Static Analysis of Concurrent Programs

MPRI 2–6: Abstract Interpretation, application to verification and static analysis

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Concurrent programming

Idea:

Decompose a program into a set of (loosely) interacting processes.

Why concurrent programs?

- exploit parallelism in current computers
  (multi-processors, multi-cores, hyper-threading)

  "Free lunch is over"
  change in Moore's law ($\times$2 transistors every 2 years)

- exploit several computers
  (distributed computing)

- ease of programming
  (GUI, network code, reactive programs)

Models of concurrent programs

Many models:

- process calculi: CSP, $\pi-$calculus, join calculus
- message passing
- shared memory (threads)
- transactional memory
- combination of several models

Example implementations:

- C, C++ with a thread library (POSIX threads, Win32)
- C, C++ with a message library (MPI, OpenMP)
- Java (native threading API)
- Erlang (based on $\pi-$calculus)
- JoCaml (OCaml + join calculus)
- processor-level (interrupts, test-and-set instructions)

Scope

In this course: static thread model

- implicit communication through shared memory
- explicit communication through synchronisation primitives
- fixed number of threads (no dynamic creation of threads)
- numeric programs (real-valued variables)

Goal: static analysis

- infer numeric program invariants
- discover possible run-time errors (e.g., division by 0)
- parametrized by a choice of numeric abstract domains

Outline

- State-based analyses
  - sequential programs (reminders)
  - concurrent programs
- Toward thread-modular analyses
  - detour through proof methods
    (Floyd–Hoare, Owicki–Gries, Jones)
  - rely-guarantee in abstract interpretation form
- Interference-based abstract analyses
  - denotational-style analysis
  - weakly consistent memory models
  - synchronisation

Simple structured numeric language

- finite set of (toplevel) threads: $\texttt{prog}_1$ to $\texttt{prog}_n$
- finite set of numeric program variables $\texttt{V} \in \mathbb{V}$
- finite set of statement locations $\ell \in \mathcal{L}$
- finite set of potential error locations $\omega \in \Omega$

Structured language syntax

$$\begin{array}{lcll}
\texttt{parprog} & ::= & {}^{\ell}\texttt{prog}_1{}^{\ell} \mid\mid \ldots \mid\mid {}^{\ell}\texttt{prog}_n{}^{\ell} & \textit{(parallel composition)} \\
{}^{\ell}\texttt{prog}^{\ell} & ::= & {}^{\ell}\texttt{V} := \texttt{exp}^{\ell} & \textit{(assignment)} \\
 & \mid & {}^{\ell}\texttt{if } \texttt{exp} \bowtie 0 \texttt{ then } {}^{\ell}\texttt{prog}^{\ell} \texttt{ fi}^{\ell} & \textit{(conditional)} \\
 & \mid & {}^{\ell}\texttt{while } {}^{\ell}\texttt{exp} \bowtie 0 \texttt{ do } {}^{\ell}\texttt{prog}^{\ell} \texttt{ done}^{\ell} & \textit{(loop)} \\
 & \mid & {}^{\ell}\texttt{prog}; {}^{\ell}\texttt{prog}^{\ell} & \textit{(sequence)} \\
\texttt{exp} & ::= & \texttt{V} \mid [c_1, c_2] \mid -\texttt{exp} \mid \texttt{exp} \diamond_{\omega} \texttt{exp} &
\end{array}$$

$c_1, c_2 \in \mathbb{R} \cup \{+\infty, -\infty\}$, $\diamond \in \{+, -, \times, /\}$, $\bowtie \in \{=, <, \ldots\}$

State-based analyses

Sequential program semantics (reminders)

Transition systems

Transition system: $(\Sigma, \tau, \mathcal{I})$

- Σ: set of program states
- $\tau \subseteq \Sigma \times \Sigma$: transition relation
  we note $(\sigma, \sigma') \in \tau$ as $\sigma \to_\tau \sigma'$
- $\mathcal{I} \subseteq \Sigma$: set of initial states

Traces: sequences of states $\sigma_0, \ldots, \sigma_n, \ldots$

- Σ^*: finite traces
- Σ^ω: infinite countable traces
- $\Sigma^\infty \stackrel{\text{def}}{=} \Sigma^* \cup \Sigma^\omega$: finite or infinite countable traces
- $u \preceq t$: u is a prefix of t

We view program semantics and properties as sets of traces.

Traces of a transition system

Maximal trace semantics: $\mathcal{M}_\infty \in \mathcal{P}(\Sigma^\infty)$

- set of total executions $\sigma_0, \ldots, \sigma_n, \ldots$
  - starting in an initial state $\sigma_0 \in \mathcal{I}$ and either
  - ending in a blocking state in $\mathcal{B} \stackrel{\text{def}}{=} \{\, \sigma \,|\, \forall \sigma' : \sigma \not\to_\tau \sigma' \,\}$
  - or infinite

$$\mathcal{M}_\infty \stackrel{\text{def}}{=} \{\, \sigma_0, \ldots, \sigma_n \,|\, \sigma_0 \in \mathcal{I} \wedge \sigma_n \in \mathcal{B} \wedge \forall i < n : \sigma_i \to_\tau \sigma_{i+1} \,\} \cup \{\, \sigma_0, \ldots, \sigma_n \ldots \,|\, \sigma_0 \in \mathcal{I} \wedge \forall i < \omega : \sigma_i \to_\tau \sigma_{i+1} \,\}$$

- able to express many properties of programs, e.g.:
  - state safety: $\mathcal{M}_\infty \subseteq S^\infty$ (executions stay in S)
  - ordering: $\mathcal{M}_\infty \subseteq S_1^\infty \cdot S_2^\infty$ (S_2 can only occur after S_1)
  - termination: $\mathcal{M}_\infty \subseteq \Sigma^*$ (executions are finite)
  - inevitability: $\mathcal{M}_\infty \subseteq \Sigma^* \cdot S \cdot \Sigma^\infty$ (executions pass through S)

Traces of a transition system

Finite prefix trace semantics: $\mathcal{T}_p \in \mathcal{P}(\Sigma^*)$

- set of finite prefixes of executions:
  $\mathcal{T}_p \stackrel{\text{def}}{=} \{\, \sigma_0, \ldots, \sigma_n \mid n \geq 0,\ \sigma_0 \in \mathcal{I},\ \forall i < n: \sigma_i \rightarrow_\tau \sigma_{i+1} \,\}$
- $\mathcal{T}_p$ is an abstraction of the maximal trace semantics
  $\mathcal{T}_p = \alpha_{*\preceq}(\mathcal{M}_\infty)$ where $\alpha_{*\preceq}(X) \stackrel{\text{def}}{=} \{\, t \in \Sigma^* \mid \exists u \in X: t \preceq u \,\}$
- $\mathcal{T}_p$ can prove state safety properties: $\mathcal{T}_p \subseteq S^*$
  (executions stay in S)

  $\mathcal{T}_p$ can prove ordering properties: $\mathcal{T}_p \subseteq S_1^* \cdot S_2^*$
  (if S_1 and S_2 occur, S_2 occurs after S_1)
- $\mathcal{T}_p$ cannot prove termination nor inevitability properties
- fixpoint characterisation: $\mathcal{T}_p = \text{lfp}\, F_p$ where
  $F_p(X) = \mathcal{I} \cup \{\, \sigma_0, \ldots, \sigma_{n+1} \mid \sigma_0, \ldots, \sigma_n \in X \wedge \sigma_n \rightarrow_\tau \sigma_{n+1} \,\}$

State abstraction

Reachable state semantics: $\mathcal{R} \in \mathcal{P}(\Sigma)$

- set of states reachable in any execution:
  $\mathcal{R} \stackrel{\text{def}}{=} \{\, \sigma \mid \exists n \geq 0,\ \sigma_0, \ldots, \sigma_n \colon \sigma_0 \in \mathcal{I},\ \forall i < n \colon \sigma_i \rightarrow_\tau \sigma_{i+1} \wedge \sigma = \sigma_n \,\}$
- $\mathcal{R}$ is an abstraction of the finite trace semantics: $\mathcal{R} = \alpha_p(\mathcal{T}_p)$
  where $\alpha_p(X) \stackrel{\text{def}}{=} \{\, \sigma \mid \exists \sigma_0, \ldots, \sigma_n \in X \colon \sigma = \sigma_n \,\}$
- $\mathcal{R}$ can prove state safety properties: $\mathcal{R} \subseteq S$
  (executions stay in S)
  $\mathcal{R}$ cannot prove ordering, termination, inevitability properties
- fixpoint characterisation: $\mathcal{R} = \text{lfp}\, F_{\mathcal{R}}$ where
  $F_{\mathcal{R}}(X) = \mathcal{I} \cup \{\, \sigma \mid \exists \sigma' \in X \colon \sigma' \rightarrow_\tau \sigma \,\}$

States of a sequential program

Simple sequential numeric programs: $\texttt{parprog} ::= {}^{\ell^i}\texttt{prog}^{\ell^x}$.

Program states: $\Sigma \stackrel{\text{def}}{=} (\mathcal{L} \times \mathcal{E}) \cup \Omega$

- a control state in $\mathcal{L}$, and
- either a memory state: an environment in $\mathcal{E} \stackrel{\text{def}}{=} \mathbb{V} \rightarrow \mathbb{R}$
- or an error state in Ω

Initial states:

start at the first control point ℓ^i with variables set to 0:

$\mathcal{I} \stackrel{\text{def}}{=} \{ (\ell^i, \lambda \mathbb{V}.0) \}$

Note that $\mathcal{P}(\Sigma) \simeq (\mathcal{L} \rightarrow \mathcal{P}(\mathcal{E})) \times \mathcal{P}(\Omega)$:

- a state property in $\mathcal{P}(\mathcal{E})$ at each program point in $\mathcal{L}$
- and a set of errors in $\mathcal{P}(\Omega)$

Expression semantics with errors

Expression semantics: $\mathsf{E}[\![\, \text{exp} \,]\!] : \mathcal{E} \to (\mathcal{P}(\mathbb{R}) \times \mathcal{P}(\Omega))$

$$\mathsf{E}[\![\, \mathtt{V} \,]\!]\,\rho \stackrel{\text{def}}{=} \langle \{ \rho(\mathtt{V}) \}, \emptyset \rangle$$

$$\mathsf{E}[\![\, [c_1, c_2] \,]\!]\,\rho \stackrel{\text{def}}{=} \langle \{ x \in \mathbb{R} \mid c_1 \leq x \leq c_2 \}, \emptyset \rangle$$

$$\mathsf{E}[\![\, -e \,]\!]\,\rho \stackrel{\text{def}}{=} \text{let } \langle V, O \rangle = \mathsf{E}[\![\, e \,]\!]\,\rho \text{ in } \langle \{ -v \mid \in V \}, O \rangle$$

$$\mathsf{E}[\![\, e_1 \diamond_{\omega} e_2 \,]\!]\,\rho \stackrel{\text{def}}{=} \begin{array}{l} \text{let } \langle V_1, O_1 \rangle = \mathsf{E}[\![\, e_1 \,]\!]\,\rho \text{ in} \\ \text{let } \langle V_2, O_2 \rangle = \mathsf{E}[\![\, e_2 \,]\!]\,\rho \text{ in} \\ \langle \{ v_1 \diamond v_2 \mid v_i \in V_i, \diamond \neq / \vee v_2 \neq 0 \}, \\ \ O_1 \cup O_2 \cup \{ \omega \text{ if } \diamond = / \wedge 0 \in V_2 \} \rangle \end{array}$$

- defined by structural induction on the syntax
- evaluates in an environment ρ to a set of values
- also returns a set of accumulated errors
  (here, only divisions by zero)

Reminders: semantics in equational form

Principle: (without handling errors in Ω)

- see $\mathrm{lfp}\, f$ as the least solution of an equation $x = f(x)$
- partition states by control: $\mathcal{P}(\mathcal{L} \times \mathcal{E}) \simeq \mathcal{L} \rightarrow \mathcal{P}(\mathcal{E})$

  $\mathcal{X}_\ell \in \mathcal{P}(\mathcal{E})$: invariant at $\ell \in \mathcal{L}$

  $\forall \ell \in \mathcal{L}$: $\mathcal{X}_\ell \stackrel{\text{def}}{=} \{\, m \in \mathcal{E} \mid (\ell, m) \in \mathcal{R} \,\}$

  $\Longrightarrow$ set of (recursive) equations on $\mathcal{X}_\ell$

Example:

```
ℓ1 i:=2;
ℓ2 n:=[−∞,+∞];
ℓ3 while ℓ4 i < n do
      ℓ5 if [0,1] = 0 then
            ℓ6 i:=i+1
      fi
ℓ7 done
ℓ8
```

$$
\begin{aligned}
\mathcal{X}_1 &= \mathcal{I} \\
\mathcal{X}_2 &= \mathsf{C}[\![\, \mathtt{i} := 2 \,]\!]\, \mathcal{X}_1 \\
\mathcal{X}_3 &= \mathsf{C}[\![\, \mathtt{n} := [-\infty, +\infty] \,]\!]\, \mathcal{X}_2 \\
\mathcal{X}_4 &= \mathcal{X}_3 \cup \mathcal{X}_7 \\
\mathcal{X}_5 &= \mathsf{C}[\![\, \mathtt{i} < \mathtt{n} \,]\!]\, \mathcal{X}_4 \\
\mathcal{X}_6 &= \mathcal{X}_5 \\
\mathcal{X}_7 &= \mathcal{X}_5 \cup \mathsf{C}[\![\, \mathtt{i} := \mathtt{i} + 1 \,]\!]\, \mathcal{X}_6 \\
\mathcal{X}_8 &= \mathsf{C}[\![\, \mathtt{i} \geq \mathtt{n} \,]\!]\, \mathcal{X}_4
\end{aligned}
$$

Semantics in denotational form

Input-output function $\mathsf{C}[\![\, \texttt{prog} \,]\!]$

$$\mathsf{C}[\![\, \texttt{prog} \,]\!] : (\mathcal{P}(\mathcal{E}) \times \mathcal{P}(\Omega)) \to (\mathcal{P}(\mathcal{E}) \times \mathcal{P}(\Omega))$$

$$\mathsf{C}[\![\, \texttt{X} := e \,]\!] \langle\, R,\, O \,\rangle \stackrel{\text{def}}{=} \langle\, \emptyset,\, O \,\rangle \sqcup \bigsqcup_{\rho \in R} \langle\, \{\, \rho[\texttt{X} \mapsto v] \mid v \in V_\rho \,\},\, O_\rho \,\rangle$$

$$\mathsf{C}[\![\, e \bowtie 0? \,]\!] \langle\, R,\, O \,\rangle \stackrel{\text{def}}{=} \langle\, \emptyset,\, O \,\rangle \sqcup \bigsqcup_{\rho \in R} \langle\, \{\, \rho \mid \exists v \in V_\rho : v \bowtie 0 \,\},\, O_\rho \,\rangle$$

where $\langle\, V_\rho,\, O_\rho \,\rangle \stackrel{\text{def}}{=} \mathsf{E}[\![\, e \,]\!]\, \rho$

$$\mathsf{C}[\![\, \texttt{if } e \bowtie 0 \texttt{ then } s \texttt{ fi} \,]\!]\, X \stackrel{\text{def}}{=} (\mathsf{C}[\![\, s \,]\!] \circ \mathsf{C}[\![\, e \bowtie 0? \,]\!])X \sqcup \mathsf{C}[\![\, e \not\bowtie 0? \,]\!]\, X$$

$$\mathsf{C}[\![\, \texttt{while } e \bowtie 0 \texttt{ do } s \texttt{ done} \,]\!]\, X \stackrel{\text{def}}{=}$$
$$\mathsf{C}[\![\, e \not\bowtie 0? \,]\!] (\mathrm{lfp}\, \lambda Y. X \sqcup (\mathsf{C}[\![\, s \,]\!] \circ \mathsf{C}[\![\, e \bowtie 0? \,]\!])Y)$$

$$\mathsf{C}[\![\, s_1;\, s_2 \,]\!] \stackrel{\text{def}}{=} \mathsf{C}[\![\, s_2 \,]\!] \circ \mathsf{C}[\![\, s_1 \,]\!]$$

- mutate memory states in $\mathcal{E}$, accumulate errors in Ω

  ($\sqcup$ is the element-wise $\cup$ in $\mathcal{P}(\mathcal{E}) \times \mathcal{P}(\Omega)$)
- structured: nested loops yield nested fixpoints
- big-step: forget information on intermediate locations ℓ

Abstract semantics in denotational form

Extend a numeric abstract domain $\mathcal{E}^\sharp$ abstracting $\mathcal{P}(\mathcal{E})$ to $\mathcal{D}^\sharp \stackrel{\text{def}}{=} \mathcal{E}^\sharp \times \mathcal{P}(\Omega)$.

$\mathsf{C}^\sharp[\![\, \texttt{prog} \,]\!] : \mathcal{D}^\sharp \to \mathcal{D}^\sharp$

$\mathsf{C}^\sharp[\![\, \texttt{X} := e \,]\!]\langle R^\sharp, O \rangle$ and $\mathsf{C}^\sharp[\![\, e \bowtie 0? \,]\!]\langle R^\sharp, O \rangle$ are given

$\mathsf{C}^\sharp[\![\, \texttt{if}\ e \bowtie 0\ \texttt{then}\ s\ \texttt{fi} \,]\!] X^\sharp \stackrel{\text{def}}{=}$
$(\mathsf{C}^\sharp[\![\, s \,]\!] \circ \mathsf{C}^\sharp[\![\, e \bowtie 0? \,]\!]) X^\sharp \sqcup^\sharp \mathsf{C}^\sharp[\![\, e \not\bowtie 0? \,]\!] X^\sharp$

$\mathsf{C}^\sharp[\![\, \texttt{while}\ e \bowtie 0\ \texttt{do}\ s\ \texttt{done} \,]\!] X^\sharp \stackrel{\text{def}}{=}$
$\mathsf{C}^\sharp[\![\, e \not\bowtie 0? \,]\!] (\lim \lambda Y^\sharp . Y^\sharp \triangledown (X^\sharp \sqcup (\mathsf{C}^\sharp[\![\, s \,]\!] \circ \mathsf{C}^\sharp[\![\, e \bowtie 0? \,]\!]) Y^\sharp))$

$\mathsf{C}^\sharp[\![\, s_1;\ s_2 \,]\!] \stackrel{\text{def}}{=} \mathsf{C}^\sharp[\![\, s_2 \,]\!] \circ \mathsf{C}^\sharp[\![\, s_1 \,]\!]$

- the abstract interpreter mimicks an actual interpreter
- efficient in memory (intermediate invariants are not kept)
- less flexibility in the iteration scheme
  (iteration order, widening points, etc.)

Concurrent program semantics

Labelled transition systems

Labelled transition system: $(\Sigma, \mathcal{A}, \tau, \mathcal{I})$

- Σ: set of program states
- $\mathcal{A}$: set of actions
- $\tau \subseteq \Sigma \times \mathcal{A} \times \Sigma$: transition relation
  we note $(\sigma, a, \sigma') \in \tau$ as $\sigma \xrightarrow{a}_\tau \sigma'$
- $\mathcal{I} \subseteq \Sigma$: set of initial states

Labelled traces: sequences of states interspersed with actions

denoted as $\sigma_0 \xrightarrow{a_0} \sigma_1 \xrightarrow{a_1} \cdots \sigma_n \xrightarrow{a_n} \sigma_{n+1}$

From concurrent programs to labelled transition systems

Notations:

- concurrent program:
  $\texttt{parprog} ::= {}^{\ell^i_1}\texttt{prog}_1{}^{\ell^x_1} \mid\mid \cdots \mid\mid {}^{\ell^i_n}\texttt{prog}_n{}^{\ell^x_n}$
- threads identifiers: $\mathbb{T} \stackrel{\text{def}}{=} \{1, \ldots, n\}$

Program states: $\Sigma \stackrel{\text{def}}{=} ((\mathbb{T} \to \mathcal{L}) \times \mathcal{E}) \cup \Omega$

- a control state $L(t) \in \mathcal{L}$ for each thread $t \in \mathbb{T}$ and
- a single shared memory state $\rho \in \mathcal{E}$
- or an error state $\omega \in \Omega$

Initial states:

threads start at their first control point ℓ^i_t, variables are set to 0:
$\mathcal{I} \stackrel{\text{def}}{=} \{ (\lambda t.\ell^i_t, \lambda \mathbb{V}.0) \}$

Actions: thread identifiers: $\mathcal{A} \stackrel{\text{def}}{=} \mathbb{T}$

From concurrent programs to labelled transition systems

Transition relation: $\tau \subseteq \Sigma \times \mathcal{A} \times \Sigma$

$$(L,\rho) \xrightarrow{t}_{\tau} (L',\rho') \overset{\text{def}}{\iff} (L(t),\rho) \rightarrow_{\tau[\mathtt{prog}_t]} (L'(t),\rho') \wedge \forall u \neq t\colon L(u) = L'(u)$$

$$(L,\rho) \xrightarrow{t}_{\tau} \omega \overset{\text{def}}{\iff} (L(t),\rho) \rightarrow_{\tau[\mathtt{prog}_t]} \omega$$

- based on the transition relation of individual threads seen as sequential processes $\mathtt{prog}_t$:
  $\tau[\mathtt{prog}] \subseteq (\mathcal{L} \times \mathcal{E}) \times ((\mathcal{L} \times \mathcal{E}) \cup \Omega)$
  - choose a thread t to run
  - update ρ and $L(t)$
  - leave $L(u)$ intact for $u \neq t$

  (See course 3 for the full definition of $\tau[\mathtt{prog}]$.)

- each $\sigma \to \sigma'$ in $\tau[\mathtt{prog}_t]$ leads to many transitions in τ!

Interleaved trace semantics

Maximal and finite prefix trace semantics are as before:

Blocking states: $\mathcal{B} \stackrel{\text{def}}{=} \{\, \sigma \mid \forall \sigma' : \forall t : \sigma \stackrel{t}{\not\to}_\tau \sigma' \,\}$

Maximal traces: $\mathcal{M}_\infty$ (finite or infinite)

$$\mathcal{M}_\infty \stackrel{\text{def}}{=} \{\, \sigma_0 \stackrel{t_0}{\to} \cdots \stackrel{t_{n-1}}{\to} \sigma_n \mid n \geq 0 \wedge \sigma_0 \in \mathcal{I} \wedge \sigma_n \in \mathcal{B} \wedge \forall i < n : \sigma_i \stackrel{t_i}{\to}_\tau \sigma_{i+1} \,\} \cup \{\, \sigma_0 \stackrel{t_0}{\to} \sigma_1 \ldots \mid n \geq 0 \wedge \sigma_0 \in \mathcal{I} \wedge \forall i < \omega : \sigma_i \stackrel{t_i}{\to}_\tau \sigma_{i+1} \,\}$$

Finite prefix traces: $\mathcal{T}_p$

$$\mathcal{T}_p \stackrel{\text{def}}{=} \{\, \sigma_0 \stackrel{t_0}{\to} \cdots \stackrel{t_{n-1}}{\to} \sigma_n \mid n \geq 0 \wedge \sigma_0 \in \mathcal{I} \wedge \forall i < n : \sigma_i \stackrel{t_i}{\to}_\tau \sigma_{i+1} \,\}$$

Fixpoint form: $\mathcal{T}_p = \text{lfp}\, F_p$ where

$$F_p(X) = \mathcal{I} \cup \{\, \sigma_0 \stackrel{t_0}{\to} \cdots \stackrel{t_n}{\to} \sigma_{n+1} \mid n \geq 0 \wedge \sigma_0 \stackrel{t_0}{\to} \cdots \stackrel{t_{n-1}}{\to} \sigma_n \in X \wedge \sigma_n \stackrel{t_n}{\to}_\tau \sigma_{n+1} \,\}$$

Abstraction: $\mathcal{T}_p = \alpha_{*\preceq}(\mathcal{M}_\infty)$

Fairness

Fairness conditions: avoid threads being denied to run

Given $\mathit{enabled}(\sigma, t) \overset{\text{def}}{\iff} \exists \sigma' \in \Sigma: \sigma \xrightarrow{t}_{\tau} \sigma'$,
an infinite trace $\sigma_0 \xrightarrow{t_0} \cdots \sigma_n \xrightarrow{t_n} \cdots$ is:

- weakly fair if $\forall t \in \mathbb{T}$:
  $$(\exists i: \forall j \geq i: \mathit{enabled}(\sigma_j, t)) \implies (\forall i: \exists j \geq i: a_j = t)$$
  (no thread can be continuously enabled without running)
- strongly fair if $\forall t \in \mathbb{T}$:
  $$(\forall i: \exists j \geq i: \mathit{enabled}(\sigma_j, t)) \implies (\forall i: \exists j \geq i: a_j = t)$$
  (no thread can be infinitely often enabled without running)

Proofs under fairness conditions given:

- the maximal traces $\mathcal{M}_\infty$ of a program
- a property X to prove (as a set of traces)
- the set F of all (weakly or strongly) fair and of finite traces

$\implies$ prove $\mathcal{M}_\infty \cap F \subseteq X$ instead of $\mathcal{M}_\infty \subseteq X$

Fairness (cont.)

Example: `while x` $\geq$ `0 do x:=x+1 done` $||$ `x:=-1`

- may not terminate without fairness
- always terminates under weak and strong fairness

Finite prefix trace abstraction

$\mathcal{M}_\infty \cap F \subseteq X$ is abstracted into testing $\alpha_{*\preceq}(\mathcal{M}_\infty \cap F) \subseteq \alpha_{*\preceq}(X)$

for all fairness conditions F, $\alpha_{*\preceq}(\mathcal{M}_\infty \cap F) = \alpha_{*\preceq}(\mathcal{M}_\infty) = \mathcal{T}_p$

$\Longrightarrow$ fairness-dependent properties cannot be proved with finite prefixes only

In the following, we ignore fairness conditions.

(see [Cous85])

Equational state semantics

State abstraction $\mathcal{R}$: as before

- $\mathcal{R} \stackrel{\text{def}}{=} \{\, \sigma \mid \exists n \geq 0, \sigma_0 \stackrel{t_0}{\rightarrow} \cdots \sigma_n \colon \sigma_0 \in \mathcal{I}\ \forall i < n \colon \sigma_i \stackrel{t_i}{\rightarrow}_\tau \sigma_{i+1} \wedge \sigma = \sigma_n \,\}$
- $\mathcal{R} = \alpha_p(\mathcal{T}_p)$ where $\alpha_p(X) \stackrel{\text{def}}{=} \{\, \sigma \mid \exists n \geq 0, \sigma_0 \stackrel{t_0}{\rightarrow} \cdots \sigma_n \in X \colon \sigma = \sigma_n \,\}$
- $\mathcal{R} = \text{lfp}\, F_\mathcal{R}$ where $F_\mathcal{R}(X) = \mathcal{I} \cup \{\, \sigma \mid \exists \sigma' \in X, t \in \mathbb{T} \colon \sigma' \stackrel{t}{\rightarrow}_\tau \sigma \,\}$

Equational form: (without handling errors in Ω)

- for each $L \in \mathbb{T} \rightarrow \mathcal{L}$, a variable $\mathcal{X}_L$ with value in $\mathcal{E}$
- equations are derived from thread equations $eq(\texttt{prog}_t)$ as:

$$\begin{aligned}\mathcal{X}_{L_1} = \bigcup_{t \in \mathbb{T}} \{\, & F(\mathcal{X}_{L_2}, \ldots, X_{L_N}) \mid \\ & \exists (\mathcal{X}_{\ell_1} = F(\mathcal{X}_{\ell_2}, \ldots, \mathcal{X}_{\ell_N})) \in eq(\texttt{prog}_t) \colon \\ & \forall i \leq N \colon L_i(t) = \ell_i,\ \forall u \neq t \colon L_i(u) = L_1(u) \,\}\end{aligned}$$

Join with $\cup$ equations from $eq(\texttt{prog}_t)$ updating a single thread $t \in \mathbb{T}$.

(See course 3 for the full definition of $eq(\texttt{prog})$.)

Equational state semantics (example)

Example: inferring $0 \leq x \leq y \leq 102$

t_1	t_2
`while` $^{\ell 1}0 = 0$ `do`$^{\ell 2}$	`while` $^{\ell 4}0 = 0$ `do`$^{\ell 5}$
`if x<y then`	`if y<100 then`
$^{\ell 3}$`x:=x+1`	$^{\ell 6}$`y:=y+[1,3]`
`fi`	`fi`
`done`	`done`

Equational state semantics (example)

Example: inferring $0 \leq x \leq y \leq 102$

t_1	t_2
`while` $^{\ell 1}0 = 0$ `do`$^{\ell 2}$	`while` $^{\ell 4}0 = 0$ `do`$^{\ell 5}$
`if x<y then`	`if y<100 then`
$^{\ell 3}$`x:=x+1`	$^{\ell 6}$`y:=y+[1,3]`
`fi`	`fi`
`done`	`done`

(Simplified) equation system:

$$
\begin{aligned}
\mathcal{X}_{1,4} &= \mathcal{I} \cup \mathsf{C}[\![\, x := x + 1 \,]\!]\, \mathcal{X}_{3,4} \cup \mathsf{C}[\![\, x \geq y \,]\!]\, \mathcal{X}_{2,4} \\
&\quad \cup \mathsf{C}[\![\, y := y + [1,3] \,]\!]\, \mathcal{X}_{1,6} \cup \mathsf{C}[\![\, y \geq 100 \,]\!]\, \mathcal{X}_{1,5} \\
\mathcal{X}_{2,4} &= \mathcal{X}_{1,4} \cup \mathsf{C}[\![\, y := y + [1,3] \,]\!]\, \mathcal{X}_{2,6} \cup \mathsf{C}[\![\, y \geq 100 \,]\!]\, \mathcal{X}_{2,5} \\
\mathcal{X}_{3,4} &= \mathsf{C}[\![\, x < y \,]\!]\, \mathcal{X}_{2,4} \cup \mathsf{C}[\![\, y := y + [1,3] \,]\!]\, \mathcal{X}_{3,6} \cup \mathsf{C}[\![\, y \geq 100 \,]\!]\, \mathcal{X}_{3,5} \\
\mathcal{X}_{1,5} &= \mathsf{C}[\![\, x := x + 1 \,]\!]\, \mathcal{X}_{3,5} \cup \mathsf{C}[\![\, x \geq y \,]\!]\, \mathcal{X}_{2,5} \cup \mathcal{X}_{1,4} \\
\mathcal{X}_{2,5} &= \mathcal{X}_{1,5} \cup \mathcal{X}_{2,4} \\
\mathcal{X}_{3,5} &= \mathsf{C}[\![\, x < y \,]\!]\, \mathcal{X}_{2,5} \cup \mathcal{X}_{3,4} \\
\mathcal{X}_{1,6} &= \mathsf{C}[\![\, x := x + 1 \,]\!]\, \mathcal{X}_{3,6} \cup \mathsf{C}[\![\, x \geq y \,]\!]\, \mathcal{X}_{2,6} \cup \mathsf{C}[\![\, y < 100 \,]\!]\, \mathcal{X}_{1,5} \\
\mathcal{X}_{2,6} &= \mathcal{X}_{1,6} \cup \mathsf{C}[\![\, y < 100 \,]\!]\, \mathcal{X}_{2,5} \\
\mathcal{X}_{3,6} &= \mathsf{C}[\![\, x < y \,]\!]\, \mathcal{X}_{2,6} \cup \mathsf{C}[\![\, y < 100 \,]\!]\, \mathcal{X}_{3,5}
\end{aligned}
$$

Equational state semantics (example)

Example: inferring $0 \le x \le y \le 102$

t_1	t_2
`while` $^{\ell 1}0 = 0$ `do`$^{\ell 2}$	`while` $^{\ell 4}0 = 0$ `do`$^{\ell 5}$
`if x<y then`	`if y<100 then`
$^{\ell 3}$`x:=x+1`	$^{\ell 6}$`y:=y+[1,3]`
`fi`	`fi`
`done`	`done`

Pros:

- easy to construct
- easy to further abstract in an abstract domain $\mathcal{E}^\sharp$

Cons:

- explosion of the number of variables and equations
- explosion of the size of equations

  $\Longrightarrow$ efficiency issues
- the equation system does *not* reflect the program structure

  (not defined by induction on the concurrent program)

Wish-list

We would like to:

- keep information attached to syntactic program locations
  (control points in $\mathcal{L}$, not control point tuples in $\mathbb{T} \to \mathcal{L}$)
- be able to abstract away control information
  (precision/cost trade-off control)
- avoid duplicating thread instructions
- have a computation structure based on the program syntax
  (denotational style)

Ideally: thread-modular denotational-style semantics

(analyze each thread independently by induction on its syntax)

Detour through proof methods

Floyd–Hoare logic

Logic to prove properties about sequential programs [Hoar69].

Hoare triples: $\{P\}\,\texttt{prog}\,\{Q\}$

- annotate programs with logic assertions $\{P\}\,\texttt{prog}\,\{Q\}$
  (if P holds before prog, then Q holds after prog)
- check that $\{P\}\texttt{prog}\{Q\}$ is derivable with the following rules
  (the assertions are program invariants)

$$\frac{}{\{P[e/\texttt{X}]\}\ \texttt{X} := \texttt{e}\ \{P\}} \qquad \frac{\{P \wedge e \bowtie 0\}\ s\ \{Q\} \quad P \wedge e \not\bowtie 0 \Rightarrow Q}{\{P\}\ \texttt{if}\ e \bowtie 0\ \texttt{then}\ s\ \texttt{fi}\ \{Q\}}$$

$$\frac{\{P\}\ s_1\ \{Q\} \quad \{Q\}\ s_2\ \{R\}}{\{P\}\ s_1; s_2\ \{R\}} \qquad \frac{\{P \wedge e \bowtie 0\}\ s\ \{P\}}{\{P\}\ \texttt{while}\ e \bowtie 0\ \texttt{do}\ s\ \texttt{done}\ \{P \wedge e \not\bowtie 0\}}$$

$$\frac{\{P'\}\ s\ \{Q'\} \quad P \Rightarrow P' \quad Q' \Rightarrow Q}{\{P\}\ s\ \{Q\}}$$

Floyd–Hoare logic as abstract interpretation

Link with the equational state semantics:

Correspondence between ${}^{\ell}\mathtt{prog}^{\ell'}$ and $\{P\}\,\mathtt{prog}\,\{Q\}$:

- if P (resp. Q) models exactly the points in $\mathcal{X}_\ell$ (resp. $\mathcal{X}_{\ell'}$) then $\{P\}\,\mathtt{prog}\,\{Q\}$ is a derivable Hoare triple
- if $\{P\}\,\mathtt{prog}\,\{Q\}$ is derivable, then $\mathcal{X}_\ell \models P$ and $\mathcal{X}_{\ell'} \models Q$
  (all the points in $\mathcal{X}_\ell$ (resp. $\mathcal{X}_{\ell'}$) satisfy P (resp. Q))

$\Longrightarrow \mathcal{X}_\ell$ provide the most precise Hoare assertions in a constructive form

- $\gamma(\mathcal{X}^\sharp)$ provide (less precise) Hoare assertions in a computable form

Link with the denotational semantics:

both $\mathsf{C}[\![\, \mathtt{prog} \,]\!]$ and the proof tree for $\{P\}\,\mathtt{prog}\,\{Q\}$ reflect the syntactic structure of $\mathtt{prog}$

(compositional methods)

Owicki–Gries proof method

Extension of Floyd–Hoare to concurrent programs [Owic76].

Principle: add a new rule, for $||$

$$\frac{\{P_1\}\, s_1\, \{Q_1\} \quad \{P_2\}\, s_2\, \{Q_2\}}{\{P_1 \wedge P_2\}\, s_1 \,||\, s_2\, \{Q_1 \wedge Q_2\}}$$

Owicki–Gries proof method

Extension of Floyd–Hoare to concurrent programs [Owic76].

Principle: add a new rule, for $||$

$$\frac{\{P_1\}\, s_1\, \{Q_1\} \quad \{P_2\}\, s_2\, \{Q_2\}}{\{P_1 \wedge P_2\}\, s_1 \,||\, s_2\, \{Q_1 \wedge Q_2\}}$$

This rule is not always sound!

e.g., we have $\{\mathtt{X}=0,\mathtt{Y}=0\}\ \mathtt{X} := 1\ \{\mathtt{X}=1,\mathtt{Y}=0\}$

and $\{\mathtt{X}=0,\mathtt{Y}=0\}\ \mathtt{if}\ \mathtt{X}=0\ \mathtt{then}\ \mathtt{Y} := 1\ \mathtt{fi}\ \{\mathtt{X}=0,\mathtt{Y}=1\}$

but not $\{\mathtt{X}=0,\mathtt{Y}=0\}\ \mathtt{X} := 1 \,||\, \mathtt{if}\ \mathtt{X}=0\ \mathtt{then}\ \mathtt{Y} := 1\ \mathtt{fi}\ \{\mathit{false}\}$

$\Longrightarrow$ we need a side-condition to the rule:
$\{P_1\}\, s_1\, \{Q_1\}$ and $\{P_2\}\, s_2\, \{Q_2\}$ must not interfere

Owicki–Gries proof method (cont.)

interference freedom

given proofs Δ_1 and Δ_2 of $\{P_1\}\,s_1\,\{Q_1\}$ and $\{P_2\}\,s_2\,\{Q_2\}$

Δ_1 does not interfere with Δ_2 if:
for any Φ appearing before a statement in Δ_1
for any $\{P'_2\}\,s'_2\,\{Q'_2\}$ appearing in Δ_2
$\{\Phi \wedge P'_2\}\,s'_2\,\{\Phi\}$ holds
and moreover $\{Q_1 \wedge P'_2\}\,s'_2\,\{Q_1\}$

i.e.: the assertions used to prove $\{P_1\}\,s_1\,\{Q_1\}$ are stable by s_2

e.g., $\{X = 0, Y \in [0,1]\}\ X := 1\ \{X = 1, Y \in [0,1]\}$
$\{X \in [0,1], Y = 0\}\ \text{if } X = 0 \text{ then } Y := 1 \text{ fi}\ \{X \in [0,1], Y \in [0,1]\}$
$\Longrightarrow$ $\{X = 0, Y = 0\}\ X := 1 \mid\mid \text{if } X = 0 \text{ then } Y := 1 \text{ fi}\ \{X = 1, Y \in [0,1]\}$

Summary:

- pros: the invariants are local to threads
- cons: the proof is not compositional

  (proving one thread requires delving into the proof of other threads)

$\Longrightarrow$ not satisfactory

Jones' rely-guarantee proof method

Idea: explicit interferences with (more) annotations [Jone81].

Rely-guarantee "quintuples": $R, G \vdash \{P\}\,\mathtt{prog}\,\{Q\}$

- if P is true before `prog` is executed
- and the effect of other threads is included in R (rely)
- then Q is true after `prog`
- and the effect of `prog` is included in G (guarantee)

where:

- P and Q are assertions on states (in $\mathcal{P}(\Sigma)$)
- R and G are assertions on transitions (in $\mathcal{P}(\Sigma \times \mathcal{A} \times \Sigma)$)

The parallel composition rule becomes:

$$\frac{R \vee G_2, G_1 \vdash \{P_1\}\, s_1\, \{Q_1\} \quad R \vee G_1, G_2 \vdash \{P_2\}\, s_2\, \{Q_2\}}{R, G_1 \vee G_2 \vdash \{P_1 \wedge P_2\}\, s_1 \mid\mid s_2\, \{Q_1 \wedge Q_2\}}$$

Rely-guarantee example

Example: proving $0 \le x \le y \le 102$

checking t_1

```
while ℓ1 0 = 0 do ℓ2
  if x<y then
    ℓ3 x:=x+1
  fi
done
```

at $\ell 1, \ell 2$: $x, y \in [0, 102]$, $x \le y$
at $\ell 3$: $x \in [0, 101]$, $y \in [1, 102]$, $x < y$

checking t_2

```
while ℓ4 0 = 0 do ℓ5
  if y<100 then
    ℓ6 y:=y+[1,3]
  fi
done
```

at $\ell 4, \ell 5$: $x, y \in [0, 102]$, $x \le y$
at $\ell 6$: $x \in [0, 99]$, $y \in [0, 99]$, $x \le y$

Rely-guarantee example

Example: proving $0 \leq x \leq y \leq 102$

checking t_1

```
while ℓ1 0 = 0 do ℓ2
  if x<y then
    ℓ3 x:=x+1
  fi
done
```

x unchanged
y incremented
$0 \leq y \leq 102$

at $\ell 1, \ell 2$: $x, y \in [0, 102]$, $x \leq y$
at $\ell 3$: $x \in [0, 101]$, $y \in [1, 102]$, $x < y$

checking t_2

y unchanged
$0 \leq x \leq y$

```
while ℓ4 0 = 0 do ℓ5
  if y<100 then
    ℓ6 y:=y+[1,3]
  fi
done
```

at $\ell 4, \ell 5$: $x, y \in [0, 102]$, $x \leq y$
at $\ell 6$: $x \in [0, 99]$, $y \in [0, 99]$, $x \leq y$

In this example:

- guarantee exactly what is relied on $(R_1 = G_1$ and $R_2 = G_2)$
- rely and guarantee are global assertions

Benefits of rely-guarantee:

- invariants are still local to threads
- checking a thread does not require looking at other threads, only at an abstraction of their semantics

Auxiliary variables

Example

t_1	t_2
$^{\ell 1}$ $\texttt{x} := \texttt{x} + 1$ $^{\ell 2}$	$^{\ell 3}$ $\texttt{x} := \texttt{x} + 1$ $^{\ell 4}$

Goal: prove $\{\texttt{x} = 0\}\ t_1 \mid\mid t_2\ \{\texttt{x} = 2\}$.

Auxiliary variables

Example

t_1	t_2
$^{\ell 1}$ $\texttt{x} := \texttt{x} + 1$ $^{\ell 2}$	$^{\ell 3}$ $\texttt{x} := \texttt{x} + 1$ $^{\ell 4}$

Goal: prove $\{\texttt{x} = 0\}\ t_1 \mid\mid t_2\ \{\texttt{x} = 2\}$.

we must rely on and guarantee that
each thread increments x exactly once!

Solution: auxiliary variables
do not change the semantics but store extra information:

- past values of variables (history of the computation)
- program counter of other threads (pc_t)

Example: for t_1:
$$\{(pc_2 = \ell 3 \wedge \texttt{x} = 0) \vee (pc_2 = \ell 4 \wedge \texttt{x} = 1)\}$$
$$\texttt{x} := \texttt{x} + 1$$
$$\{(pc_2 = \ell 3 \wedge \texttt{x} = 1) \vee (pc_2 = \ell 4 \wedge \texttt{x} = 2)\}$$

Rely-guarantee as abstract interpretation

Local states

State projection: on a thread $t \in \mathbb{T}$

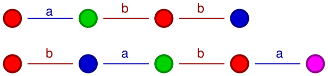
- add auxiliary variables $\mathbb{V}_t \stackrel{\text{def}}{=} \mathbb{V} \cup \{\, pc_{t'} \mid t' \in \mathbb{T},\ t' \neq t \,\}$
- enriched environments for t: $\mathcal{E}_t \stackrel{\text{def}}{=} \mathbb{V}_t \to \mathbb{R}$
  (for simplicity, $pc_{t'}$ are numeric variables, i.e., $\mathcal{L} \subseteq \mathbb{R}$)
- local states: $\Sigma_t \stackrel{\text{def}}{=} (\mathcal{L} \times \mathcal{E}_t) \cup \Omega$

  recall that $\Sigma \stackrel{\text{def}}{=} ((\mathbb{T} \to \mathcal{L}) \times \mathcal{E}) \cup \Omega$
  Σ_t has a simpler, sequential control state
- projection: $\pi_t(L, \rho) \stackrel{\text{def}}{=} (L(t), \rho[\forall t' \neq t\colon pc_{t'} \mapsto L(t')])$

  from Σ to Σ_t: shift control state to auxiliary variables
  extended naturally to $\pi_t : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma_t)$

  π_t is a bijection, no information is lost

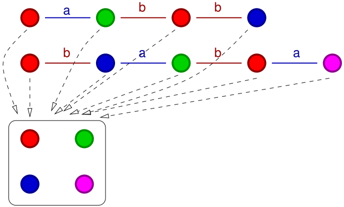
Local invariants



Abstraction steps to local reachable states:

- concrete (prefix) labelled trace semantics: $\mathcal{T}_p$

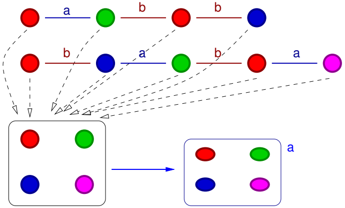
Local invariants



Abstraction steps to local reachable states:

- concrete (prefix) labelled trace semantics: $\mathcal{T}_p$
- state reachability abstraction: $\mathcal{R} = \alpha_p(\mathcal{T}_p) \in \mathcal{P}(\Sigma)$

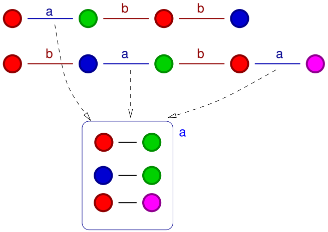
Local invariants



Abstraction steps to local reachable states:

- concrete (prefix) labelled trace semantics: $\mathcal{T}_p$
- state reachability abstraction: $\mathcal{R} = \alpha_p(\mathcal{T}_p) \in \mathcal{P}(\Sigma)$
- local state reachability: $\mathcal{R}l(t) \stackrel{\text{def}}{=} \pi_t(\mathcal{R}) \in \mathcal{P}(\Sigma_t)$
  thread's view of reachable states

Interferences



Interference: $A \in \mathbb{T} \rightarrow \mathcal{P}(\Sigma \times \Sigma)$ caused by a thread $t \in \mathbb{T}$

$A(t) \stackrel{\text{def}}{=} \alpha^{itf}(\mathcal{T}_p)(t)$

where $\alpha^{itf}(X)(t) \stackrel{\text{def}}{=} \{\, (\sigma, \sigma') \,|\, \exists \cdots \sigma \stackrel{t}{\rightarrow} \sigma' \cdots \in X \,\}$

Subset of the transition system τ:

- spawned by t
- and actually observed in some execution trace (in $\mathcal{T}_p$)

Fixpoint form

Local state fixpoint:

- we express $\mathcal{R}I(t)$ as a function of A and thread $t \in \mathbb{T}$:

$$\mathcal{R}I(t) = \text{lfp}\, R_t(A) \text{ where}$$

$$R_t : (\mathbb{T} \to \mathcal{P}(\Sigma \times \Sigma)) \to \mathcal{P}(\Sigma_t) \to \mathcal{P}(\Sigma_t)$$

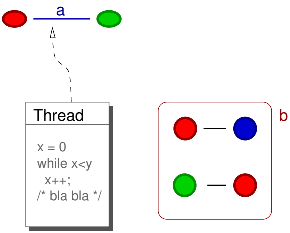
$$R_t(Y)(X) \stackrel{\text{def}}{=} \pi_t(\mathcal{I}) \cup \{\, \pi_t(\sigma') \mid \exists \pi_t(\sigma) \in X : \sigma \xrightarrow{t}_\tau \sigma' \lor \exists u \neq t : (\sigma, \sigma') \in Y(u) \,\}$$

A state is reachable if it is initial,
or reachable by transitions from t or from the environment A.

R_t only looks into the syntax of thread t.
R_t is parameterized by the interferences from other threads Y.

Fixpoint form

Local state fixpoint: illustration

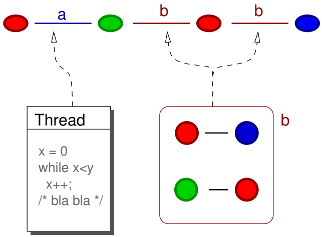


lfp $R_t(A)$ interleaves:

- transitions in π_t from thread t

Fixpoint form

Local state fixpoint: illustration

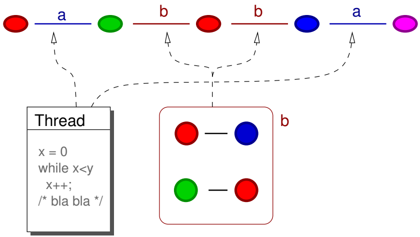


lfp $R_t(A)$ interleaves:

- transitions in π_t from thread t
- transitions in A from interferences

Fixpoint form

Local state fixpoint: illustration

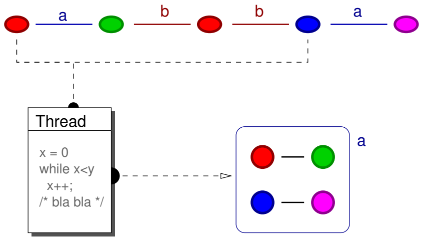


lfp $R_t(A)$ interleaves:

- transitions in π_t from thread t
- transitions in A from interferences

Fixpoint form (cont.)

Interferences:



- we express $A(t)$ as a function of $\mathcal{R}I$ and thread $t \in \mathbb{T}$:

$$A(t) = B(\mathcal{R}I)(t) \text{ where}$$

$$B : (\prod_{t \in \mathbb{T}} \{t\} \to \mathcal{P}(\Sigma_t)) \to \mathbb{T} \to \mathcal{P}(\Sigma \times \Sigma)$$

$$B(Z)(t) \stackrel{\text{def}}{=} \{\, (\sigma, \sigma') \,|\, \pi_t(\sigma) \in Z(t) \wedge \sigma \xrightarrow{t}_\tau \sigma' \,\}$$

Collect transitions starting from reachable states.

No fixpoint needed.

Fixpoint form (cont.)

Nested fixpoint characterization:

1. $\mathcal{R}I(t) = \text{lfp } R_t(A)$
2. $A(t) = B(\mathcal{R}I)(t)$
3. mutual dependency between $\mathcal{R}I$ and A

Fixpoint form (cont.)

Nested fixpoint characterization:

1. $\mathcal{RI}(t) = \text{lfp } R_t(A)$
2. $A(t) = B(\mathcal{RI})(t)$
3. mutual dependency between $\mathcal{RI}$ and A
   $\implies$ solved using a fixpoint:

   $\mathcal{RI} = \text{lfp } H$ where

   $H : (\prod_{t \in \mathbb{T}} \{t\} \to \mathcal{P}(\Sigma_t)) \to (\prod_{t \in \mathbb{T}} \{t\} \to \mathcal{P}(\Sigma_t))$

   $H(Z)(t) \stackrel{\text{def}}{=} \text{lfp } R_t(B(Z))$

Fixpoint form (cont.)

Nested fixpoint characterization:

1. $\mathcal{R}I(t) = \text{lfp } R_t(A)$
2. $A(t) = B(\mathcal{R}I)(t)$
3. mutual dependency between $\mathcal{R}I$ and A
   $\Longrightarrow$ solved using a fixpoint:
   $\mathcal{R}I = \text{lfp } H$ where
   $H : (\prod_{t \in \mathbb{T}} \{t\} \to \mathcal{P}(\Sigma_t)) \to (\prod_{t \in \mathbb{T}} \{t\} \to \mathcal{P}(\Sigma_t))$
   $H(Z)(t) \stackrel{\text{def}}{=} \text{lfp } R_t(B(Z))$

Completeness: $\forall t: \mathcal{R}I(t) \simeq \mathcal{R}$ (π_t is bijective thanks to auxiliary variables)

Fixpoint form (cont.)

Constructive fixpoint form:

Use Kleene's iteration to construct fixpoints:

- $\mathcal{R}I = \text{lfp } H = \bigsqcup_{n\in\mathbb{N}} H^n(\lambda t.\emptyset)$
  in the pointwise powerset lattice $\prod_{t\in\mathbb{T}} \{t\} \to \mathcal{P}(\Sigma_t)$
- $H(Z)(t) = \text{lfp } R_t(B(Z)) = \bigcup_{n\in\mathbb{N}}(R_t(B(Z)))^n(\emptyset)$
  in the powerset lattice $\mathcal{P}(\Sigma_t)$

  (similar to the sequential semantics of thread t in isolation)

$\Longrightarrow$ nested iterations

Abstract rely-guarantee

Suggested algorithm: nested iterations with acceleration

once abstract domains for states and interferences are chosen

- start from $\mathcal{RI}_0^\sharp \stackrel{\text{def}}{=} A_0^\sharp \stackrel{\text{def}}{=} \lambda t.\bot^\sharp$
- while $A_n^\sharp$ is not stable
  - compute $\forall t \in \mathbb{T}: \mathcal{RI}_{n+1}^\sharp(t) \stackrel{\text{def}}{=} \text{lfp } R_t^\sharp(A_n^\sharp)$
    by iteration with widening $\triangledown$

    ($\simeq$ separate analysis of each thread)
  - compute $A_{n+1}^\sharp \stackrel{\text{def}}{=} A_n^\sharp \triangledown B^\sharp(\mathcal{RI}_{n+1}^\sharp)$
- when $A_n^\sharp = A_{n+1}^\sharp$, return $\mathcal{RI}_n^\sharp$

$\Longrightarrow$ thread-modular analysis
parameterized by abstract domains
able to easily reuse existing sequential analyses

Flow-insensitive abstraction

Idea:

- reduce as much control information as possible
- but keep flow-sensitivity on each thread's control location

Local state abstraction: remove auxiliary variables

$$\alpha^{nf}_{\mathcal{R}} : \mathcal{P}(\Sigma_t) \to \mathcal{P}((\mathcal{L} \times \mathcal{E}) \cup \Omega)$$
$$\alpha^{nf}_{\mathcal{R}}(X) \stackrel{\text{def}}{=} \{ (\ell, \rho_{|\mathbb{V}}) \mid (\ell, \rho) \in X \} \cup (X \cap \Omega)$$

Interference abstraction: remove all control state

$$\alpha^{nf}_{A} : \mathcal{P}(\Sigma \times \Sigma) \to \mathcal{P}(\mathcal{E} \times \mathcal{E})$$
$$\alpha^{nf}_{A}(Y) \stackrel{\text{def}}{=} \{ (\rho, \rho') \mid \exists L, L' \in \mathbb{T} \to \mathcal{L} : ((L, \rho), (L', \rho')) \in Y \}$$

Flow-insensitive abstraction (cont.)

Flow-insensitive fixpoint semantics: (omitting errors Ω)

We apply $\alpha^{nf}_{\mathcal{R}}$ and α^{nf}_{A} to the nested fixpoint semantics.

$\mathcal{RI}^{nf} \stackrel{\text{def}}{=} \text{lfp}\, \lambda Z.\lambda t.\, \text{lfp}\, {R^{nf}}_t(B^{nf}(Z))$, where

$B^{nf}(Z)(t) \stackrel{\text{def}}{=} \{\, (\rho, \rho') \mid \exists \ell, \ell' \in \mathcal{L}\colon (\ell, \rho) \in Z(t) \wedge (\ell, \rho) \to_t (\ell', \rho') \,\}$
(extract interferences from reachable states)

$R^{nf}_t(Y)(X) \stackrel{\text{def}}{=} R^{loc}_t(X) \cup A^{nf}_t(Y)(X)$ (interleave steps)

$R^{loc}_t(X) \stackrel{\text{def}}{=} \{(\ell^i_t, \lambda \mathbb{V}.0)\} \cup \{\, (\ell', \rho') \mid \exists (\ell, \rho) \in X\colon (\ell, \rho) \to_t (\ell', \rho') \,\}$ (thread step)

$A^{nf}_t(Y)(X) \stackrel{\text{def}}{=} \{\, (\ell, \rho') \mid \exists \rho,\, u \neq t\colon (\ell, \rho) \in X \wedge (\rho, \rho') \in Y(u) \,\}$ (interference step)

where $\to_t$ is the transition relation for thread t alone: $\tau[\texttt{prog}_t]$

Cost/precision trade-off:

- less variables
  $\Longrightarrow$ subsequent numeric abstractions are more efficient
- sufficient to analyze our first example (slide 26)
- insufficient to analyze $\texttt{x} := \texttt{x} + 1 \,||\, \texttt{x} := \texttt{x} + 1$ (slide 35)

Non-relational interference abstraction

Idea: simplify further flow-insensitive interferences

- numeric relations are more costly than numeric sets
  $\implies$ remove input sensitivity
- relational domains are more costly than non-relational
  $\implies$ abstract the interference on each variable separately

Non-relational interference abstraction:

$$\alpha_A^{nr} : \mathcal{P}(\mathcal{E} \times \mathcal{E}) \to (\mathbb{V} \to \mathcal{P}(\mathbb{R}))$$

$$\alpha_A^{nr}(Y) \stackrel{\text{def}}{=} \lambda \mathtt{V}.\{\, x \in \mathbb{V} \mid \exists (\rho, \rho') \in Y : \rho(\mathtt{V}) \neq x \wedge \rho'(\mathtt{V}) = x \,\}$$

(remember which variables are modified and their new values)

To apply interferences, we get, in the nested fixpoint form:

$$A_t^{nr}(Y)(X) \stackrel{\text{def}}{=} \{\, (\ell, \rho[\mathtt{V} \mapsto v]) \mid (\ell, \rho) \in X, \mathtt{V} \in \mathbb{V}, \exists u \neq t : v \in Y(u)(\mathtt{V}) \,\}$$

A note on unbounded threads

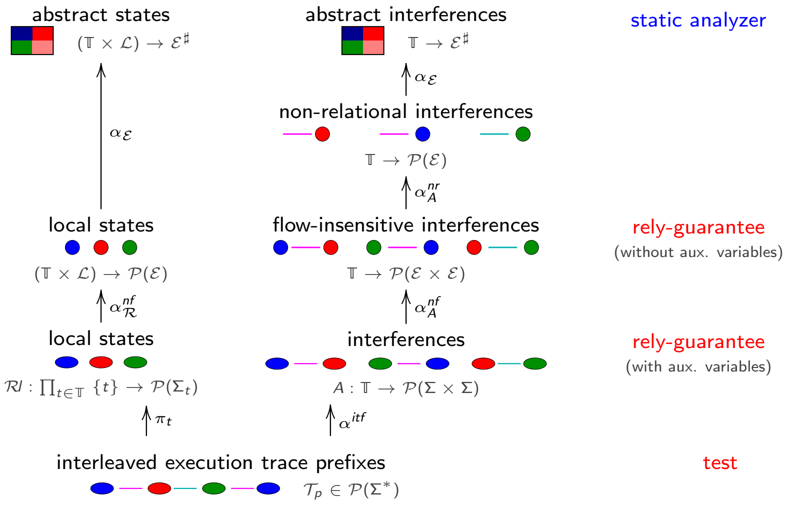
Extension: relax the finiteness constraint on $\mathbb{T}$

- there is still a finite syntactic set of threads $\mathbb{T}_s$
- some threads $\mathbb{T}_\infty \subseteq \mathbb{T}_s$ can have several instances
  (possibly an unbounded number)

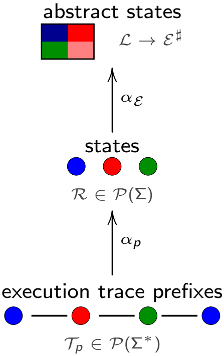
Flow-insensitive analysis:

- local state and interference domains have finite dimensions
  ($\mathcal{E}_t$ and $(\mathcal{L} \times \mathcal{E}) \times (\mathcal{L} \times \mathcal{E})$, as opposed to $\mathcal{E}$ and $\mathcal{E} \times \mathcal{E}$)
- all instances of a thread $t \in \mathbb{T}_s$ are isomorphic
  $\Longrightarrow$ iterate the analysis on the finite set $\mathbb{T}_s$ (instead of $\mathbb{T}$)
- we must handle self-interferences for threads in $\mathbb{T}_\infty$:
  $A_t^{nf}(Y)(X) \stackrel{\text{def}}{=}$
  $\{\, (\ell, \rho') \mid \exists \rho,\, u: (u \neq t \vee t \in \mathbb{T}_\infty) \wedge (\ell, \rho) \in X \wedge (\rho, \rho') \in Y(u) \,\}$

From traces to thread-modular analyses



Compare with sequential analyses



static analyzer

reachability

test

Construction of an interference-based analysis

Reminder: sequential analysis in denotational form

Expression semantics: $\mathsf{E}[\![\, \texttt{exp} \,]\!] : \mathcal{E} \to (\mathcal{P}(\mathbb{R}) \times \mathcal{P}(\Omega))$

$$\mathsf{E}[\![\, \texttt{X} \,]\!]\, \rho \stackrel{\text{def}}{=} \langle \{ \rho(\texttt{X}) \}, \emptyset \rangle$$

$$\mathsf{E}[\![\, [c_1, c_2] \,]\!]\, \rho \stackrel{\text{def}}{=} \langle \{ x \in \mathbb{R} \mid c_1 \leq x \leq c_2 \}, \emptyset \rangle$$

$$\mathsf{E}[\![\, -e \,]\!]\, \rho \stackrel{\text{def}}{=} \text{let } \langle V, O \rangle = \mathsf{E}[\![\, e \,]\!]\, \rho \text{ in } \langle \{ -v \mid v \in V \}, O \rangle$$

$$\mathsf{E}[\![\, e_1 \diamond_\omega e_2 \,]\!]\, \rho \stackrel{\text{def}}{=}$$
$$\quad \text{let } \langle V_1, O_1 \rangle = \mathsf{E}[\![\, e_1 \,]\!]\, \rho \text{ in}$$
$$\quad \text{let } \langle V_2, O_2 \rangle = \mathsf{E}[\![\, e_2 \,]\!]\, \rho \text{ in}$$
$$\quad \langle \{ v_1 \diamond v_2 \mid v_i \in V_i, \diamond \neq / \vee v_2 \neq 0 \}, O_1 \cup O_2 \cup \{ \omega \text{ if } \diamond = / \wedge 0 \in V_2 \} \rangle$$

Statement semantics: $\mathsf{C}[\![\, \texttt{prog} \,]\!] : (\mathcal{P}(\mathcal{E}) \times \mathcal{P}(\Omega)) \to (\mathcal{P}(\mathcal{E}) \times \mathcal{P}(\Omega))$

$$\mathsf{C}[\![\, \texttt{X} := \texttt{e} \,]\!] \langle R, O \rangle \stackrel{\text{def}}{=} \langle \emptyset, O \rangle \sqcup \bigsqcup_{\rho \in R} \langle \{ \rho[\texttt{X} \mapsto v] \mid v \in V_\rho \}, O_\rho \rangle$$

$$\mathsf{C}[\![\, e \bowtie 0? \,]\!] \langle R, O \rangle \stackrel{\text{def}}{=} \langle \emptyset, O \rangle \sqcup \bigsqcup_{\rho \in R} \langle \{ \rho \mid \exists v \in V_\rho : v \bowtie 0 \}, O_\rho \rangle$$

$$\mathsf{C}[\![\, \texttt{if } e \bowtie 0 \texttt{ then } s \texttt{ fi} \,]\!] X \stackrel{\text{def}}{=} (\mathsf{C}[\![\, s \,]\!] \circ \mathsf{C}[\![\, e \bowtie 0? \,]\!])X \sqcup \mathsf{C}[\![\, e \not\bowtie 0? \,]\!] X$$

$$\mathsf{C}[\![\, \texttt{while } e \bowtie 0 \texttt{ do } s \texttt{ done} \,]\!] X \stackrel{\text{def}}{=}$$
$$\quad \mathsf{C}[\![\, e \not\bowtie 0? \,]\!] (\text{lfp}\, \lambda Y. X \sqcup (\mathsf{C}[\![\, s \,]\!] \circ \mathsf{C}[\![\, e \bowtie 0? \,]\!]) Y)$$

$$\mathsf{C}[\![\, s_1;\, s_2 \,]\!] \stackrel{\text{def}}{=} \mathsf{C}[\![\, s_2 \,]\!] \circ \mathsf{C}[\![\, s_1 \,]\!]$$

where $\langle V_\rho, O_\rho \rangle \stackrel{\text{def}}{=} \mathsf{E}[\![\, e \,]\!]\, \rho$

Denotational semantics with interferences

Interferences in $\mathbb{I} \stackrel{\text{def}}{=} \mathbb{T} \times \mathbb{V} \times \mathbb{R}$

$\langle\, t,\, \mathtt{X},\, v \,\rangle$ means: t can store the value v into the variable $\mathtt{X}$

We define the analysis of a thread t
with respect to a set of interferences $I \subseteq \mathbb{I}$.

Expressions with interference: for thread t

$$\mathsf{E_t}[\![\, \mathrm{exp} \,]\!] : (\mathcal{E} \times \mathcal{P}(\mathbb{I})) \to (\mathcal{P}(\mathbb{R}) \times \mathcal{P}(\Omega))$$

- Apply interferences to read variables:
  $\mathsf{E_t}[\![\, \mathtt{X} \,]\!]\,\langle\, \rho,\, I \,\rangle \stackrel{\text{def}}{=} \langle\, \{\, \rho(\mathtt{X}) \,\} \cup \{\, v \mid \exists u \neq t \colon \langle\, u,\, \mathtt{X},\, v \,\rangle \in I \,\},\, \emptyset \,\rangle$
- Pass recursively I down to sub-expressions:
  $\mathsf{E_t}[\![\, -e \,]\!]\,\langle\, \rho,\, I \,\rangle \stackrel{\text{def}}{=}$
  $\quad \mathbf{let}\ \langle\, V,\, O \,\rangle = \mathsf{E_t}[\![\, e \,]\!]\,\langle\, \rho,\, I \,\rangle\ \mathbf{in}\ \langle\, \{\, -v \mid v \in V \,\},\, O \,\rangle$

  ...

Denotational semantics with interferences (cont.)

Statements with interference: for thread t

$$\mathsf{C_t}[\![\, \texttt{prog} \,]\!] : (\mathcal{P}(\mathcal{E}) \times \mathcal{P}(\Omega) \times \mathcal{P}(\mathbb{I})) \to (\mathcal{P}(\mathcal{E}) \times \mathcal{P}(\Omega) \times \mathcal{P}(\mathbb{I}))$$

- pass interferences to expressions
- collect new interferences due to assignments
- accumulate interferences from inner statements

$$\mathsf{C_t}[\![\, \texttt{X} := \texttt{e} \,]\!] \langle R, O, I \rangle \stackrel{\text{def}}{=}$$
$$\langle \emptyset, O, I \rangle \sqcup \bigsqcup_{\rho \in R} \langle \{ \rho[\texttt{X} \mapsto v] \mid v \in V_\rho \}, O_\rho, \{ \langle t, \texttt{X}, v \rangle \mid v \in V_\rho \} \rangle$$

$$\mathsf{C_t}[\![\, s_1;\, s_2 \,]\!] \stackrel{\text{def}}{=} \mathsf{C_t}[\![\, s_2 \,]\!] \circ \mathsf{C_t}[\![\, s_1 \,]\!]$$

...

noting $\langle V_\rho, O_\rho \rangle \stackrel{\text{def}}{=} \mathsf{E_t}[\![\, e \,]\!] \langle \rho, I \rangle$

$\sqcup$ is now the element-wise $\cup$ in $\mathcal{P}(\mathcal{E}) \times \mathcal{P}(\Omega) \times \mathcal{P}(\mathbb{I})$

Denotational semantics with interferences (cont.)

Program semantics: $\mathsf{P}[\![\,\texttt{parprog}\,]\!] \subseteq \Omega$

Given $\texttt{parprog} ::= \texttt{prog}_1 \mid\mid \cdots \mid\mid \texttt{prog}_n$, we compute:

$$\mathsf{P}[\![\,\texttt{parprog}\,]\!] \stackrel{\text{def}}{=} \left[\mathrm{lfp}\, \lambda\langle\, O,\, I\,\rangle. \bigsqcup_{t\in\mathbb{T}} \, [\mathsf{C_t}[\![\,\texttt{prog}_t\,]\!]\,\langle\, \mathcal{E}_0,\, \emptyset,\, I\,\rangle]_{\Omega,\mathbb{I}}\right]_{\Omega}$$

- each thread analysis starts in an initial environment set $\mathcal{E}_0 \stackrel{\text{def}}{=} \{\, \lambda \mathtt{V}.0\, \}$
- $[\mathtt{X}]_{\Omega,\mathbb{I}}$ projects $\mathtt{X} \in \mathcal{P}(\mathcal{E}) \times \mathcal{P}(\Omega) \times \mathcal{P}(\mathbb{I})$ on $\mathcal{P}(\Omega) \times \mathcal{P}(\mathbb{I})$ and interferences and errors from all threads are joined

  (the output environments are ignored)
- $\mathsf{P}[\![\,\texttt{parprog}\,]\!]$ only outputs the set of possible run-time errors

Example

Example

t_1	t_2
`while` $^{\ell 1}0=0$ `do`$^{\ell 2}$	`while` $^{\ell 4}0=0$ `do`$^{\ell 5}$
`if x<y then`	`if y<100 then`
$^{\ell 3}$`x:=x+1`	$^{\ell 6}$`y:=y+[1,3]`
`fi`	`fi`
`done`	`done`

Concrete interference semantics:

iteration 1

$I = \emptyset$

$\ell 1 : \mathrm{x} = 0, \mathrm{y} = 0$

$\ell 4 : \mathrm{x} = 0, \mathrm{y} \in [0, 102]$

new $I = \{ \langle t_2, \mathrm{y}, 1 \rangle, \ldots, \langle t_2, \mathrm{y}, 102 \rangle \}$

Example

Example

t_1	t_2
`while` $^{\ell 1}0 = 0$ `do`$^{\ell 2}$	`while` $^{\ell 4}0 = 0$ `do`$^{\ell 5}$
`if x<y then`	`if y<100 then`
$^{\ell 3}$`x:=x+1`	$^{\ell 6}$`y:=y+[1,3]`
`fi`	`fi`
`done`	`done`

Concrete interference semantics:

iteration 2

$I = \{ \langle t_2, \mathrm{y}, 1 \rangle, \ldots, \langle t_2, \mathrm{y}, 102 \rangle \}$

$\ell 1 : \mathrm{x} \in [0, 102], \mathrm{y} = 0$

$\ell 4 : \mathrm{x} = 0, \mathrm{y} \in [0, 102]$

new $I = \{ \langle t_1, \mathrm{x}, 1 \rangle, \ldots, \langle t_1, \mathrm{x}, 102 \rangle, \langle t_2, \mathrm{y}, 1 \rangle, \ldots, \langle t_2, \mathrm{y}, 102 \rangle \}$

Example

Example

t_1	t_2

while $^{\ell 1}0 = 0$ do$^{\ell 2}$
  if x<y then
    $^{\ell 3}$x:=x+1
  fi
done
	while $^{\ell 4}0 = 0$ do$^{\ell 5}$
	  if y<100 then
	    $^{\ell 6}$y:=y+[1,3]
	  fi
	done

Concrete interference semantics:

iteration 3

$I = \{ \langle t_1, \mathrm{x}, 1 \rangle, \ldots, \langle t_1, \mathrm{x}, 102 \rangle, \langle t_2, \mathrm{y}, 1 \rangle, \ldots, \langle t_2, \mathrm{y}, 102 \rangle \}$

$\ell 1 : \mathrm{x} \in [0, 102], \mathrm{y} = 0$

$\ell 4 : \mathrm{x} = 0, \mathrm{y} \in [0, 102]$

new $I = \{ \langle t_1, \mathrm{x}, 1 \rangle, \ldots, \langle t_1, \mathrm{x}, 102 \rangle, \langle t_2, \mathrm{y}, 1 \rangle, \ldots, \langle t_2, \mathrm{y}, 102 \rangle \}$

Example

Example

t_1	t_2

while ${}^{\ell 1}0 = 0$ do${}^{\ell 2}$
  if x<y then
    ${}^{\ell 3}$x:=x+1
  fi
done
	while ${}^{\ell 4}0 = 0$ do${}^{\ell 5}$
	  if y<100 then
	    ${}^{\ell 6}$y:=y+[1,3]
	  fi
	done

Concrete interference semantics:

iteration 3

$I = \{ \langle t_1, \mathrm{x}, 1 \rangle, \ldots, \langle t_1, \mathrm{x}, 102 \rangle, \langle t_2, \mathrm{y}, 1 \rangle, \ldots, \langle t_2, \mathrm{y}, 102 \rangle \}$

$\ell 1 : \mathrm{x} \in [0, 102], \mathrm{y} = 0$

$\ell 4 : \mathrm{x} = 0, \mathrm{y} \in [0, 102]$

new $I = \{ \langle t_1, \mathrm{x}, 1 \rangle, \ldots, \langle t_1, \mathrm{x}, 102 \rangle, \langle t_2, \mathrm{y}, 1 \rangle, \ldots, \langle t_2, \mathrm{y}, 102 \rangle \}$

Note: we don't get that $\mathrm{x} \leq \mathrm{y}$ at $\ell 1$, only that $\mathrm{x}, \mathrm{y} \in [0, 102]$

Interference abstraction

Abstract interferences $\mathbb{I}^\sharp$

$\mathcal{P}(\mathbb{I}) \stackrel{\text{def}}{=} \mathcal{P}(\mathbb{T} \times \mathbb{V} \times \mathbb{R})$ is abstracted as $\mathbb{I}^\sharp \stackrel{\text{def}}{=} (\mathbb{T} \times \mathbb{V}) \to \mathcal{R}^\sharp$
where $\mathcal{R}^\sharp$ abstracts $\mathcal{P}(\mathbb{R})$ (e.g. intervals)

Abstract semantics with interferences $\mathsf{C}_t^\sharp[\![\, s \,]\!]$

derived from $\mathsf{C}^\sharp[\![\, s \,]\!]$ in a generic way:

Example: $\mathsf{C}_t^\sharp[\![\, \mathtt{X} := \mathsf{e} \,]\!] \langle\, R^\sharp,\, \Omega,\, I^\sharp \,\rangle$

- for each $\mathtt{Y}$ in e, get its interference $\mathtt{Y}^\sharp_{\mathcal{R}} = \bigsqcup^\sharp_{\mathcal{R}} \{\, I^\sharp\langle\, u,\, \mathtt{Y} \,\rangle \mid u \neq t \,\}$
- if $\mathtt{Y}^\sharp_{\mathcal{R}} \neq \bot^\sharp_{\mathcal{R}}$, replace $\mathtt{Y}$ in e with $get\langle\, \mathtt{Y},\, R^\sharp \,\rangle \sqcup^\sharp_{\mathcal{R}} \mathtt{Y}^\sharp_{\mathcal{R}}$
  (where $get(\mathtt{Y}, R^\sharp)$ extracts the abstract values in $\mathcal{R}^\sharp$ of a variable $\mathtt{Y}$ from $R^\sharp \in \mathcal{E}^\sharp$)
- compute $\langle\, R^{\sharp\prime},\, O' \,\rangle = \mathsf{C}^\sharp[\![\, e \,]\!] \langle\, R^\sharp,\, O \,\rangle$
- enrich $I^\sharp\langle\, t,\, \mathtt{X} \,\rangle$ with $get(\mathtt{X}, R^{\sharp\prime})$

Static analysis with interferences

Abstract analysis

$$\mathsf{P}^\sharp[\![\,\texttt{parprog}\,]\!] \stackrel{\text{def}}{=} \left[\lim \lambda\langle\, O,\, I^\sharp\,\rangle.\langle\, O,\, I^\sharp\,\rangle \triangledown \bigsqcup^\sharp_{t\in\mathbb{T}} \left[\mathsf{C}^\sharp_t[\![\,\texttt{prog}_t\,]\!]\langle\, \mathcal{E}^\sharp_0,\, \emptyset,\, I^\sharp\,\rangle\right]_{\Omega,\mathbb{I}^\sharp}\right]_\Omega$$

- effective analysis by structural induction
- termination ensured by a widening
- parametrized by a choice of abstract domains $\mathcal{R}^\sharp$, $\mathcal{E}^\sharp$

- interferences are flow-insensitive and non-relational in $\mathcal{R}^\sharp$
- thread analysis remains flow-sensitive and relational in $\mathcal{E}^\sharp$

(reminder: $[X]_\Omega$, $[Y]_{\Omega,\mathbb{I}^\sharp}$ keep only X's component in Ω, Y's components in Ω and $\mathbb{I}^\sharp$)

Path-based semantics

Control paths

$atomic ::= \mathtt{X} := \mathtt{exp} \mid \mathtt{exp} \bowtie 0?$

Control paths

$\pi : \mathtt{prog} \to \mathcal{P}(atomic^*)$

$\pi(\mathtt{X} := e) \stackrel{\text{def}}{=} \{\, \mathtt{X} := e \,\}$

$\pi(\mathtt{if}\ e \bowtie 0\ \mathtt{then}\ s\ \mathtt{fi}) \stackrel{\text{def}}{=} (\{\, e \bowtie 0? \,\} \cdot \pi(s)) \cup \{\, e \not\bowtie 0? \,\}$

$\pi(\mathtt{while}\ e \bowtie 0\ \mathtt{do}\ s\ \mathtt{done}) \stackrel{\text{def}}{=} \left(\bigcup_{i \geq 0} (\{\, e \bowtie 0? \,\} \cdot \pi(s))^i\right) \cdot \{\, e \not\bowtie 0? \,\}$

$\pi(s_1;\, s_2) \stackrel{\text{def}}{=} \pi(s_1) \cdot \pi(s_2)$

$\pi(\mathtt{prog})$ is a (generally infinite) set of finite control paths

Path-based concrete semantics of sequential programs

Join-over-all-path semantics

$\Pi[\![P]\!] : (\mathcal{P}(\mathcal{E}) \times \mathcal{P}(\Omega)) \to (\mathcal{P}(\mathcal{E}) \times \mathcal{P}(\Omega)) \qquad P \subseteq atomic^*$

$$\Pi[\![P]\!]\langle R, O \rangle \stackrel{\text{def}}{=} \bigsqcup_{s_1 \cdot \ldots \cdot s_n \in P} (\mathsf{C}[\![s_n]\!] \circ \cdots \circ \mathsf{C}[\![s_1]\!])\langle R, O \rangle$$

Semantic equivalence

$$\mathsf{C}[\![\texttt{prog}]\!] = \Pi[\![\pi(\texttt{prog})]\!]$$

(not true in the abstract)

Advantages:

- easily extended to concurrent programs (path interleavings)
- able to model program transformations (weak memory models)

Path-based concrete semantics of concurrent programs

Concurrent control paths

$$\pi_* \stackrel{\text{def}}{=} \{ \text{interleavings of } \pi(\mathtt{prog}_t),\ t \in \mathbb{T} \}$$
$$= \{ p \in atomic^* \mid \forall t \in \mathbb{T},\ proj_t(p) \in \pi(\mathtt{prog}_t) \}$$

Interleaving program semantics

$$\mathsf{P}_*[\![\mathtt{parprog}]\!] \stackrel{\text{def}}{=} [\, \Pi[\![\pi_*]\!] \langle \mathcal{E}_0, \emptyset \rangle \,]_\Omega$$

($proj_t(p)$ keeps only the atomic statement in p coming from thread t)

Soundness of the interference semantics

Soundness theorem

$$\mathsf{P}_*[\![\, \mathtt{parprog} \,]\!] \subseteq \mathsf{P}[\![\, \mathtt{parprog} \,]\!]$$

Proof sketch:

- define $\Pi_t[\![\, P \,]\!]\, X \stackrel{\text{def}}{=} \bigsqcup \{ \mathsf{C_t}[\![\, s_1; \ldots; s_n \,]\!]\, X \mid s_1 \cdot \ldots \cdot s_n \in P \}$,
  then $\Pi_t[\![\, \pi(s) \,]\!] = \mathsf{C_t}[\![\, s \,]\!]$;
- given the interference fixpoint $I \subseteq \mathbb{I}$ from $\mathsf{P}[\![\, \mathtt{parprog} \,]\!]$,
  prove by recurrence on the length of $p \in \pi_*$ that:
  - $\forall t \in \mathbb{T}, \forall \rho \in [\Pi[\![\, p \,]\!]\langle\, \mathcal{E}_0,\, \emptyset \,\rangle]_{\mathcal{E}}$,
    $\exists \rho' \in [\Pi_t[\![\, proj_t(p) \,]\!]\langle\, \mathcal{E}_0,\, \emptyset,\, I \,\rangle]_{\mathcal{E}}$ such that
    $\forall \mathtt{X} \in \mathbb{V},\ \rho(\mathtt{X}) = \rho'(\mathtt{X})$ or $\langle\, u,\, \mathtt{X},\, \rho(\mathtt{X}) \,\rangle \in I$ for some $u \neq t$.
  - $[\Pi[\![\, p \,]\!]\langle\, \mathcal{E}_0,\, \emptyset \,\rangle]_\Omega \subseteq \bigcup_{t \in \mathbb{T}} [\Pi_t[\![\, proj_t(p) \,]\!]\langle\, \mathcal{E}_0,\, \emptyset,\, I \,\rangle]_\Omega$

Note: sound but not complete

Weakly consistent memories

Issues with weak consistency

program written

$F_1 := 1$;
`if` $F_2 = 0$ `then`
  S_1
`fi`
	$F_2 := 1$;
	`if` $F_1 = 0$ `then`
	  S_2
	`fi`

(simplified Dekker mutual exclusion algorithm)

S_1 and S_2 cannot execute simultaneously.

Issues with weak consistency

program written

$F_1 := 1$;
`if` $F_2 = 0$ `then`
  S_1
`fi`
	$F_2 := 1$;
	`if` $F_1 = 0$ `then`
	  S_2
	`fi`

$\longrightarrow$

program executed

`if` $F_2 = 0$ `then`
  $F_1 := 1$;
  S_1
`fi`
	`if` $F_1 = 0$ `then`
	  $F_2 := 1$;
	  S_2
	`fi`

(simplified Dekker mutual exclusion algorithm)

S_1 and S_2 can execute simultaneously.
Not a sequentially consistent behavior!

Caused by:

- write FIFOs, caches, distributed memory
- hardware or compiler optimizations, transformations
- ...

behavior accepted by Java [Mans05]

Out of thin air principle

original program

$R_1 := X;$	$R_2 := Y;$
$Y := R_1$	$X := R_2$

(example from causality test case #4 for Java by Pugh et al.)

We should not have $R_1 = 42$.

Out of thin air principle

original program

$R_1 := X;$ $Y := R_1$	$R_2 := Y;$ $X := R_2$

$\longrightarrow$

“optimized” program

$Y := 42;$ $R_1 := X;$ $Y := R_1$	$R_2 := Y;$ $X := R_2$

(example from causality test case #4 for Java by Pugh et al.)

We should not have $R_1 = 42$.

Possible if we allow speculative writes!
$\Longrightarrow$ we disallow this kind of program transformations.

(also forbidden in Java)

Atomicity and granularity

original program

$\texttt{X} := \texttt{X} + 1 \;\Big|\; \texttt{X} := \texttt{X} + 1$

We assumed that assignments are atomic. . .

Atomicity and granularity

original program

$\texttt{X} := \texttt{X} + 1 \mid \texttt{X} := \texttt{X} + 1$

$\longrightarrow$

executed program

$r_1 := \texttt{X} + 1$	$r_2 := \texttt{X} + 1$
$\texttt{X} := r_1$	$\texttt{X} := r_2$

We assumed that assignments are atomic...
but that may not be the case

The second program admits more behaviors
e.g.: $\texttt{X} = 1$ at the end of the program

[Reyn04]

Path-based definition of weak consistency

Acceptable control path transformations: $p \rightsquigarrow q$

only reduce interferences and errors

- Reordering: $\texttt{X}_1 := e_1 \cdot \texttt{X}_2 := e_2 \rightsquigarrow \texttt{X}_2 := e_2 \cdot \texttt{X}_1 := e_1$
  (if $\texttt{X}_1 \notin var(e_2)$, $\texttt{X}_2 \notin var(e_1)$, and e_1 does not stop the program)
- Propagation: $\texttt{X} := e \cdot s \rightsquigarrow \texttt{X} := e \cdot s[e/\texttt{X}]$
  (if $\texttt{X} \notin var(e)$, $var(e)$ are thread-local, and e is deterministic)
- Factorization: $s_1 \cdot \ldots \cdot s_n \rightsquigarrow \texttt{X} := e \cdot s_1[\texttt{X}/e] \cdot \ldots \cdot s_n[\texttt{X}/e]$
  (if $\texttt{X}$ is fresh, $\forall i$, $var(e) \cap lval(s_i) = \emptyset$, and e has no error)
- Decomposition: $\texttt{X} := e_1 + e_2 \rightsquigarrow \texttt{T} := e_1 \cdot \texttt{X} := \texttt{T} + e_2$
  (change of granularity)
- ...

but **NOT:**

- "out-of-thin-air" writes: $\texttt{X} := e \rightsquigarrow \texttt{X} := 42 \cdot \texttt{X} := e$

Soundness of the interference semantics

Interleaving semantics of transformed programs $\mathsf{P}'_*[\![\,\texttt{parprog}\,]\!]$

- $\pi'(s) \stackrel{\text{def}}{=} \{\, p \mid \exists p' \in \pi(s) \colon p' \rightsquigarrow^{*} p \,\}$
- $\pi'_* \stackrel{\text{def}}{=} \{\, \text{interleavings of } \pi'(\texttt{prog}_t),\ t \in \mathbb{T} \,\}$
- $\mathsf{P}'_*[\![\,\texttt{parprog}\,]\!] \stackrel{\text{def}}{=} [\, \Pi[\![\, \pi'_* \,]\!] \langle\, \mathcal{E}_0,\, \emptyset \,\rangle \,]_\Omega$

Soundness theorem

$\mathsf{P}'_*[\![\,\texttt{parprog}\,]\!] \subseteq \mathsf{P}[\![\,\texttt{parprog}\,]\!]$

$\Longrightarrow$ the interference semantics is sound
wrt. weakly consistent memories and changes of granularity

Synchronisation

Scheduling

Synchronization primitives

$$\begin{array}{lcl} \texttt{prog} & ::= & \texttt{lock}(m) \\ & | & \texttt{unlock}(m) \end{array}$$

$m \in \mathbb{M}$: finite set of non-recursive mutexes

Scheduling

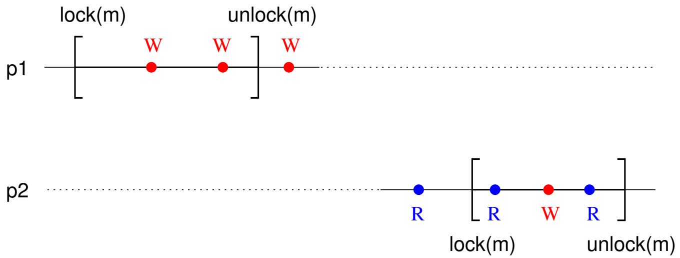
- mutexes ensure mutual exclusion

  at each time, each mutex can be locked by a single thread

- mutexes enforce memory consistency and atomicity

  no optimization across `lock` and `unlock` instructions
  memory caches and buffer are flushed

Mutual exclusion



Interleaving semantics $P_*[\![\, \texttt{parprog} \,]\!]$:

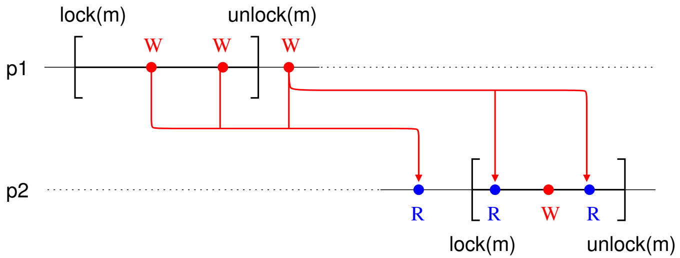
restrict interleavings of control paths

Interference semantics $P[\![\, \texttt{parprog} \,]\!]$, $P^\sharp[\![\, \texttt{parprog} \,]\!]$:

partition wrt. an abstract local view of the scheduler $\mathbb{C}$

- $\mathcal{E} \rightsquigarrow \mathcal{E} \times \mathbb{C}$, $\quad \mathcal{E}^\sharp \rightsquigarrow \mathbb{C} \to \mathcal{E}^\sharp$
- $\mathbb{I} \stackrel{\text{def}}{=} \mathbb{T} \times \mathbb{V} \times \mathbb{R} \rightsquigarrow \mathbb{I} \stackrel{\text{def}}{=} \mathbb{T} \times \mathbb{C} \times \mathbb{V} \times \mathbb{R}$,
  $\mathbb{I}^\sharp \stackrel{\text{def}}{=} (\mathbb{T} \times \mathbb{V}) \to \mathcal{R}^\sharp \rightsquigarrow \mathbb{I}^\sharp \stackrel{\text{def}}{=} (\mathbb{T} \times \mathbb{C} \times \mathbb{V}) \to \mathcal{R}^\sharp$

Mutual exclusion

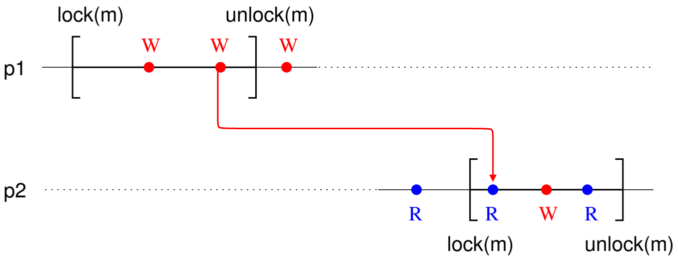


Data-race effects

Partition wrt. mutexes $M \subseteq \mathbb{M}$ held by the current thread t

- $\mathsf{C_t}[\![\, \mathtt{X} := e \,]\!] \langle \rho, M, I \rangle$ adds $\{ \langle t, M, \mathtt{X}, v \rangle \mid v \in \mathsf{E_t}[\![\, X \,]\!] \langle \rho, M, I \rangle \}$ to I
- $\mathsf{E_t}[\![\, \mathtt{X} \,]\!] \langle \rho, M, I \rangle =$ $\{ \rho(\mathtt{X}) \} \cup \{ v \mid \langle t', M', \mathtt{X}, v \rangle \in I,\ t \neq t',\ M \cap M' = \emptyset \}$
- flow-insensitive, subject to weak memory consistency

Mutual exclusion



Well-synchronized effects

- last write before unlock affects first read after lock
- partition interferences wrt. a protecting mutex m (and M)
- $C_t[\![\, \mathtt{unlock}(m) \,]\!]\langle\, \rho,\ M,\ I \,\rangle$ stores $\rho(\mathtt{X})$ into I
- $C_t[\![\, \mathtt{lock}(m) \,]\!]\langle\, \rho,\ M,\ I \,\rangle$ imports values form I into ρ
- imprecision: non-relational, largely flow-insensitive

Example analysis

abstract consumer/producer

t_1	t_2

`while 0=0 do`
`  lock(m);`$^{\ell 1}$
`  if X>0 then `$^{\ell 2}$`X:=X-1 fi;`
`  unlock(m);`
`  `$^{\ell 3}$`Y:=X`
`done`
	`while 0=0 do`
	`  lock(m);`
	`  X:=X+1;`
	`  if X>10 then X:=10 fi;`
	`  unlock(m)`
	`done`

- at $\ell 1$, the unlock-lock effect from t_2 imports $\{\texttt{X}\} \times [1,10]$
- at $\ell 2$, $\texttt{X} \in [1,10]$, no effect from t_2: X:=X-1 is safe
- at $\ell 3$, $\texttt{X} \in [0,9]$, and t_2 has the effects $\{\texttt{X}\} \times [1,10]$
  so, $\texttt{X} \in [0,10]$

Limitations of the interference abstraction

Lack of relational lock invariants

a difficult example

$\mathcal{E}_0$: X = Y = 5

`while 0=0 do`
`    lock(m);`
`    if X>0 then`
`        X:=X-1;`
`        Y:=Y-1;`
`    fi;`
`    unlock(m)`
`done`
	`while 0=0 do`
	`    lock(m);`
	`    if X<10 then`
	`        X:=X+1;`
	`        Y:=Y+1;`
	`    fi;`
	`    unlock(m)`
	`done`

Our analysis finds X $\in [0, 10]$, but no bound on Y.

Actually Y $\in [0, 10]$.

To prove this, we would need to infer the relational invariant X = Y at lock boundaries.

Lack of inter-process flow-sensitivity

a more difficult example

`while 0=0 do`
`    lock(m);`
`    X:=X+1;`
`    unlock(m);`
`    lock(m);`
`    X:=X-1;`
`    unlock(m)`
`done`
	`while 0=0 do`
	`    lock(m);`
	`    X:= X+1;`
	`    unlock(m);`
	`    lock(m);`
	`    X:=X-1;`
	`    unlock(m)`
	`done`

Our analysis finds no bound on X.

Actually $\texttt{X} \in [-2, 2]$ at all program points.

To prove this we need to infer an invariant on the history of interleaved executions:

no more than two incrementation (resp. decrementation) can occur without a decrementation (resp. incrementation).

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