

# Correction of exercices from course 03

MPRI 2–6: Abstract Interpretation,  
application to verification and static analysis

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## Question 1: $\mathcal{S}[T]$

$(\Sigma, \tau)$  is a transition system.

The partial finite traces generated by  $\tau$  are:

$$\mathcal{T}[\tau] \stackrel{\text{def}}{=} \{ (\sigma_0, \dots, \sigma_n) \in \Sigma^+ \mid \forall i < n: (\sigma_i, \sigma_{i+1}) \in \tau \}$$

The smallest transition system that generates  $T$  is:

$$\mathcal{S}[T] \stackrel{\text{def}}{=} \{ (\sigma, \sigma') \in \Sigma^2 \mid \\ \exists (\sigma_0, \dots, \sigma_n) \in T \wedge i < n: \sigma = \sigma_i \wedge \sigma' = \sigma_{i+1} \}$$

$\mathcal{S}[T]$  is the set of transitions appearing within any trace in  $T$

## Question 2: Galois connection

Recall that:

$$\mathcal{T}[\tau] \stackrel{\text{def}}{=} \{ (\sigma_0, \dots, \sigma_n) \in \Sigma^+ \mid \forall i < n: (\sigma_i, \sigma_{i+1}) \in \tau \}$$

$$\mathcal{S}[T] \stackrel{\text{def}}{=} \{ (\sigma, \sigma') \in \Sigma^2 \mid \exists (\sigma_0, \dots, \sigma_n) \in T \wedge i < n: \sigma = \sigma_i \wedge \sigma' = \sigma_{i+1} \}$$

We have  $(\mathcal{P}(\Sigma^+), \subseteq) \stackrel{\mathcal{T}}{\longleftarrow} \stackrel{\mathcal{S}}{\longrightarrow} (\mathcal{P}(\Sigma \times \Sigma), \subseteq)$ .

proof:

$$\mathcal{S}[T] \subseteq \tau$$

$$\iff \forall (\sigma, \sigma') \in \mathcal{S}[T]: (\sigma, \sigma') \in \tau$$

$$\iff \forall (\sigma, \sigma'): (\exists (\sigma_0, \dots, \sigma_n) \in T \wedge i < n: \sigma = \sigma_i \wedge \sigma' = \sigma_{i+1}) \implies (\sigma, \sigma') \in \tau$$

$$\iff \forall (\sigma_0, \dots, \sigma_n) \in T \wedge i < n: (\sigma_i, \sigma_{i+1}) \in \tau$$

$$\iff \forall (\sigma_0, \dots, \sigma_n) \in T: (\forall i < n: (\sigma_i, \sigma_{i+1}) \in \tau)$$

$$\iff \forall (\sigma_0, \dots, \sigma_n) \in T: (\sigma_0, \dots, \sigma_n) \in \mathcal{T}[\tau]$$

$$\iff T \subseteq \mathcal{T}[\tau]$$

As a consequence  $\forall T: T \subseteq (\mathcal{T} \circ \mathcal{S})[T]$  and  $\forall \tau: (\mathcal{S} \circ \mathcal{T})[\tau] \subseteq \tau$ .

In fact, we have a **Galois embedding**:  $\forall \tau: (\mathcal{S} \circ \mathcal{T})[\tau] = \tau$ .

proof:  $\mathcal{S}$  is onto as  $\forall \tau: \mathcal{S}[\tau] = \tau$ .

## Question 3: Approximation

Recall that:

$$\mathcal{T}[\tau] \stackrel{\text{def}}{=} \{(\sigma_0, \dots, \sigma_n) \in \Sigma^+ \mid \forall i < n: (\sigma_i, \sigma_{i+1}) \in \tau\}$$

$$\mathcal{S}[T] \stackrel{\text{def}}{=} \{(\sigma, \sigma') \in \Sigma^2 \mid \exists(\sigma_0, \dots, \sigma_n) \in T \wedge i < n: \sigma = \sigma_i \wedge \sigma' = \sigma_{i+1}\}$$

- $T \stackrel{\text{def}}{=} \{a, aa\}$  is not generated by any transition system
- $\mathcal{S}[T] = \{(a, a)\}$

which generates:  $(\mathcal{T} \circ \mathcal{S})[T] \stackrel{\text{def}}{=} a^+ \supsetneq T$

(if a transition appears once in  $T$ , it can appear any number of times in  $(\mathcal{T} \circ \mathcal{S})[T]$ )

## Question 4: Exactness conditions

Recall that:

$$\mathcal{T}[\tau] \stackrel{\text{def}}{=} \{ (\sigma_0, \dots, \sigma_n) \in \Sigma^+ \mid \forall i < n: (\sigma_i, \sigma_{i+1}) \in \tau \}$$

$$\mathcal{S}[T] \stackrel{\text{def}}{=} \{ (\sigma, \sigma') \in \Sigma^2 \mid \exists (\sigma_0, \dots, \sigma_n) \in T \wedge i < n: \sigma = \sigma_i \wedge \sigma' = \sigma_{i+1} \}$$

### Necessary and sufficient conditions for $(\mathcal{T} \circ \mathcal{S})[T] = T$

- Assume that  $T = \mathcal{T}[\tau]$  for some  $\tau$ , then
  - $\forall (\sigma_0, \dots, \sigma_n) \in T: (\sigma_0, \dots, \sigma_{n-1}) \in T$
  - $\forall (\sigma_0, \dots, \sigma_n) \in T: (\sigma_1, \dots, \sigma_n) \in T$
  - $\forall (\sigma_0, \dots, \sigma_n) \in T, (\sigma_n, \dots, \sigma_m) \in T: (\sigma_0, \dots, \sigma_m) \in T$
  - $\Sigma \subseteq T$ $\implies T$  is closed by prefix, suffix and junction, and  $\Sigma \subseteq T$
- Assume that  $T$  is closed by prefix, suffix, junction and  $\Sigma \subseteq T$ 
  - by prefix and suffix:  $\forall (\sigma_0, \dots, \sigma_n) \in T: \forall i < n: (\sigma_i, \sigma_{i+1}) \in T$   
i.e.,  $\mathcal{S}[T] \subseteq T$ ; as  $\mathcal{S}[T] \subseteq \Sigma^2$ , we get  $\mathcal{S}[T] \subseteq T \cap \Sigma^2$
  - by junction:  $\forall i < n: (\sigma_i, \sigma_{i+1}) \in T \implies (\sigma_0, \dots, \sigma_n) \in T$   
together with  $\Sigma \subseteq T$ , we get  $\mathcal{T}[T \cap \Sigma^2] \subseteq T$ $\implies (\mathcal{T} \circ \mathcal{S})[T] \subseteq T$ , hence  $(\mathcal{T} \circ \mathcal{S})[T] = T$

## Question 5: Galois connection

$$\mathcal{T}_\infty[\tau] \stackrel{\text{def}}{=} \mathcal{T}[\tau] \cup \{(\sigma_0, \dots) \in \Sigma^\omega \mid \forall i: (\sigma_i, \sigma_{i+1}) \in \tau\}$$

$$\mathcal{S}_\infty[T] \stackrel{\text{def}}{=} \{(\sigma, \sigma') \in \Sigma^2 \mid \\ \exists(\sigma_0, \dots, \sigma_n) \in T \cap \Sigma^+ : \exists i < n: \sigma = \sigma_i \wedge \sigma' = \sigma_{i+1} \vee \\ \exists(\sigma_0, \dots) \in T \cap \Sigma^\omega : \exists i: \sigma = \sigma_i \wedge \sigma' = \sigma_{i+1}\}$$

We have  $(\mathcal{P}(\Sigma^\infty), \subseteq) \begin{matrix} \xleftarrow{\mathcal{T}_\infty} \\ \xrightarrow{\mathcal{S}_\infty} \end{matrix} (\mathcal{P}(\Sigma \times \Sigma), \subseteq)$ .

proof: very similar to question 2

$$\mathcal{S}_\infty[T] \subseteq \tau$$

$$\iff \forall(\sigma, \sigma') \in \mathcal{S}_\infty[T]: (\sigma, \sigma') \in \tau$$

$$\iff \forall(\sigma_0, \dots, \sigma_n) \in T \cap \Sigma^+ : \forall i < n: (\sigma_i, \sigma_{i+1}) \in \tau \\ \wedge \forall(\sigma_0, \dots) \in T \cap \Sigma^\omega : \forall i: (\sigma_i, \sigma_{i+1}) \in \tau$$

$$\iff \forall(\sigma_0, \dots, \sigma_n) \in T \cap \Sigma^+ : (\sigma_0, \dots, \sigma_n) \in \mathcal{T}[\tau] \\ \wedge \forall(\sigma_0, \dots) \in T \cap \Sigma^\omega : (\sigma_0, \dots) \in \mathcal{T}[\tau]$$

$$\iff T \cap \Sigma^+ \subseteq \mathcal{T}[\tau] \wedge T \cap \Sigma^\omega \subseteq \mathcal{T}[\tau]$$

$$\iff T \subseteq \mathcal{T}[\tau]$$

We also have a Galois embedding.

## Question 6: Approximation

Recall that:

$$\mathcal{T}_\infty[\tau] \stackrel{\text{def}}{=} \mathcal{T}[\tau] \cup \{(\sigma_0, \dots) \in \Sigma^\omega \mid \forall i: (\sigma_i, \sigma_{i+1}) \in \tau\}$$

$$\mathcal{S}_\infty[T] \stackrel{\text{def}}{=} \{(\sigma, \sigma') \in \Sigma^2 \mid \\ \exists(\sigma_0, \dots, \sigma_n) \in T \cap \Sigma^+ : \exists i < n: \sigma = \sigma_i \wedge \sigma' = \sigma_{i+1} \vee \\ \exists(\sigma_0, \dots) \in T \cap \Sigma^\omega : \exists i: \sigma = \sigma_i \wedge \sigma' = \sigma_{i+1}\}$$

Consider  $T \stackrel{\text{def}}{=} a^+$  (with  $\Sigma \stackrel{\text{def}}{=} \{a\}$ ).

$T$  is closed by prefix, suffix and junction, and  $\Sigma \subseteq T$ .

We have  $\mathcal{S}_\infty[T] = \{(a, a)\}$ .

But then,  $(\mathcal{T}_\infty \circ \mathcal{S}_\infty)[T] = a^\infty \supsetneq a^+ = T$ .

( $\mathcal{T}_\infty \circ \mathcal{S}_\infty$  adds infinite traces to sets of finite traces)

## Question 7: Exactness conditions

Necessary and sufficient conditions for  $(\mathcal{T}_\infty \circ \mathcal{S}_\infty)[T] = T$

- $T$  must be closed by prefix, suffix, junction and contain  $\Sigma$
- and  $T$  must be **closed by limit**:

given  $(\sigma_0, \dots) \in \Sigma^\omega$ ,  $\forall n: (\sigma_0, \dots, \sigma_n) \in T \implies (\sigma_0, \dots) \in T$

proof:

$\forall T: \mathcal{T}_\infty[T]$  is closed by limit, so, it is a necessary condition.

Assume now that  $T$  is closed by prefix, suffix, junction and contain  $\Sigma$ , then, by question 4:  $(\mathcal{T}_\infty \circ \mathcal{S}_\infty)[T] \cap \Sigma^+ = T \cap \Sigma^+$ .

We denote by  $\text{lim} : \mathcal{P}(\Sigma^\infty) \rightarrow \mathcal{P}(\Sigma^\infty)$  the closure by limit.

Note that  $(\mathcal{T}_\infty \circ \mathcal{S}_\infty)[T] = \text{lim}((\mathcal{T}_\infty \circ \mathcal{S}_\infty)[T] \cap \Sigma^+)$ .

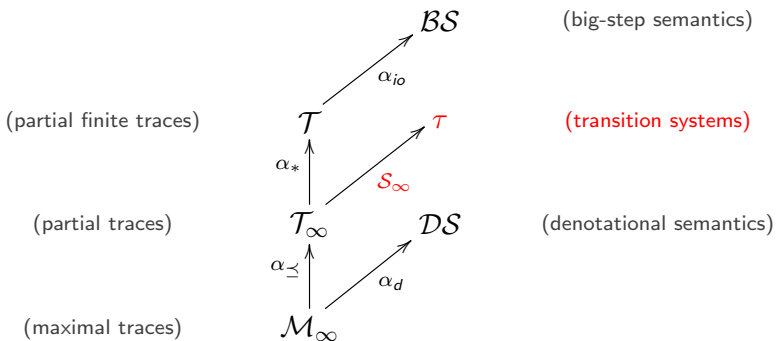
By hypothesis,  $\text{lim}(T) = T$ ; by monotonicity of  $\text{lim}$ ,  $\text{lim}(T \cap \Sigma^+) \subseteq \text{lim}(T)$ , hence  $\text{lim}(T \cap \Sigma^+) \subseteq T$ .

In general, the equality does not hold ( $T$  may have infinite traces that are not limits of finite ones); however, as  $T$  is closed by prefix,  $T \cap \Sigma^+$  contains all finite prefixes of traces in  $T \cap \Sigma^\omega$ , hence  $\text{lim}(T \cap \Sigma^+) = T$ .

Hence,  $(\mathcal{T}_\infty \circ \mathcal{S}_\infty)[T] = T$ .



# Note: Hierarchy of semantics



Transition systems are (relational) abstractions of traces semantics.