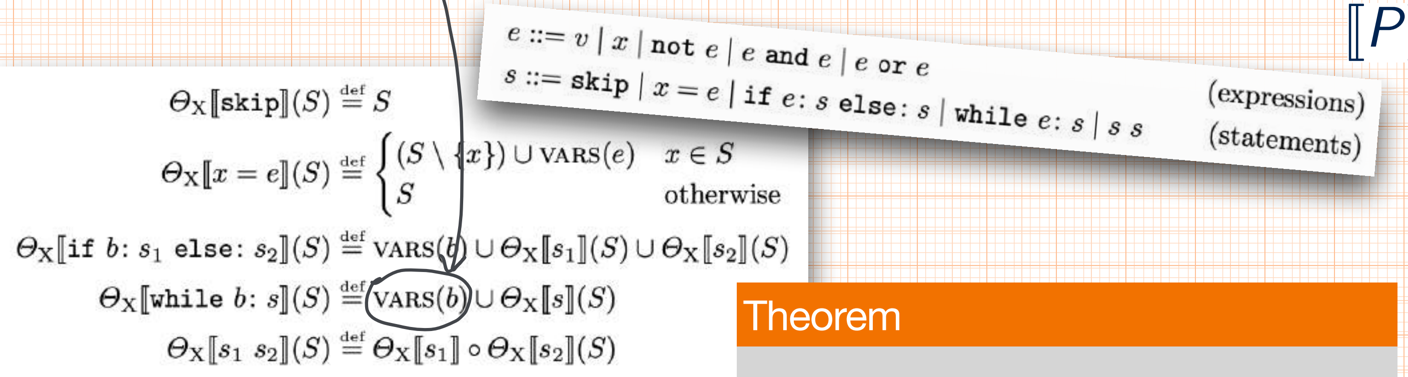
- **it is used in a statement other than an assignment**



$e ::= v \mid x \mid \texttt{not}\ e \mid e\ \texttt{and}\ e \mid e\ \texttt{or}\ e$ (expressions)

$s ::= \texttt{skip} \mid x = e \mid \texttt{if}\ e\texttt{:}\ s\ \texttt{else:}\ s \mid \texttt{while}\ e\texttt{:}\ s \mid s\ s$ (statements)

$$\Theta_X[\![\texttt{skip}]\!](S) \overset{\text{def}}{=} S$$

$$\Theta_X[\![x = e]\!](S) \overset{\text{def}}{=} \begin{cases} (S \setminus \{x\}) \cup \text{VARS}(e) & x \in S \\ S & \text{otherwise} \end{cases}$$

$$\Theta_X[\![\texttt{if}\ b\texttt{:}\ s_1\ \texttt{else:}\ s_2]\!](S) \overset{\text{def}}{=} \text{VARS}(b) \cup \Theta_X[\![s_1]\!](S) \cup \Theta_X[\![s_2]\!](S)$$

$$\Theta_X[\![\texttt{while}\ b\texttt{:}\ s]\!](S) \overset{\text{def}}{=} \boxed{\text{VARS}(b)} \cup \Theta_X[\![s]\!](S)$$

$$\Theta_X[\![s_1\ s_2]\!](S) \overset{\text{def}}{=} \Theta_X[\![s_1]\!] \circ \Theta_X[\![s_2]\!](S)$$

$[\![P]\!]_X$

Theorem

$$P \models \mathcal{N}_J \Leftarrow \gamma_{\rightsquigarrow}(\gamma_X([\![P]\!]_X)) \subseteq \mathcal{N}_J$$

sound even for non-terminating programs

```
passing = True
if not english:
    english = False
if not math:
    passing = False or bonus
if not math:
    passing = False or bonus
```

{ bonus, math, english }
{ bonus, math, english }
{ bonus, math }
{ bonus, math }
{ bonus, math }
{ bonus, math }
{ bonus }
{ passing }

passing = True
while not english:
    english = False

Syntactic (Non-)Usage

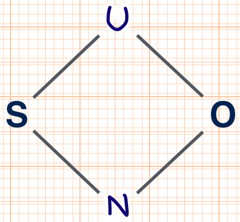
x ⇢ U

y ⇢ S | y ⇢ U

t ⇢ N

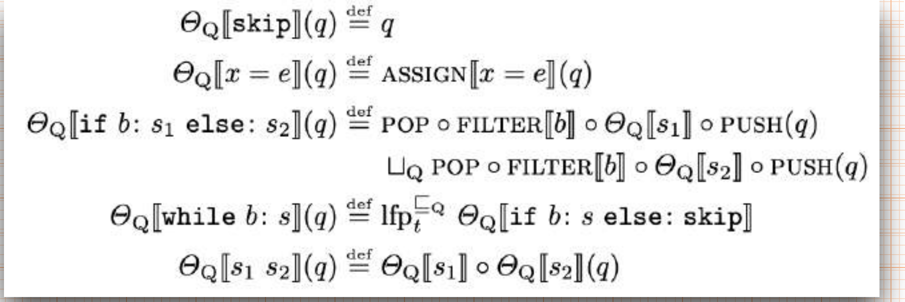
z ⇢ N

w ⇢ O | w ⇢ U



- **U**: used in the current scope (or an inner scope)
- **S**: used in an outer scope
- **O**: used in an outer scope and overridden in the current scope
- **N**: not used

$\llbracket P \rrbracket_{\mathrm{U}}$



$$\Theta_{\mathrm{Q}}\llbracket \texttt{skip} \rrbracket(q) \overset{\text{def}}{=} q$$

$$\Theta_{\mathrm{Q}}\llbracket x = e \rrbracket(q) \overset{\text{def}}{=} \mathrm{ASSIGN}\llbracket x = e \rrbracket(q)$$

$$\Theta_{\mathrm{Q}}\llbracket \texttt{if}\ b\texttt{:}\ s_1\ \texttt{else:}\ s_2 \rrbracket(q) \overset{\text{def}}{=} \mathrm{POP} \circ \mathrm{FILTER}\llbracket b \rrbracket \circ \Theta_{\mathrm{Q}}\llbracket s_1 \rrbracket \circ \mathrm{PUSH}(q)$$
$$\sqcup_{\mathrm{Q}}\ \mathrm{POP} \circ \mathrm{FILTER}\llbracket b \rrbracket \circ \Theta_{\mathrm{Q}}\llbracket s_2 \rrbracket \circ \mathrm{PUSH}(q)$$

$$\Theta_{\mathrm{Q}}\llbracket \texttt{while}\ b\texttt{:}\ s \rrbracket(q) \overset{\text{def}}{=} \mathrm{lfp}_t^{\sqsubseteq_{\mathrm{Q}}}\ \Theta_{\mathrm{Q}}\llbracket \texttt{if}\ b\texttt{:}\ s\ \texttt{else:}\ \texttt{skip} \rrbracket$$

$$\Theta_{\mathrm{Q}}\llbracket s_1\ s_2 \rrbracket(q) \overset{\text{def}}{=} \Theta_{\mathrm{Q}}\llbracket s_1 \rrbracket \circ \Theta_{\mathrm{Q}}\llbracket s_2 \rrbracket(q)$$

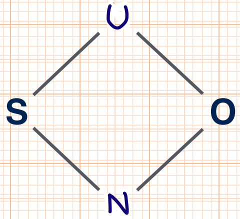
```
passing = True
if not english:
    english = False

if not math:
    passing = False or bonus

if not math:
    passing = False or bonus
●
```

passing ⇢ U (any other variable maps to N)

Syntactic (Non-)Usage



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- **N**: not used

x ⇢ U

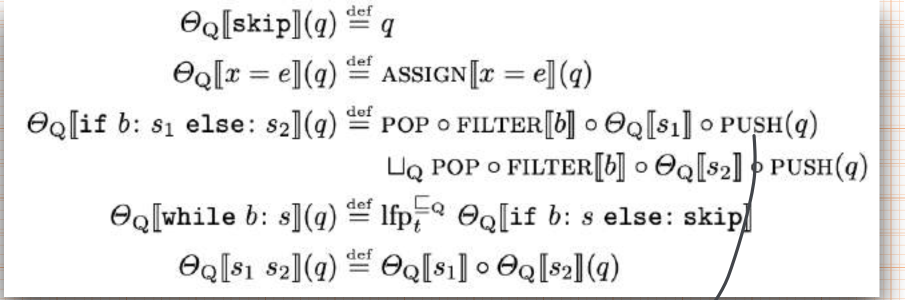
y ⇢ S | y ⇢ U

t ⇢ N

z ⇢ N

w ⇢ O | w ⇢ U

$[\![P]\!]_{\mathrm{U}}$



$$\Theta_Q[\![\texttt{skip}]\!](q) \overset{\text{def}}{=} q$$

$$\Theta_Q[\![x = e]\!](q) \overset{\text{def}}{=} \text{ASSIGN}[\![x = e]\!](q)$$

$$\Theta_Q[\![\texttt{if } b\texttt{: } s_1 \texttt{ else: } s_2]\!](q) \overset{\text{def}}{=} \text{POP} \circ \text{FILTER}[\![b]\!] \circ \Theta_Q[\![s_1]\!] \circ \text{PUSH}(q) \sqcup_Q \text{POP} \circ \text{FILTER}[\![b]\!] \circ \Theta_Q[\![s_2]\!] \circ \text{PUSH}(q)$$

$$\Theta_Q[\![\texttt{while } b\texttt{: } s]\!](q) \overset{\text{def}}{=} \text{lfp}_t^{\sqsubseteq_Q} \Theta_Q[\![\texttt{if } b\texttt{: } s \texttt{ else: skip}]\!]$$

$$\Theta_Q[\![s_1\ s_2]\!](q) \overset{\text{def}}{=} \Theta_Q[\![s_1]\!] \circ \Theta_Q[\![s_2]\!](q)$$

```
passing = True
if not english:
    english = False

if not math:
    passing = False or bonus

if not math:
    passing = False or bonus
```

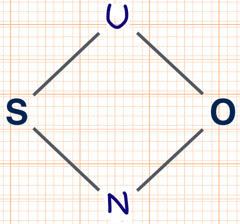
passing ⇢ S | passing ⇢ U

passing ⇢ U

x → U => x → S
x → O => x → N

domain elements are stacks of maps matching nesting level of analyzed statements

Syntactic (Non-)Usage



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- **S**: used in an outer scope
- **O**: used in an outer scope and overridden in the current scope
- **N**: not used

x ⇢ U

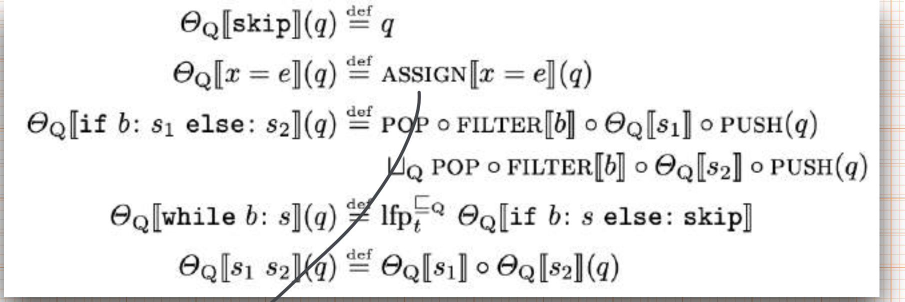
y ⇢ S | y ⇢ U

t ⇢ N

z ⇢ N

w ⇢ O | w ⇢ U

$[\![P]\!]_{\mathrm{U}}$



$$\Theta_{\mathrm{Q}}[\![\texttt{skip}]\!](q) \overset{\text{def}}{=} q$$

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$$\sqcup_{\mathrm{Q}}\ \text{POP} \circ \text{FILTER}[\![b]\!] \circ \Theta_{\mathrm{Q}}[\![s_2]\!] \circ \text{PUSH}(q)$$

$$\Theta_{\mathrm{Q}}[\![\texttt{while}\ b\texttt{:}\ s]\!](q) \overset{\text{def}}{=} \text{lfp}_t^{\sqsubseteq_{\mathrm{Q}}}\ \Theta_{\mathrm{Q}}[\![\texttt{if}\ b\texttt{:}\ s\ \texttt{else:}\ \texttt{skip}]\!]$$

$$\Theta_{\mathrm{Q}}[\![s_1\ s_2]\!](q) \overset{\text{def}}{=} \Theta_{\mathrm{Q}}[\![s_1]\!] \circ \Theta_{\mathrm{Q}}[\![s_2]\!](q)$$

```
passing = True

if not english:

    english = False


if not math:

    passing = False or bonus


if not math:

    passing = False or bonus

```

if the assigned variable was used (U or S) it becomes overwritten (O) if not also used in the expression being assigned; otherwise it becomes freshly used (U)

bonus ⇢ U, passing ⇢ O | passing ⇢ U

passing ⇢ S | passing ⇢ U

passing ⇢ U

Syntactic (Non-)Usage

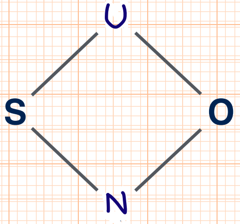
x ⇢ U

y ⇢ S | y ⇢ U

t ⇢ N

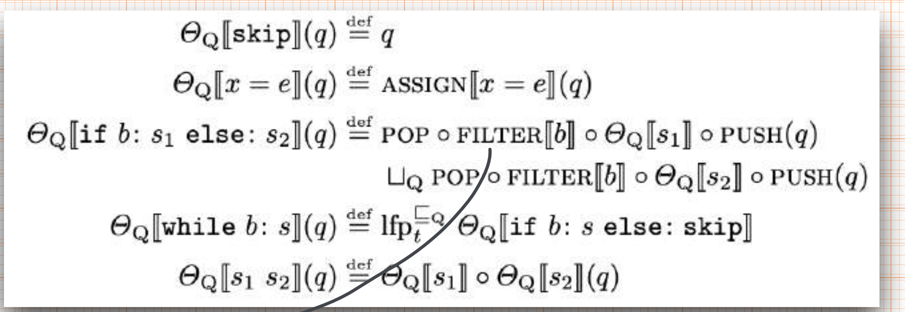
z ⇢ N

w ⇢ O | w ⇢ U



- **U**: used in the current scope (or an inner scope)
- **S**: used in an outer scope
- **O**: used in an outer scope and overridden in the current scope
- **N**: not used

$[\![P]\!]_U$



$$\Theta_Q[\![\texttt{skip}]\!](q) \stackrel{\text{def}}{=} q$$

$$\Theta_Q[\![x = e]\!](q) \stackrel{\text{def}}{=} \text{ASSIGN}[\![x = e]\!](q)$$

$$\Theta_Q[\![\texttt{if } b\texttt{: } s_1 \texttt{ else: } s_2]\!](q) \stackrel{\text{def}}{=} \text{POP} \circ \text{FILTER}[\![b]\!] \circ \Theta_Q[\![s_1]\!] \circ \text{PUSH}(q)$$
$$\sqcup_Q \text{POP} \circ \text{FILTER}[\![b]\!] \circ \Theta_Q[\![s_2]\!] \circ \text{PUSH}(q)$$

$$\Theta_Q[\![\texttt{while } b\texttt{: } s]\!](q) \stackrel{\text{def}}{=} \text{lfp}_t^{\sqsubseteq_Q} \Theta_Q[\![\texttt{if } b\texttt{: } s \texttt{ else: skip}]\!]$$

$$\Theta_Q[\![s_1 \; s_2]\!](q) \stackrel{\text{def}}{=} \Theta_Q[\![s_1]\!] \circ \Theta_Q[\![s_2]\!](q)$$

a variable becomes used (U)
if it appears in the boolean condition
of a statement that uses (U) or modifies (O)
another variable

```
passing = True

if not english:

    english = False


if not math:

    passing = False or bonus
```

math, bonus, passing ⇢ U

```
if not math:
```

bonus ⇢ U, passing ⇢ O | passing ⇢ U

```
    passing = False or bonus
```

passing ⇢ S | passing ⇢ U

passing ⇢ U

Syntactic (Non-)Usage

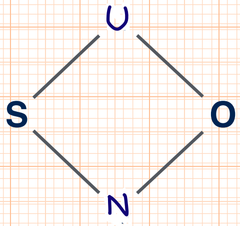
x ⇢ U

y ⇢ S | y ⇢ U

t ⇢ N

z ⇢ N

w ⇢ O | w ⇢ U



- **U**: used in the current scope (or an inner scope)
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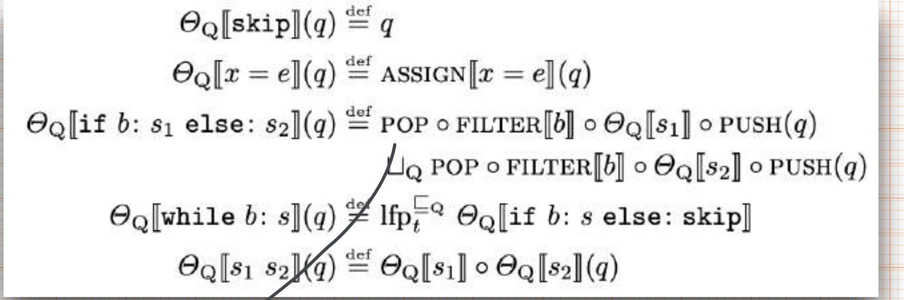
$[\![P]\!]_{\mathrm{U}}$



$$\Theta_{\mathrm{Q}}[\![\texttt{skip}]\!](q) \stackrel{\text{def}}{=} q$$

$$\Theta_{\mathrm{Q}}[\![x = e]\!](q) \stackrel{\text{def}}{=} \text{ASSIGN}[\![x = e]\!](q)$$

$$\Theta_{\mathrm{Q}}[\![\texttt{if } b\texttt{: } s_1 \texttt{ else: } s_2]\!](q) \stackrel{\text{def}}{=} \text{POP} \circ \text{FILTER}[\![b]\!] \circ \Theta_{\mathrm{Q}}[\![s_1]\!] \circ \text{PUSH}(q) \sqcup_{\mathrm{Q}} \text{POP} \circ \text{FILTER}[\![b]\!] \circ \Theta_{\mathrm{Q}}[\![s_2]\!] \circ \text{PUSH}(q)$$

$$\Theta_{\mathrm{Q}}[\![\texttt{while } b\texttt{: } s]\!](q) \stackrel{\text{def}}{=} \text{lfp}_t^{\sqsubseteq_{\mathrm{Q}}} \Theta_{\mathrm{Q}}[\![\texttt{if } b\texttt{: } s \texttt{ else: skip}]\!]$$

$$\Theta_{\mathrm{Q}}[\![s_1 \; s_2]\!](q) \stackrel{\text{def}}{=} \Theta_{\mathrm{Q}}[\![s_1]\!] \circ \Theta_{\mathrm{Q}}[\![s_2]\!](q)$$

restores the previous value of variables (before increasing the nesting level) if it has not changed since

```
passing = True

if not english:

    english = False


if not math:

    passing = False or bonus
```

math, bonus, passing ⇢ U

math, bonus ↦ U, passing ↦ O

```
if not math:
```

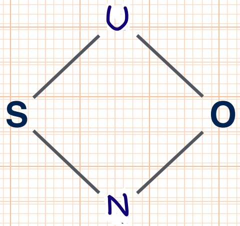
bonus ⇢ U, passing ⇢ O | passing ⇢ U

```
    passing = False or bonus
```

passing ⇢ S | passing ⇢ U

passing ⇢ U

Syntactic (Non-)Usage



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- **S**: used in an outer scope
- **O**: used in an outer scope and overridden in the current scope
- **N**: not used

$x \dashrightarrow U$

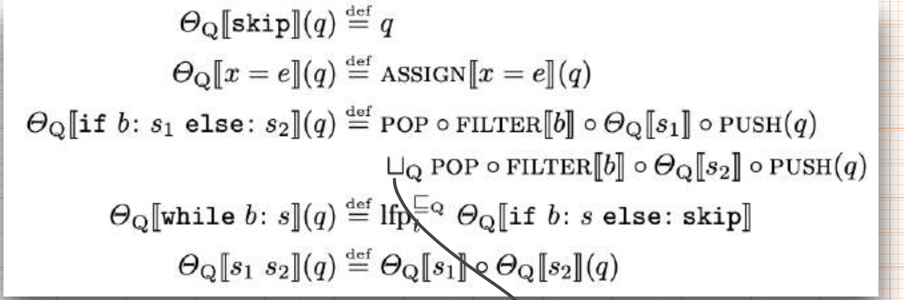
$y \dashrightarrow S \mid y \dashrightarrow U$

$t \dashrightarrow N$

$z \dashrightarrow N$

$w \dashrightarrow O \mid w \dashrightarrow U$

$[\![P]\!]_{\mathrm{U}}$



$$\Theta_{\mathrm{Q}}[\![\texttt{skip}]\!](q) \stackrel{\text{def}}{=} q$$

$$\Theta_{\mathrm{Q}}[\![x = e]\!](q) \stackrel{\text{def}}{=} \text{ASSIGN}[\![x = e]\!](q)$$

$$\Theta_{\mathrm{Q}}[\![\texttt{if } b\texttt{: } s_1 \texttt{ else: } s_2]\!](q) \stackrel{\text{def}}{=} \text{POP} \circ \text{FILTER}[\![b]\!] \circ \Theta_{\mathrm{Q}}[\![s_1]\!] \circ \text{PUSH}(q) \sqcup_{\mathrm{Q}} \text{POP} \circ \text{FILTER}[\![b]\!] \circ \Theta_{\mathrm{Q}}[\![s_2]\!] \circ \text{PUSH}(q)$$

$$\Theta_{\mathrm{Q}}[\![\texttt{while } b\texttt{: } s]\!](q) \stackrel{\text{def}}{=} \text{lfp}_q^{\sqsubseteq_{\mathrm{Q}}} \Theta_{\mathrm{Q}}[\![\texttt{if } b\texttt{: } s \texttt{ else: skip}]\!]$$

$$\Theta_{\mathrm{Q}}[\![s_1 \; s_2]\!](q) \stackrel{\text{def}}{=} \Theta_{\mathrm{Q}}[\![s_1]\!] \circ \Theta_{\mathrm{Q}}[\![s_2]\!](q)$$

```
passing = True

if not english:

    english = False


if not math:

    passing = False or bonus

●
if not math:
    ●
    passing = False or bonus
    ●
●
```

$\text{math, bonus, passing} \dashrightarrow U$

$\text{math, bonus} \mapsto U, \text{passing} \mapsto O \sqcup_Q \text{passing} \mapsto U$

$\text{bonus} \dashrightarrow U, \text{passing} \dashrightarrow O \mid \text{passing} \dashrightarrow U$

$\text{passing} \dashrightarrow S \mid \text{passing} \dashrightarrow U$

$\text{passing} \dashrightarrow U$

Syntactic (Non-)Usage

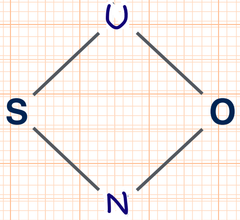
x ⇢ U

y ⇢ S | y ⇢ U

t ⇢ N

z ⇢ N

w ⇢ O | w ⇢ U



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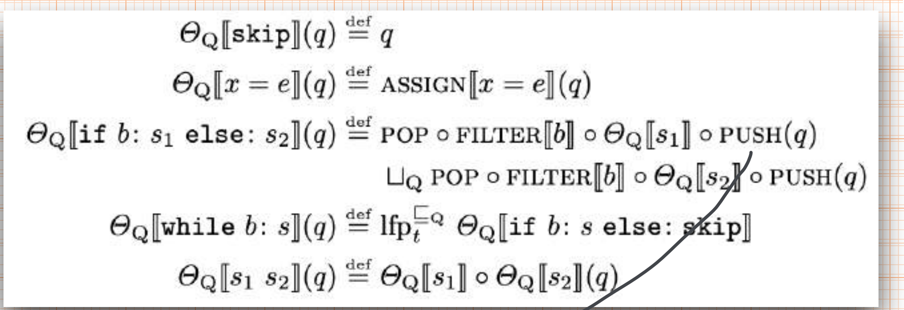
$\llbracket P \rrbracket_{\mathrm{U}}$



$$\Theta_{\mathrm{Q}}\llbracket \texttt{skip} \rrbracket(q) \stackrel{\text{def}}{=} q$$

$$\Theta_{\mathrm{Q}}\llbracket x = e \rrbracket(q) \stackrel{\text{def}}{=} \mathrm{ASSIGN}\llbracket x = e \rrbracket(q)$$

$$\Theta_{\mathrm{Q}}\llbracket \texttt{if } b\texttt{: } s_1 \texttt{ else: } s_2 \rrbracket(q) \stackrel{\text{def}}{=} \mathrm{POP} \circ \mathrm{FILTER}\llbracket b \rrbracket \circ \Theta_{\mathrm{Q}}\llbracket s_1 \rrbracket \circ \mathrm{PUSH}(q) \sqcup_{\mathrm{Q}} \mathrm{POP} \circ \mathrm{FILTER}\llbracket b \rrbracket \circ \Theta_{\mathrm{Q}}\llbracket s_2 \rrbracket \circ \mathrm{PUSH}(q)$$

$$\Theta_{\mathrm{Q}}\llbracket \texttt{while } b\texttt{: } s \rrbracket(q) \stackrel{\text{def}}{=} \mathrm{lfp}_t^{\sqsubseteq_{\mathrm{Q}}} \Theta_{\mathrm{Q}}\llbracket \texttt{if } b\texttt{: } s \texttt{ else: skip} \rrbracket$$

$$\Theta_{\mathrm{Q}}\llbracket s_1 \; s_2 \rrbracket(q) \stackrel{\text{def}}{=} \Theta_{\mathrm{Q}}\llbracket s_1 \rrbracket \circ \Theta_{\mathrm{Q}}\llbracket s_2 \rrbracket(q)$$

```
passing = True

if not english:

    english = False


if not math:

    passing = False or bonus

if not math:

    passing = False or bonus

```

math, bonus, passing ⇢ S | math, bonus, passing ⇢ U

math, bonus, passing ⇢ U

bonus ⇢ U, passing ⇢ O | passing ⇢ U

passing ⇢ S | passing ⇢ U

passing ⇢ U

x ⇢ U

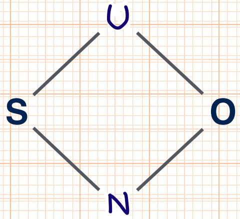
y ⇢ S | y ⇢ U

t ⇢ N

z ⇢ N

w ⇢ O | w ⇢ U

Syntactic (Non-)Usage



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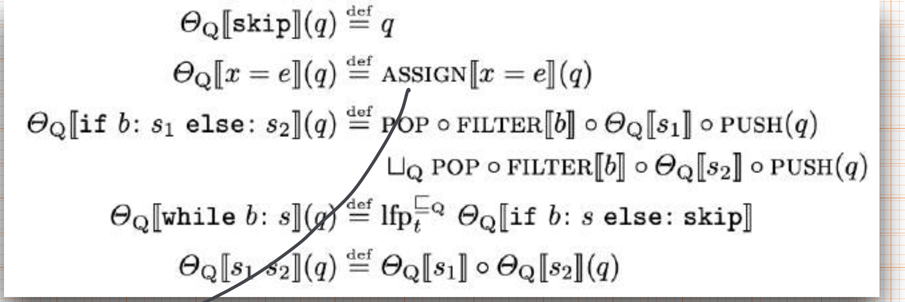
$[\![P]\!]_{\mathrm{U}}$



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$$\Theta_{\mathrm{Q}}[\![\texttt{while } b\texttt{: } s]\!](q) \stackrel{\text{def}}{=} \text{lfp}_t^{\sqsubseteq_{\mathrm{Q}}} \; \Theta_{\mathrm{Q}}[\![\texttt{if } b\texttt{: } s \texttt{ else: skip}]\!]$$

$$\Theta_{\mathrm{Q}}[\![s_1 \; s_2]\!](q) \stackrel{\text{def}}{=} \Theta_{\mathrm{Q}}[\![s_1]\!] \circ \Theta_{\mathrm{Q}}[\![s_2]\!](q)$$

```
passing = True

if not english:

    english = False


if not math:
    ●
    passing = False or bonus
    ●
●
if not math:
    ●
    passing = False or bonus
    ●
●
```

math ⇢ S, bonus ⇢ U, passing ⇢ O | …

math, bonus, passing ⇢ S | math, bonus, passing ⇢ U

math, bonus, passing ⇢ U

bonus ⇢ U, passing ⇢ O | passing ⇢ U

passing ⇢ S | passing ⇢ U

passing ⇢ U

x ⇢ U

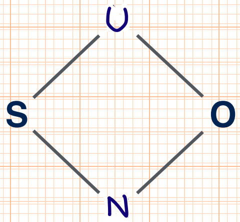
y ⇢ S | y ⇢ U

t ⇢ N

z ⇢ N

w ⇢ O | w ⇢ U

Syntactic (Non-)Usage



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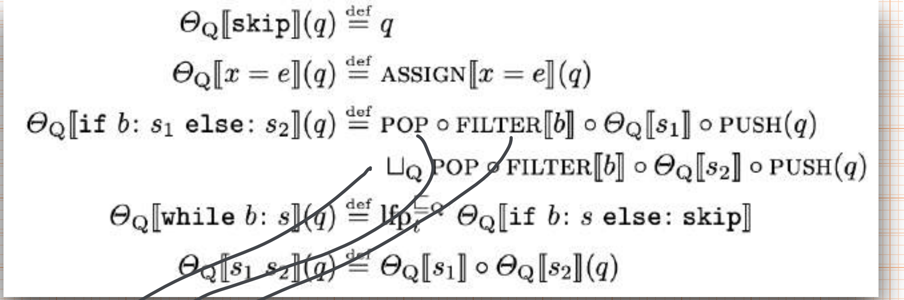
$[\![P]\!]_{\mathrm{U}}$



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$$\sqcup_{\mathrm{Q}} \mathrm{POP} \circ \mathrm{FILTER}[\![b]\!] \circ \Theta_{\mathrm{Q}}[\![s_2]\!] \circ \mathrm{PUSH}(q)$$

$$\Theta_{\mathrm{Q}}[\![\texttt{while } b\texttt{: } s]\!](q) \overset{\text{def}}{=} \mathrm{lfp}_{\iota}^{\sqsubseteq_{\mathrm{Q}}} \Theta_{\mathrm{Q}}[\![\texttt{if } b\texttt{: } s \texttt{ else: skip}]\!]$$

$$\Theta_{\mathrm{Q}}[\![s_1\ s_2]\!](q) \overset{\text{def}}{=} \Theta_{\mathrm{Q}}[\![s_1]\!] \circ \Theta_{\mathrm{Q}}[\![s_2]\!](q)$$

```
passing = True

if not english:

    english = False

●
if not math:
    ●
    passing = False or bonus
    ●
●
if not math:
    ●
    passing = False or bonus
    ●
●
```

math, bonus, passing ⇢ U

math ⇢ S, bonus ⇢ U, passing ⇢ O | …

math, bonus, passing ⇢ S | math, bonus, passing ⇢ U

math, bonus, passing ⇢ U

bonus ⇢ U, passing ⇢ O | passing ⇢ U

passing ⇢ S | passing ⇢ U

passing ⇢ U

Syntactic (Non-)Usage

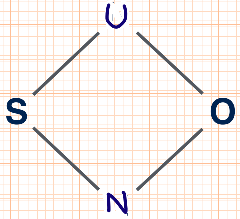
x ⇢ U

y ⇢ S | y ⇢ U

t ⇢ N

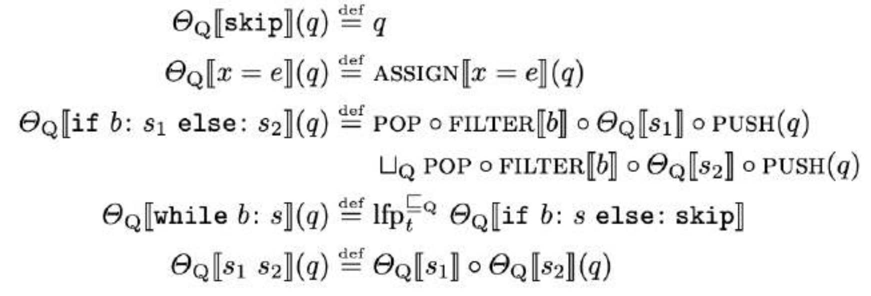
z ⇢ N

w ⇢ O | w ⇢ U



- **U**: used in the current scope (or an inner scope)
- **S**: used in an outer scope
- **O**: used in an outer scope and overridden in the current scope
- **N**: not used

$[\![P]\!]_{\mathrm{U}}$



$$\Theta_{\mathrm{Q}}[\![\texttt{skip}]\!](q) \overset{\text{def}}{=} q$$

$$\Theta_{\mathrm{Q}}[\![x = e]\!](q) \overset{\text{def}}{=} \textsc{assign}[\![x = e]\!](q)$$

$$\Theta_{\mathrm{Q}}[\![\texttt{if } b\texttt{: } s_1 \texttt{ else: } s_2]\!](q) \overset{\text{def}}{=} \textsc{pop} \circ \textsc{filter}[\![b]\!] \circ \Theta_{\mathrm{Q}}[\![s_1]\!] \circ \textsc{push}(q)$$

$$\sqcup_{\mathrm{Q}} \textsc{pop} \circ \textsc{filter}[\![b]\!] \circ \Theta_{\mathrm{Q}}[\![s_2]\!] \circ \textsc{push}(q)$$

$$\Theta_{\mathrm{Q}}[\![\texttt{while } b\texttt{: } s]\!](q) \overset{\text{def}}{=} \mathrm{lfp}_t^{\sqsubseteq_{\mathrm{Q}}} \Theta_{\mathrm{Q}}[\![\texttt{if } b\texttt{: } s \texttt{ else: skip}]\!]$$

$$\Theta_{\mathrm{Q}}[\![s_1 \; s_2]\!](q) \overset{\text{def}}{=} \Theta_{\mathrm{Q}}[\![s_1]\!] \circ \Theta_{\mathrm{Q}}[\![s_2]\!](q)$$

```
passing = True

if not english:

    english = False
    ●·········· math, bonus, passing ⇢ S | math, bonus, passing ⇢ U
●·········· math, bonus, passing ⇢ U
if not math:
    ●·········· math ⇢ S, bonus ⇢ U, passing ⇢ O | …
    passing = False or bonus
    ●·········· math, bonus, passing ⇢ S | math, bonus, passing ⇢ U
●·········· math, bonus, passing ⇢ U
if not math:
    ●·········· bonus ⇢ U, passing ⇢ O | passing ⇢ U
    passing = False or bonus
    ●·········· passing ⇢ S | passing ⇢ U
●·········· passing ⇢ U
```

Syntactic (Non-)Usage

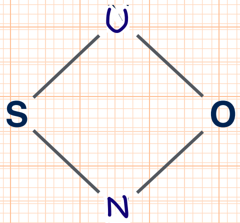
x ⇢ U

y ⇢ S | y ⇢ U

t ⇢ N

z ⇢ N

w ⇢ O | w ⇢ U



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- **O**: used in an outer scope and overridden in the current scope
- **N**: not used

$[\![P]\!]_{\mathrm{U}}$

```
passing = True

if not english:
    ●
    english = False
    ●
●
if not math:
    ●
    passing = False or bonus
    ●
●
if not math:
    ●
    passing = False or bonus
    ●
●
```

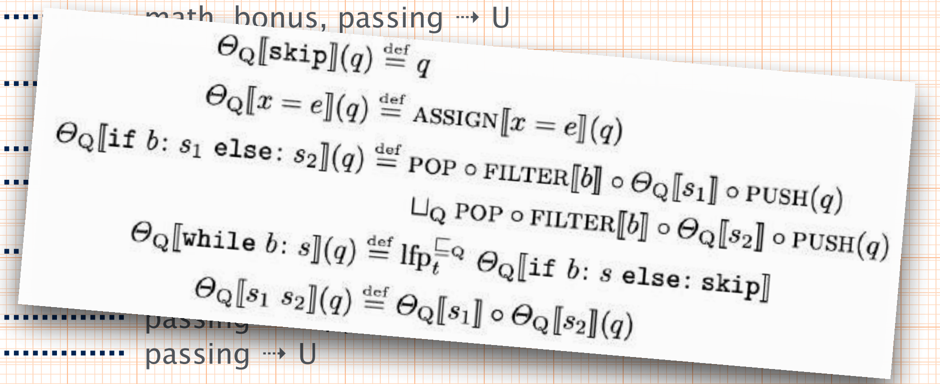
math, bonus, passing ⇢ S | math, bonus, passing ⇢ U

math, bonus, passing ⇢ S | math, bonus, passing ⇢ U

math, bonus, passing ⇢ U

passing

passing ⇢ U



$$\Theta_{\mathrm{Q}}[\![\texttt{skip}]\!](q) \overset{\text{def}}{=} q$$

$$\Theta_{\mathrm{Q}}[\![x = e]\!](q) \overset{\text{def}}{=} \mathrm{ASSIGN}[\![x = e]\!](q)$$

$$\begin{aligned}\Theta_{\mathrm{Q}}[\![\texttt{if } b\texttt{: } s_1 \texttt{ else: } s_2]\!](q) \overset{\text{def}}{=} & \ \mathrm{POP} \circ \mathrm{FILTER}[\![b]\!] \circ \Theta_{\mathrm{Q}}[\![s_1]\!] \circ \mathrm{PUSH}(q) \\ & \sqcup_{\mathrm{Q}} \mathrm{POP} \circ \mathrm{FILTER}[\![b]\!] \circ \Theta_{\mathrm{Q}}[\![s_2]\!] \circ \mathrm{PUSH}(q)\end{aligned}$$

$$\Theta_{\mathrm{Q}}[\![\texttt{while } b\texttt{: } s]\!](q) \overset{\text{def}}{=} \mathrm{lfp}_t^{\sqsubseteq_{\mathrm{Q}}} \Theta_{\mathrm{Q}}[\![\texttt{if } b\texttt{: } s \texttt{ else: skip}]\!]$$

$$\Theta_{\mathrm{Q}}[\![s_1 \ s_2]\!](q) \overset{\text{def}}{=} \Theta_{\mathrm{Q}}[\![s_1]\!] \circ \Theta_{\mathrm{Q}}[\![s_2]\!](q)$$

Syntactic (Non-)Usage

x ⇢ U

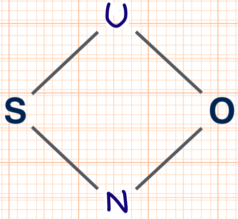
y ⇢ S | y ⇢ U

t ⇢ N

z ⇢ N

w ⇢ O | w ⇢ U

$[\![P]\!]_{\mathrm{U}}$



- **U**: used in the current scope (or an inner scope)
- **S**: used in an outer scope
- **O**: used in an outer scope and overridden in the current scope
- **N**: not used

```
passing = True
●
if not english:
    ●
    english = False
    ●
●
if not math:
    ●
    passing = False or bonus
    ●
●
if not math:
    ●
    passing = False or bonus
    ●
●
```

math, bonus, passing ⇢ U

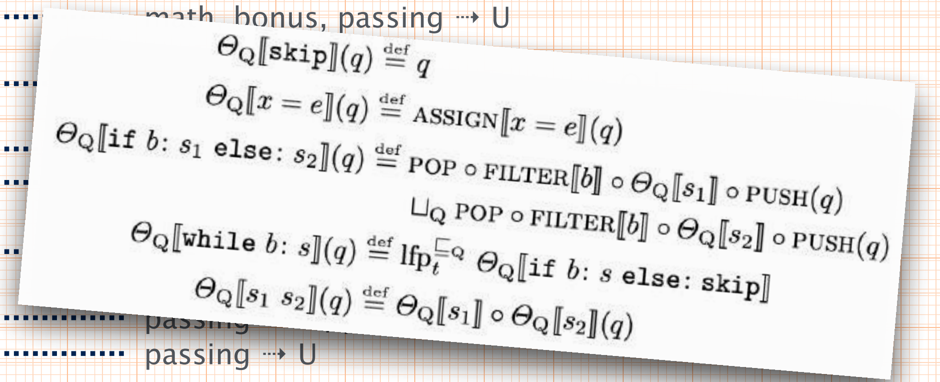
math, bonus, passing ⇢ S | math, bonus, passing ⇢ U

math, bonus, passing ⇢ S | math, bonus, passing ⇢ U

math, bonus, passing ⇢ U

passing

passing ⇢ U



$$\Theta_{\mathrm{Q}}[\![\texttt{skip}]\!](q) \stackrel{\text{def}}{=} q$$

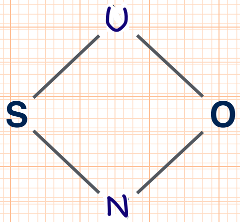
$$\Theta_{\mathrm{Q}}[\![x = e]\!](q) \stackrel{\text{def}}{=} \mathrm{ASSIGN}[\![x = e]\!](q)$$

$$\Theta_{\mathrm{Q}}[\![\texttt{if } b\texttt{: } s_1 \texttt{ else: } s_2]\!](q) \stackrel{\text{def}}{=} \mathrm{POP} \circ \mathrm{FILTER}[\![b]\!] \circ \Theta_{\mathrm{Q}}[\![s_1]\!] \circ \mathrm{PUSH}(q)$$
$$\sqcup_{\mathrm{Q}} \mathrm{POP} \circ \mathrm{FILTER}[\![b]\!] \circ \Theta_{\mathrm{Q}}[\![s_2]\!] \circ \mathrm{PUSH}(q)$$

$$\Theta_{\mathrm{Q}}[\![\texttt{while } b\texttt{: } s]\!](q) \stackrel{\text{def}}{=} \mathrm{lfp}_t^{\sqsubseteq_{\mathrm{Q}}} \, \Theta_{\mathrm{Q}}[\![\texttt{if } b\texttt{: } s \texttt{ else: skip}]\!]$$

$$\Theta_{\mathrm{Q}}[\![s_1 \; s_2]\!](q) \stackrel{\text{def}}{=} \Theta_{\mathrm{Q}}[\![s_1]\!] \circ \Theta_{\mathrm{Q}}[\![s_2]\!](q)$$

Syntactic (Non-)Usage



- **U**: used in the current scope (or an inner scope)
- **S**: used in an outer scope
- **O**: used in an outer scope and overridden in the current scope
- **N**: not used

x ⇢ U

y ⇢ S | y ⇢ U

t ⇢ N

z ⇢ N

w ⇢ O | w ⇢ U

$[\![P]\!]_{\mathrm{U}}$

the input variables english and science are definetly not used by the program

```
●
passing = True
●
if not english:
    ●
    english = False
    ●
●
if not math:
    ●
    passing = False or bonus
    ●
●
if not math:
    ●
    passing = False or bonus
    ●
●
```

math, bonus ⇢ U, passing ⇢ O

math, bonus, passing ⇢ U

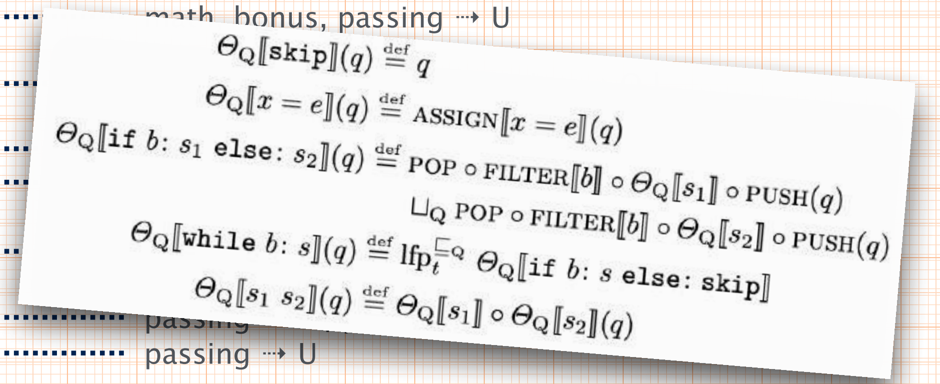
math, bonus, passing ⇢ S | math, bonus, passing ⇢ U

math, bonus, passing ⇢ S | math, bonus, passing ⇢ U

math, bonus, passing ⇢ U

passing

passing ⇢ U



$$\Theta_{\mathrm{Q}}[\![\texttt{skip}]\!](q) \stackrel{\text{def}}{=} q$$

$$\Theta_{\mathrm{Q}}[\![x = e]\!](q) \stackrel{\text{def}}{=} \mathrm{ASSIGN}[\![x = e]\!](q)$$

$$\Theta_{\mathrm{Q}}[\![\texttt{if } b\texttt{: } s_1 \texttt{ else: } s_2]\!](q) \stackrel{\text{def}}{=} \mathrm{POP} \circ \mathrm{FILTER}[\![b]\!] \circ \Theta_{\mathrm{Q}}[\![s_1]\!] \circ \mathrm{PUSH}(q) \sqcup_{\mathrm{Q}} \mathrm{POP} \circ \mathrm{FILTER}[\![b]\!] \circ \Theta_{\mathrm{Q}}[\![s_2]\!] \circ \mathrm{PUSH}(q)$$

$$\Theta_{\mathrm{Q}}[\![\texttt{while } b\texttt{: } s]\!](q) \stackrel{\text{def}}{=} \mathrm{lfp}_t^{\sqsubseteq_{\mathrm{Q}}} \Theta_{\mathrm{Q}}[\![\texttt{if } b\texttt{: } s \texttt{ else: skip}]\!]$$

$$\Theta_{\mathrm{Q}}[\![s_1 \ s_2]\!](q) \stackrel{\text{def}}{=} \Theta_{\mathrm{Q}}[\![s_1]\!] \circ \Theta_{\mathrm{Q}}[\![s_2]\!](q)$$

Syntactic (Non-)Usage

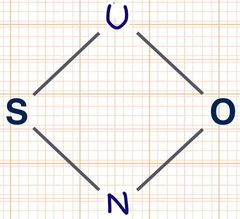
x ⇢ U

y ⇢ S | y ⇢ U

t ⇢ N

z ⇢ N

w ⇢ O | w ⇢ U



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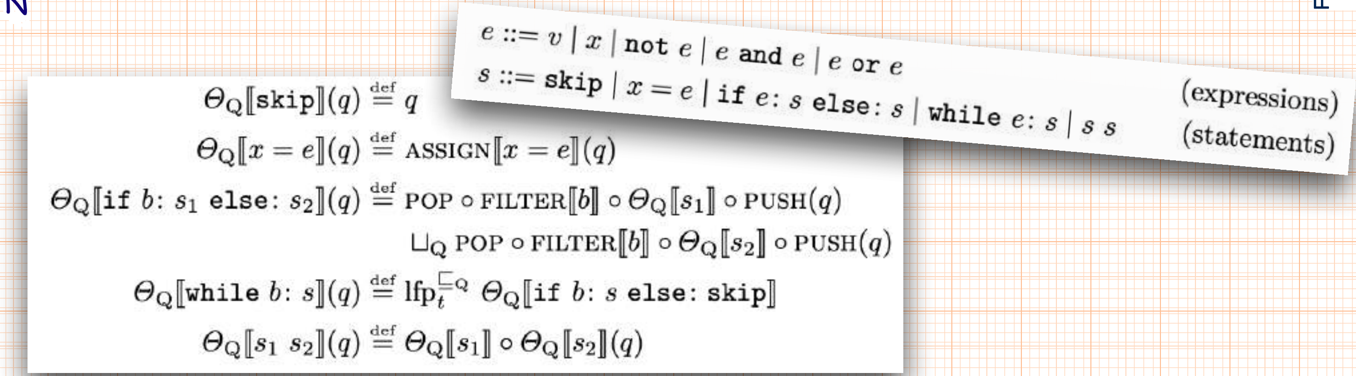
- **N**: not used

$[\![P]\!]_{\mathrm{U}}$



$$e ::= v \mid x \mid \texttt{not}\ e \mid e\ \texttt{and}\ e \mid e\ \texttt{or}\ e \qquad \text{(expressions)}$$

$$s ::= \texttt{skip} \mid x = e \mid \texttt{if}\ e\texttt{:}\ s\ \texttt{else:}\ s \mid \texttt{while}\ e\texttt{:}\ s \mid s\ s \qquad \text{(statements)}$$

$$\Theta_{\mathrm{Q}}[\![\texttt{skip}]\!](q) \stackrel{\text{def}}{=} q$$

$$\Theta_{\mathrm{Q}}[\![x = e]\!](q) \stackrel{\text{def}}{=} \text{ASSIGN}[\![x = e]\!](q)$$

$$\Theta_{\mathrm{Q}}[\![\texttt{if}\ b\texttt{:}\ s_1\ \texttt{else:}\ s_2]\!](q) \stackrel{\text{def}}{=} \text{POP} \circ \text{FILTER}[\![b]\!] \circ \Theta_{\mathrm{Q}}[\![s_1]\!] \circ \text{PUSH}(q) \sqcup_{\mathrm{Q}} \text{POP} \circ \text{FILTER}[\![b]\!] \circ \Theta_{\mathrm{Q}}[\![s_2]\!] \circ \text{PUSH}(q)$$

$$\Theta_{\mathrm{Q}}[\![\texttt{while}\ b\texttt{:}\ s]\!](q) \stackrel{\text{def}}{=} \text{lfp}_t^{\sqsubseteq_{\mathrm{Q}}}\ \Theta_{\mathrm{Q}}[\![\texttt{if}\ b\texttt{:}\ s\ \texttt{else:}\ \texttt{skip}]\!]$$

$$\Theta_{\mathrm{Q}}[\![s_1\ s_2]\!](q) \stackrel{\text{def}}{=} \Theta_{\mathrm{Q}}[\![s_1]\!] \circ \Theta_{\mathrm{Q}}[\![s_2]\!](q)$$

```
passing = True
while not english:
    english = False
```

passing ⇢ O

passing ⇢ U

passing ⇢ U

Theorem

$$P \models \mathcal{N}_J^* \Leftarrow \gamma_{\rightsquigarrow}(\gamma_Q([\![P]\!]_Q)) \subseteq \mathcal{N}_J^*$$

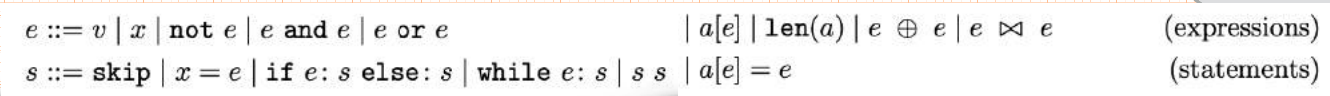
we could combine it with the termination analysis that we have seen in lecture 12 to have soundness even for non-terminating programs

Piecewise Syntactic (Non-)Usage

{0} N {i} U {i+1} O {len}

{0} N {len}

{0} S {len} | {0} U {len}



$e ::= v \mid x \mid \texttt{not}\ e \mid e\ \texttt{and}\ e \mid e\ \texttt{or}\ e \mid a[e] \mid \texttt{len}(a) \mid e \oplus e \mid e \bowtie e$ (expressions)

$s ::= \texttt{skip} \mid x = e \mid \texttt{if}\ e\texttt{:}\ s\ \texttt{else:}\ s \mid \texttt{while}\ e\texttt{:}\ s \mid s\ s \mid a[e] = e$ (statements)

$[\![P]\!]_S$

```
grades = list(map(int, input().split()))

count = 0

i = 1

while i < len(grades):

    if grades[i] < 4:
        count = count + 1


    i = i + 1


if 2 * count < len(grades):
    passing = True
else:
    passing = False

print(passing)
```

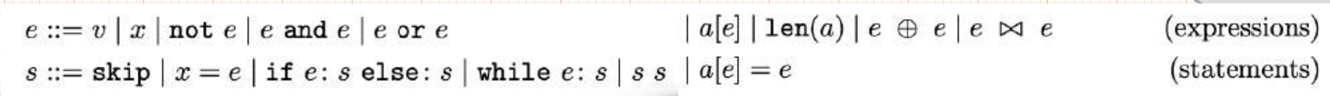
ERROR: 1 SHOULD BE 0

Piecewise Syntactic (Non-)Usage

{0} N {i} U {i+1} O {len}

{0} N {len}

{0} S {len} | {0} U {len}



$e ::= v \mid x \mid \texttt{not}\ e \mid e\ \texttt{and}\ e \mid e\ \texttt{or}\ e \mid a[e] \mid \texttt{len}(a) \mid e \oplus e \mid e \bowtie e$ (expressions)

$s ::= \texttt{skip} \mid x = e \mid \texttt{if}\ e\texttt{:}\ s\ \texttt{else:}\ s \mid \texttt{while}\ e\texttt{:}\ s \mid s\ s \mid a[e] = e$ (statements)

$\llbracket P \rrbracket_S$

```
grades = list(map(int, input().split()))

count = 0

i = 1

while i < len(grades):

    if grades[i] < 4:
        count = count + 1


    i = i + 1


if 2 * count < len(grades):
    passing = True
else:
    passing = False

print(passing)
```

ERROR: 1 SHOULD BE 0

$\forall 0 \leq x < \text{len(grades)} : x \longrightarrow N$

grades ⇢ { 0 } N { len(grades) }?

passing ⟶ U

Piecewise Syntactic (Non-)Usage

{0} N {i} U {i+1} O {len}

{0} N {len}

{0} S {len} | {0} U {len}

$$e ::= v \mid x \mid \texttt{not}\ e \mid e\ \texttt{and}\ e \mid e\ \texttt{or}\ e \mid a[e] \mid \texttt{len}(a) \mid e \oplus e \mid e \bowtie e \qquad \text{(expressions)}$$

$$s ::= \texttt{skip} \mid x = e \mid \texttt{if}\ e\texttt{:}\ s\ \texttt{else:}\ s \mid \texttt{while}\ e\texttt{:}\ s \mid s\ s \mid a[e] = e \qquad \text{(statements)}$$

$\llbracket P \rrbracket_S$

```
grades = list(map(int, input().split()))

count = 0

i = 1
```

ERROR: 1 SHOULD BE 0

```
while i < len(grades):

    if grades[i] < 4:
        count = count + 1


    i = i + 1
```

grades ⇢ {0} N {len(grades)}?

count → U

```
if 2 * count < len(grades):
    passing = True
else:
    passing = False
```

grades ⇢ { 0 } N { len(grades) }?

```
print(passing)
```

Piecewise Syntactic (Non-)Usage

{0} N {i} U {i+1} O {len}

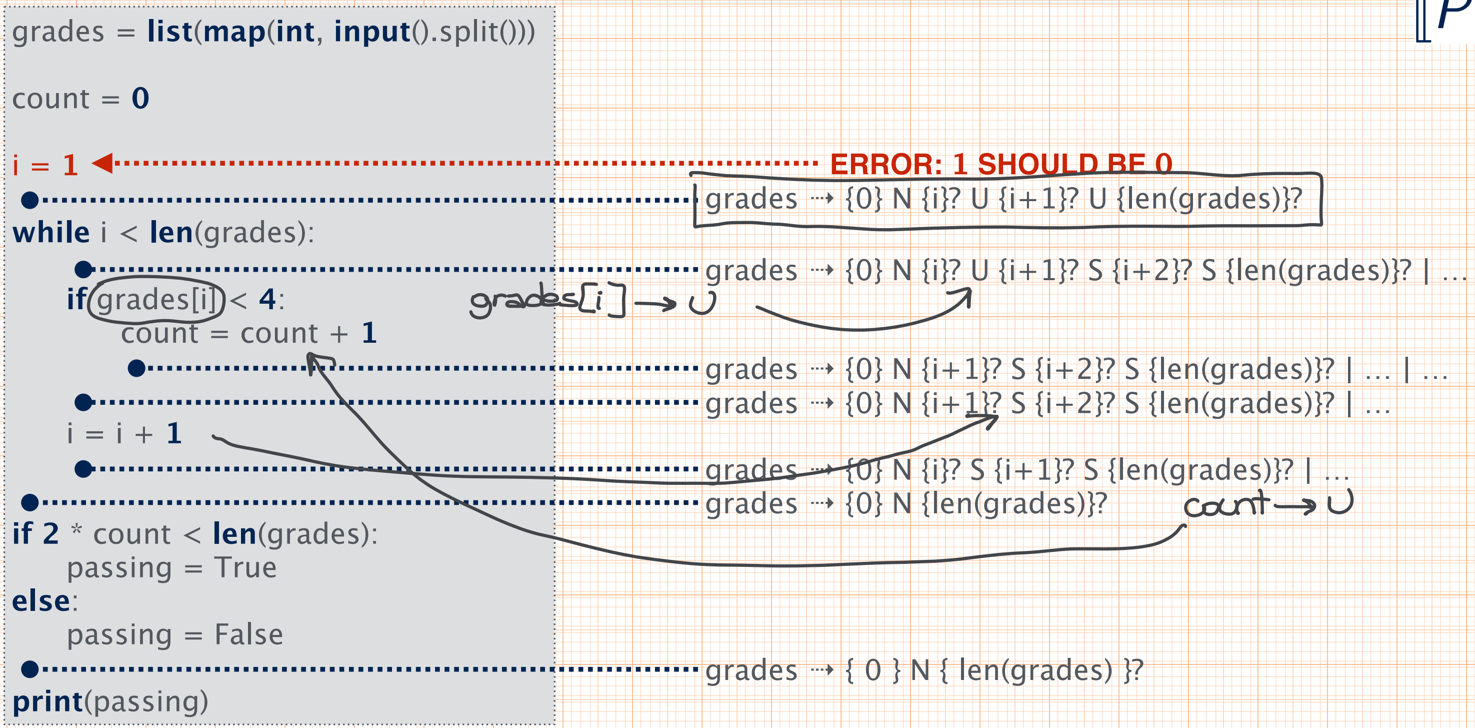
{0} N {len}

{0} S {len} | {0} U {len}

$$e ::= v \mid x \mid \texttt{not}\ e \mid e\ \texttt{and}\ e \mid e\ \texttt{or}\ e \mid a[e] \mid \texttt{len}(a) \mid e \oplus e \mid e \bowtie e \qquad \text{(expressions)}$$

$$s ::= \texttt{skip} \mid x = e \mid \texttt{if}\ e\texttt{:}\ s\ \texttt{else:}\ s \mid \texttt{while}\ e\texttt{:}\ s \mid s\ s \mid a[e] = e \qquad \text{(statements)}$$

$[\![P]\!]_S$



```
grades = list(map(int, input().split()))

count = 0

i = 1
●
while i < len(grades):
    ●
    if grades[i] < 4:
        count = count + 1
        ●
    ●
    i = i + 1
    ●
●
if 2 * count < len(grades):
    passing = True
else:
    passing = False
●
print(passing)
```

ERROR: 1 SHOULD BE 0

grades ⇢ {0} N {i}? U {i+1}? U {len(grades)}?

grades ⇢ {0} N {i}? U {i+1}? S {i+2}? S {len(grades)}? | …

grades[i] → U

grades ⇢ {0} N {i+1}? S {i+2}? S {len(grades)}? | … | …

grades ⇢ {0} N {i+1}? S {i+2}? S {len(grades)}? | …

grades ⇢ {0} N {i}? S {i+1}? S {len(grades)}? | …

grades ⇢ {0} N {len(grades)}?

count → U

grades ⇢ { 0 } N { len(grades) }?

Piecewise Syntactic (Non-)Usage

{0} N {i} U {i+1} O {len}

{0} N {len}

{0} S {len} | {0} U {len}

$e ::= v \mid x \mid \texttt{not}\ e \mid e\ \texttt{and}\ e \mid e\ \texttt{or}\ e \mid a[e] \mid \texttt{len}(a) \mid e \oplus e \mid e \bowtie e$ (expressions)

$s ::= \texttt{skip} \mid x = e \mid \texttt{if}\ e\texttt{:}\ s\ \texttt{else:}\ s \mid \texttt{while}\ e\texttt{:}\ s \mid s\ s \mid a[e] = e$ (statements)

$\llbracket P \rrbracket_S$

grades[0] is definitely not used by the program

```
grades = list(map(int, input().split()))

count = 0
```

● grades ⇢ {0} N {1}? U {2}? U {len(grades)}?

```
i = 1
```

ERROR: 1 SHOULD BE 0

● grades ⇢ {0} N {i}? U {i+1}? U {len(grades)}?

```
while i < len(grades):
```

● grades ⇢ {0} N {i}? U {i+1}? S {i+2}? S {len(grades)}? | …

```
    if grades[i] < 4:
        count = count + 1
```

● grades ⇢ {0} N {i+1}? S {i+2}? S {len(grades)}? | … | …

● grades ⇢ {0} N {i+1}? S {i+2}? S {len(grades)}? | …

```
    i = i + 1
```

● grades ⇢ {0} N {i}? S {i+1}? S {len(grades)}? | …

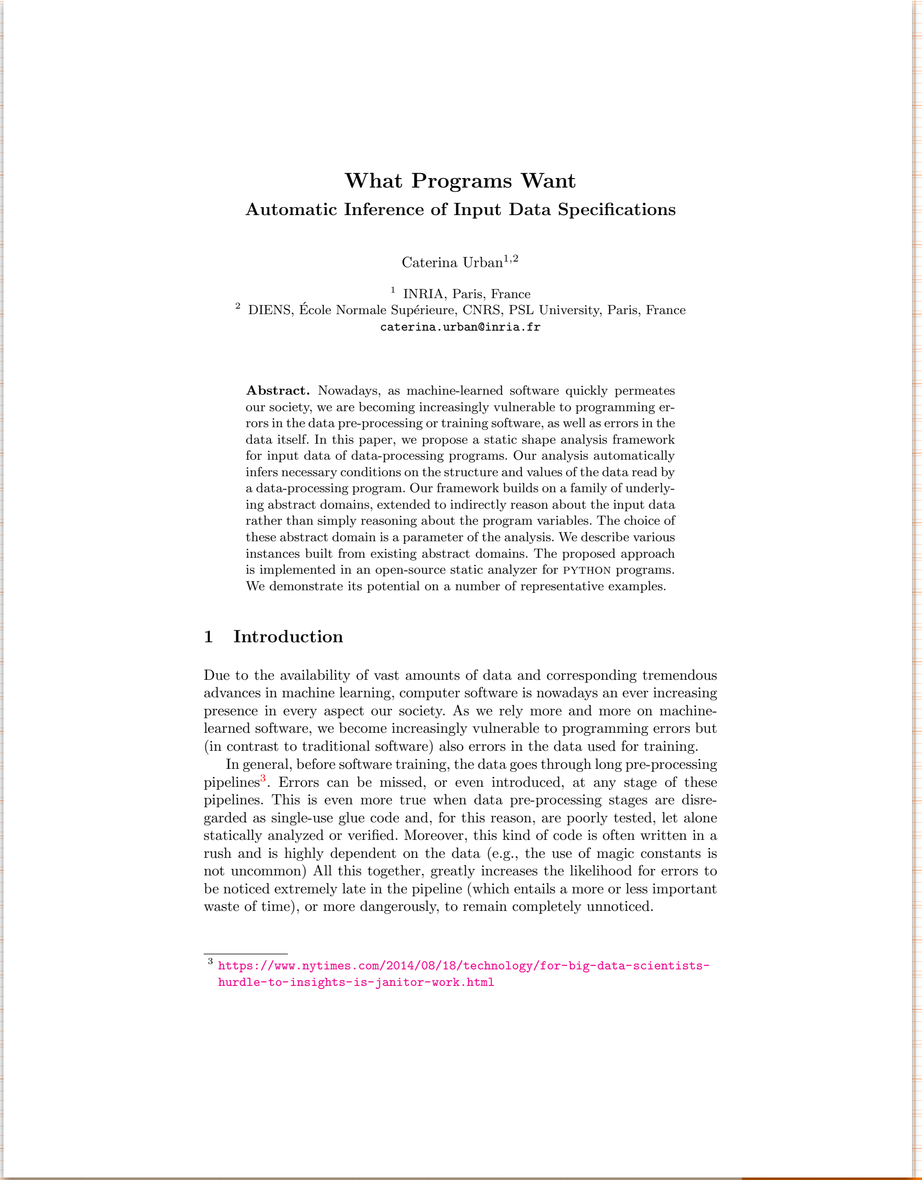
● grades ⇢ {0} N {len(grades)}?

```
if 2 * count < len(grades):
    passing = True
else:
    passing = False
```

● grades ⇢ { 0 } N { len(grades) }?

```
print(passing)
```

Implicit Assumptions on Data



What Programs Want

Automatic Inference of Input Data Specifications

Caterina Urban[1,2]

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[2] DIENS, École Normale Supérieure, CNRS, PSL University, Paris, France
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Abstract. Nowadays, as machine-learned software quickly permeates our society, we are becoming increasingly vulnerable to programming errors in the data pre-processing or training software, as well as errors in the data itself. In this paper, we propose a static shape analysis framework for input data of data-processing programs. Our analysis automatically infers necessary conditions on the structure and values of the data read by a data-processing program. Our framework builds on a family of underlying abstract domains, extended to indirectly reason about the input data rather than simply reasoning about the program variables. The choice of these abstract domain is a parameter of the analysis. We describe various instances built from existing abstract domains. The proposed approach is implemented in an open-source static analyzer for PYTHON programs. We demonstrate its potential on a number of representative examples.

1 Introduction

Due to the availability of vast amounts of data and corresponding tremendous advances in machine learning, computer software is nowadays an ever increasing presence in every aspect our society. As we rely more and more on machine-learned software, we become increasingly vulnerable to programming errors but (in contrast to traditional software) also errors in the data used for training.

In general, before software training, the data goes through long pre-processing pipelines[3]. Errors can be missed, or even introduced, at any stage of these pipelines. This is even more true when data pre-processing stages are disregarded as single-use glue code and, for this reason, are poorly tested, let alone statically analyzed or verified. Moreover, this kind of code is often written in a rush and is highly dependent on the data (e.g., the use of magic constants is not uncommon) All this together, greatly increases the likelihood for errors to be noticed extremely late in the pipeline (which entails a more or less important waste of time), or more dangerously, to remain completely unnoticed.

---

[3] https://www.nytimes.com/2014/08/18/technology/for-big-data-scientists-hurdle-to-insights-is-janitor-work.html

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