Static Analysis of Concurrent Programs

MPRI 2–6: Abstract Interpretation,
application to verification and static analysis

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Concurrent programming

Idea:

Decompose a program into a set of (loosely) interacting processes.

Why concurrent programs?

- exploit parallelism in current computers
  (multi-processors, multi-cores, hyper-threading)

  "Free lunch is over"
  change in Moore's law ($\times$2 transistors every 2 years)

- exploit several computers
  (distributed computing)

- ease of programming
  (GUI, network code, reactive programs)

Scope

In this course: static thread model

- implicit communication through shared memory
- explicit communication through synchronisation primitives
- fixed number of threads (no dynamic creation of threads)
- numeric programs (real-valued variables)

Goal: static analysis

- infer numeric program invariants
- discover possible run-time errors (e.g., division by 0)
- parametrized by a choice of numeric abstract domains

Outline

- From sequential to concurrent abstract interpreters
  - alternate sequential semantics
    (denotational semantics with errors)
  - interleaving concurrent semantics
  - (non-relational) interference-based analysis
- Abstract rely-guarantee
  - rely-guarantee proof method
  - complete modular concrete semantics
  - relational interference abstractions

Simple structured numeric language

- finite set of (toplevel) threads: $\texttt{prog}_1$ to $\texttt{prog}_n$
- finite set of numeric program variables $\texttt{V} \in \mathbb{V}$
- finite set of statement locations $\ell \in \mathcal{L}$
- finite set of potential error locations $\omega \in \Omega$

Structured language syntax

$$\begin{array}{lcll}
\texttt{parprog} & ::= & {}^{\ell}\texttt{prog}_1{}^{\ell} \mid\mid \ldots \mid\mid {}^{\ell}\texttt{prog}_n{}^{\ell} & \textit{(parallel composition)} \\
{}^{\ell}\texttt{prog}^{\ell} & ::= & {}^{\ell}\texttt{V} := \texttt{exp}^{\ell} & \textit{(assignment)} \\
 & \mid & {}^{\ell}\texttt{if exp} \bowtie 0 \texttt{ then } {}^{\ell}\texttt{prog}^{\ell} \texttt{ fi}^{\ell} & \textit{(conditional)} \\
 & \mid & {}^{\ell}\texttt{while } {}^{\ell}\texttt{exp} \bowtie 0 \texttt{ do } {}^{\ell}\texttt{prog}^{\ell} \texttt{ done}^{\ell} & \textit{(loop)} \\
 & \mid & {}^{\ell}\texttt{prog}; {}^{\ell}\texttt{prog}^{\ell} & \textit{(sequence)} \\
\texttt{exp} & ::= & \texttt{V} \mid [c_1, c_2] \mid -\texttt{exp} \mid \texttt{exp} \diamond_{\omega} \texttt{exp} &
\end{array}$$

$c_1, c_2 \in \mathbb{R} \cup \{+\infty, -\infty\}$, $\diamond \in \{+, -, \times, /\}$, $\bowtie \in \{=, <, \ldots\}$

From sequential to concurrent semantics

Sequential semantics

Reminder : transition systems

Transition system: $(\Sigma, \tau, \mathcal{I})$

- Σ: set of program states
- $\tau \subseteq \Sigma \times \Sigma$: transition relation
  we note $(\sigma, \sigma') \in \tau$ as $\sigma \to_\tau \sigma'$
- $\mathcal{I} \subseteq \Sigma$: set of initial states

Reminder : traces of a transition system

Maximal trace semantics: $\mathcal{M}_\infty \in \mathcal{P}(\Sigma^\infty)$

Set of total executions $\sigma_0, \ldots, \sigma_n, \ldots$

- starting in an initial state $\sigma_0 \in \mathcal{I}$ and either
- ending in a blocking state in $\mathcal{B} \stackrel{\text{def}}{=} \{\, \sigma \mid \forall \sigma' : \sigma \not\to_\tau \sigma' \,\}$
- or infinite

$$\mathcal{M}_\infty \stackrel{\text{def}}{=} \{\, \sigma_0, \ldots, \sigma_n \mid \sigma_0 \in \mathcal{I} \land \sigma_n \in \mathcal{B} \land \forall i < n : \sigma_i \to_\tau \sigma_{i+1} \,\} \cup \{\, \sigma_0, \ldots, \sigma_n \ldots \mid \sigma_0 \in \mathcal{I} \land \forall i < \omega : \sigma_i \to_\tau \sigma_{i+1} \,\}$$

Reminder : prefix trace abstraction

Finite prefix trace semantics: $\mathcal{T}_p \in \mathcal{P}(\Sigma^*)$

set of finite prefixes of executions:

$\mathcal{T}_p \stackrel{\text{def}}{=} \{\, \sigma_0, \ldots, \sigma_n \,|\, n \geq 0,\ \sigma_0 \in \mathcal{I},\ \forall i < n: \sigma_i \to_\tau \sigma_{i+1} \,\}$

$\mathcal{T}_p$ is an abstraction of the maximal trace semantics

$\mathcal{T}_p = \alpha_{*\preceq}(\mathcal{M}_\infty)$ where $\alpha_{*\preceq}(X) \stackrel{\text{def}}{=} \{\, t \in \Sigma^* \,|\, \exists u \in X: t \preceq u \,\}$

- can still prove safety properties
- cannot prove termination nor inevitability

fixpoint characterisation: $\mathcal{T}_p = \text{lfp}\, F_p$ where

$F_p(X) = \mathcal{I} \cup \{\, \sigma_0, \ldots, \sigma_{n+1} \,|\, \sigma_0, \ldots, \sigma_n \in X \wedge \sigma_n \to_\tau \sigma_{n+1} \,\}$

Reminder : reachable state abstraction

Reachable state semantics: $\mathcal{R} \in \mathcal{P}(\Sigma)$

set of states reachable in any execution:
$\mathcal{R} \stackrel{\text{def}}{=} \{\, \sigma \mid \exists n \geq 0,\ \sigma_0, \ldots, \sigma_n \colon \sigma_0 \in \mathcal{I},\ \forall i < n \colon \sigma_i \to_\tau \sigma_{i+1} \land \sigma = \sigma_n \,\}$

$\mathcal{R}$ is an abstraction of the finite trace semantics: $\mathcal{R} = \alpha_p(\mathcal{T}_p)$
where $\alpha_p(X) \stackrel{\text{def}}{=} \{\, \sigma \mid \exists \sigma_0, \ldots, \sigma_n \in X \colon \sigma = \sigma_n \,\}$

- $\mathcal{R}$ can prove state safety properties: $\mathcal{R} \subseteq S$
  (executions stay in S)
- $\mathcal{R}$ cannot prove ordering, termination, inevitability properties

fixpoint characterisation: $\mathcal{R} = \text{lfp}\, F_\mathcal{R}$ where
$F_\mathcal{R}(X) = \mathcal{I} \cup \{\, \sigma \mid \exists \sigma' \in X \colon \sigma' \to_\tau \sigma \,\}$

States of a sequential program, with errors

Simple sequential numeric programs: $\texttt{parprog} ::= {}^{\ell^i}\texttt{prog}^{\ell^x}$.

Program states: $\Sigma \stackrel{\text{def}}{=} (\mathcal{L} \times \mathcal{E}) \cup \Omega$

- a control state in $\mathcal{L}$, and
- either a memory state: an environment in $\mathcal{E} \stackrel{\text{def}}{=} \mathbb{V} \to \mathbb{R}$
- or an **error state**, in Ω

Initial states:

start at the first control point ℓ^i with variables set to 0:

$\mathcal{I} \stackrel{\text{def}}{=} \{\, (\ell^i, \lambda \mathbb{V}.0) \,\}$

Note that $\mathcal{P}(\Sigma) \simeq (\mathcal{L} \to \mathcal{P}(\mathcal{E})) \times \mathcal{P}(\Omega)$:

- a state property in $\mathcal{P}(\mathcal{E})$ at each program point in $\mathcal{L}$
- and a set of errors in $\mathcal{P}(\Omega)$

Expression semantics with errors

Expression semantics: $\mathsf{E}[\![\, \exp \,]\!] : \mathcal{E} \to (\mathcal{P}(\mathbb{R}) \times \mathcal{P}(\Omega))$

$$
\begin{array}{lcl}
\mathsf{E}[\![\, \mathtt{V} \,]\!]\, \rho & \stackrel{\text{def}}{=} & \langle\, \{\, \rho(\mathtt{V}) \,\},\, \emptyset \,\rangle \\
\mathsf{E}[\![\, [c_1, c_2] \,]\!]\, \rho & \stackrel{\text{def}}{=} & \langle\, \{\, x \in \mathbb{R} \mid c_1 \le x \le c_2 \,\},\, \emptyset \,\rangle \\
\mathsf{E}[\![\, -e \,]\!]\, \rho & \stackrel{\text{def}}{=} & \text{let } \langle\, V,\, O \,\rangle = \mathsf{E}[\![\, e \,]\!]\, \rho \text{ in} \\
 & & \langle\, \{\, -v \mid \in V \,\},\, O \,\rangle \\
\mathsf{E}[\![\, e_1 \diamond_\omega\ e_2 \,]\!]\, \rho & \stackrel{\text{def}}{=} & \text{let } \langle\, V_1,\, O_1 \,\rangle = \mathsf{E}[\![\, e_1 \,]\!]\, \rho \text{ in} \\
 & & \text{let } \langle\, V_2,\, O_2 \,\rangle = \mathsf{E}[\![\, e_2 \,]\!]\, \rho \text{ in} \\
 & & \langle\, \{\, v_1 \diamond v_2 \mid v_i \in V_i,\, \diamond \ne / \lor v_2 \ne 0 \,\}, \\
 & & \ O_1 \cup O_2 \cup \{\, \omega \text{ if } \diamond = / \land 0 \in V_2 \,\} \,\rangle
\end{array}
$$

- defined by structural induction on the syntax
- evaluates in an environment ρ to a set of values
- also returns a set of accumulated errors
  (here, only divisions by zero)

Reminders: semantics in equational form

Principle: (without handling errors in Ω)

- see lfp f as the least solution of an equation $x = f(x)$
- partition states by control: $\mathcal{P}(\mathcal{L} \times \mathcal{E}) \simeq \mathcal{L} \to \mathcal{P}(\mathcal{E})$

  $\mathcal{X}_\ell \in \mathcal{P}(\mathcal{E})$: invariant at $\ell \in \mathcal{L}$

  $\forall \ell \in \mathcal{L}$: $\mathcal{X}_\ell \stackrel{\text{def}}{=} \{\, m \in \mathcal{E} \mid (\ell, m) \in \mathcal{R} \,\}$

  $\Longrightarrow$ set of (recursive) equations on $\mathcal{X}_\ell$

Example:

ℓ^1 `i:=2;`	$\mathcal{X}_1 = \mathcal{I}$
ℓ^2 `n:=`$[-\infty,+\infty]$`;`	$\mathcal{X}_2 = \mathsf{C}[\![\, \mathtt{i} := 2 \,]\!]\, \mathcal{X}_1$
ℓ^3 `while` ℓ^4 `i` $<$ `n do`	$\mathcal{X}_3 = \mathsf{C}[\![\, \mathtt{n} := [-\infty,+\infty] \,]\!]\, \mathcal{X}_2$
ℓ^5 `if` $[0,1] = 0$ `then`	$\mathcal{X}_4 = \mathcal{X}_3 \cup \mathcal{X}_7$
ℓ^6 `i:=i+1`	$\mathcal{X}_5 = \mathsf{C}[\![\, \mathtt{i} < \mathtt{n} \,]\!]\, \mathcal{X}_4$
`fi`	$\mathcal{X}_6 = \mathcal{X}_5$
ℓ^7 `done`	$\mathcal{X}_7 = \mathcal{X}_5 \cup \mathsf{C}[\![\, \mathtt{i} := \mathtt{i} + 1 \,]\!]\, \mathcal{X}_6$
ℓ^8	$\mathcal{X}_8 = \mathsf{C}[\![\, \mathtt{i} \geq \mathtt{n} \,]\!]\, \mathcal{X}_4$

Semantics in denotational form

Input-output function $\mathsf{C}[\![\,\texttt{prog}\,]\!]$

$$\underline{\mathsf{C}[\![\,\texttt{prog}\,]\!] : (\mathcal{P}(\mathcal{E}) \times \mathcal{P}(\Omega)) \to (\mathcal{P}(\mathcal{E}) \times \mathcal{P}(\Omega))}$$

$$\mathsf{C}[\![\,\texttt{X} := e\,]\!]\,\langle\, R,\ O\,\rangle \stackrel{\text{def}}{=} \langle\,\emptyset,\ O\,\rangle \sqcup \bigsqcup_{\rho \in R} \langle\, \{\, \rho[\texttt{X} \mapsto v] \mid v \in V_\rho\,\},\ O_\rho\,\rangle$$

$$\mathsf{C}[\![\, e \bowtie 0?\,]\!]\,\langle\, R,\ O\,\rangle \stackrel{\text{def}}{=} \langle\,\emptyset,\ O\,\rangle \sqcup \bigsqcup_{\rho \in R} \langle\, \{\, \rho \mid \exists v \in V_\rho \colon v \bowtie 0\,\},\ O_\rho\,\rangle$$

where $\langle\, V_\rho,\ O_\rho\,\rangle \stackrel{\text{def}}{=} \mathsf{E}[\![\, e\,]\!]\,\rho$

$$\mathsf{C}[\![\,\texttt{if}\ e \bowtie 0\ \texttt{then}\ s\ \texttt{fi}\,]\!]\,X \stackrel{\text{def}}{=} (\mathsf{C}[\![\, s\,]\!] \circ \mathsf{C}[\![\, e \bowtie 0?\,]\!])X \sqcup \mathsf{C}[\![\, e \not\bowtie 0?\,]\!]\,X$$

$$\mathsf{C}[\![\,\texttt{while}\ e \bowtie 0\ \texttt{do}\ s\ \texttt{done}\,]\!]\,X \stackrel{\text{def}}{=}$$
$$\mathsf{C}[\![\, e \not\bowtie 0?\,]\!]\,(\mathrm{lfp}\,\lambda Y.X \sqcup (\mathsf{C}[\![\, s\,]\!] \circ \mathsf{C}[\![\, e \bowtie 0?\,]\!])Y)$$

$$\mathsf{C}[\![\, s_1;\ s_2\,]\!] \stackrel{\text{def}}{=} \mathsf{C}[\![\, s_2\,]\!] \circ \mathsf{C}[\![\, s_1\,]\!]$$

- mutate memory states in $\mathcal{E}$, accumulate errors in Ω

  ($\sqcup$ is the element-wise $\cup$ in $\mathcal{P}(\mathcal{E}) \times \mathcal{P}(\Omega)$)
- structured: nested loops yield nested fixpoints
- big-step: forget information on intermediate locations ℓ

Abstract semantics in denotational form

Extend a numeric abstract domain $\mathcal{E}^\sharp$ abstracting $\mathcal{P}(\mathcal{E})$ to $\mathcal{D}^\sharp \stackrel{\text{def}}{=} \mathcal{E}^\sharp \times \mathcal{P}(\Omega)$.

$\underline{\mathsf{C}^\sharp[\![\, \texttt{prog} \,]\!] : \mathcal{D}^\sharp \to \mathcal{D}^\sharp}$

$\mathsf{C}^\sharp[\![\, \texttt{X} := e \,]\!]\langle R^\sharp, O \rangle$ and $\mathsf{C}^\sharp[\![\, e \bowtie 0? \,]\!]\langle R^\sharp, O \rangle$ are given

$\mathsf{C}^\sharp[\![\, \texttt{if } e \bowtie 0 \texttt{ then } s \texttt{ fi} \,]\!] X^\sharp \stackrel{\text{def}}{=}$
$\quad (\mathsf{C}^\sharp[\![\, s \,]\!] \circ \mathsf{C}^\sharp[\![\, e \bowtie 0? \,]\!]) X^\sharp \sqcup^\sharp \mathsf{C}^\sharp[\![\, e \not\bowtie 0? \,]\!] X^\sharp$

$\mathsf{C}^\sharp[\![\, \texttt{while } e \bowtie 0 \texttt{ do } s \texttt{ done} \,]\!] X^\sharp \stackrel{\text{def}}{=}$
$\quad \mathsf{C}^\sharp[\![\, e \not\bowtie 0? \,]\!] (\lim \lambda Y^\sharp. Y^\sharp \,\triangledown\, (X^\sharp \sqcup (\mathsf{C}^\sharp[\![\, s \,]\!] \circ \mathsf{C}^\sharp[\![\, e \bowtie 0? \,]\!]) Y^\sharp))$

$\mathsf{C}^\sharp[\![\, s_1;\, s_2 \,]\!] \stackrel{\text{def}}{=} \mathsf{C}^\sharp[\![\, s_2 \,]\!] \circ \mathsf{C}^\sharp[\![\, s_1 \,]\!]$

- the abstract interpreter mimicks an actual interpreter
- less flexibility in the iteration scheme
  (iteration order, widening points, etc.)
- efficient in memory (intermediate invariants are not kept)
  $\Longrightarrow$ much more scalable for large programs

Concurrent semantics

Multi-thread execution model

t_1	t_2
$\ell 1$ `while random do`	$\ell 4$ `while random do`
$\ell 2$ `if x < y then`	$\ell 5$ `if y < 100 then`
$\ell 3$ `x := x + 1`	$\ell 6$ `y := y + [1,3]`

Execution model:

- finite number of threads
- the memory is shared (x, y)
- each thread has its own program counter
- execution interleaves steps from threads t_1 and t_2
  (assignments and tests are assumed to be atomic)

$\Longrightarrow$ we have the global invariant $0 \leq x \leq y \leq 102$

Labelled transition systems

Labelled transition system: $(\Sigma, \mathcal{A}, \tau, \mathcal{I})$

- Σ: set of program states
- $\mathcal{A}$: set of actions
- $\tau \subseteq \Sigma \times \mathcal{A} \times \Sigma$: transition relation
  we note $(\sigma, a, \sigma') \in \tau$ as $\sigma \xrightarrow{a}_\tau \sigma'$
- $\mathcal{I} \subseteq \Sigma$: set of initial states

Labelled traces: sequences of states interspersed with actions

denoted as $\sigma_0 \xrightarrow{a_0} \sigma_1 \xrightarrow{a_1} \cdots \sigma_n \xrightarrow{a_n} \sigma_{n+1}$

From concurrent programs to labelled transition systems

Notations:

- concurrent program:
  $\texttt{parprog} ::= {}^{\ell^i_1}\texttt{prog}_1{}^{\ell^x_1} \mid\mid \cdots \mid\mid {}^{\ell^i_n}\texttt{prog}_n{}^{\ell^x_n}$
- threads identifiers: $\mathbb{T} \stackrel{\text{def}}{=} \{1, \ldots, n\}$

Program states: $\Sigma \stackrel{\text{def}}{=} ((\mathbb{T} \to \mathcal{L}) \times \mathcal{E}) \cup \Omega$

- a control state $L(t) \in \mathcal{L}$ for each thread $t \in \mathbb{T}$ and
- a single shared memory state $\rho \in \mathcal{E}$
- or an error state $\omega \in \Omega$

Initial states:

threads start at their first control point ℓ^i_t, variables are set to 0:
$\mathcal{I} \stackrel{\text{def}}{=} \{ (\lambda t.\ell^i_t,\ \lambda \mathbb{V}.0) \}$

Actions: thread identifiers: $\mathcal{A} \stackrel{\text{def}}{=} \mathbb{T}$

From concurrent programs to labelled transition systems

Transition relation: $\tau \subseteq \Sigma \times \mathcal{A} \times \Sigma$

$$(L,\rho)\xrightarrow{t}_{\tau}(L',\rho') \stackrel{\text{def}}{\iff} (L(t),\rho)\rightarrow_{\tau[\texttt{prog}_t]}(L'(t),\rho') \wedge \forall u \neq t\colon L(u) = L'(u)$$

$$(L,\rho)\xrightarrow{t}_{\tau}\omega \stackrel{\text{def}}{\iff} (L(t),\rho)\rightarrow_{\tau[\texttt{prog}_t]}\omega$$

- based on the transition relation of individual threads seen as sequential processes $\texttt{prog}_t$:
  $\tau[\texttt{prog}] \subseteq (\mathcal{L} \times \mathcal{E}) \times ((\mathcal{L} \times \mathcal{E}) \cup \Omega)$
  - choose a thread t to run
  - update ρ and $L(t)$
  - leave $L(u)$ intact for $u \neq t$

  (See course 3 for the full definition of $\tau[\texttt{prog}]$.)
- each $\sigma \rightarrow \sigma'$ in $\tau[\texttt{prog}_t]$ leads to many transitions in τ!

Interleaved trace semantics

Maximal and finite prefix trace semantics are as before:

Blocking states: $\mathcal{B} \stackrel{\text{def}}{=} \{\, \sigma \mid \forall \sigma' : \forall t : \sigma \stackrel{t}{\not\rightarrow}_\tau \sigma' \,\}$

Maximal traces: $\mathcal{M}_\infty$ (finite or infinite)

$$\mathcal{M}_\infty \stackrel{\text{def}}{=} \{\, \sigma_0 \xrightarrow{t_0} \cdots \xrightarrow{t_{n-1}} \sigma_n \mid n \geq 0 \land \sigma_0 \in \mathcal{I} \land \sigma_n \in \mathcal{B} \land \forall i < n : \sigma_i \xrightarrow{t_i}_\tau \sigma_{i+1} \,\} \cup \{\, \sigma_0 \xrightarrow{t_0} \sigma_1 \ldots \mid n \geq 0 \land \sigma_0 \in \mathcal{I} \land \forall i < \omega : \sigma_i \xrightarrow{t_i}_\tau \sigma_{i+1} \,\}$$

Finite prefix traces: $\mathcal{T}_p$

$$\mathcal{T}_p \stackrel{\text{def}}{=} \{\, \sigma_0 \xrightarrow{t_0} \cdots \xrightarrow{t_{n-1}} \sigma_n \mid n \geq 0 \land \sigma_0 \in \mathcal{I} \land \forall i < n : \sigma_i \xrightarrow{t_i}_\tau \sigma_{i+1} \,\}$$

Fixpoint form: $\mathcal{T}_p = \text{lfp}\, F_p$ where

$$F_p(X) = \mathcal{I} \cup \{\, \sigma_0 \xrightarrow{t_0} \cdots \xrightarrow{t_n} \sigma_{n+1} \mid n \geq 0 \land \sigma_0 \xrightarrow{t_0} \cdots \xrightarrow{t_{n-1}} \sigma_n \in X \land \sigma_n \xrightarrow{t_n}_\tau \sigma_{n+1} \,\}$$

Abstraction: $\mathcal{T}_p = \alpha_{*\preceq}(\mathcal{M}_\infty)$

Fairness

Fairness conditions: avoid threads being denied to run

Given $enabled(\sigma, t) \stackrel{\text{def}}{\iff} \exists \sigma' \in \Sigma: \sigma \xrightarrow{t}_{\tau} \sigma'$,
an infinite trace $\sigma_0 \xrightarrow{t_0} \cdots \sigma_n \xrightarrow{t_n} \cdots$ is:

- weakly fair if $\forall t \in \mathbb{T}$:
  $(\exists i: \forall j \geq i: enabled(\sigma_j, t)) \implies (\forall i: \exists j \geq i: a_j = t)$
  (no thread can be continuously enabled without running)
- strongly fair if $\forall t \in \mathbb{T}$:
  $(\forall i: \exists j \geq i: enabled(\sigma_j, t)) \implies (\forall i: \exists j \geq i: a_j = t)$
  (no thread can be infinitely often enabled without running)

Proofs under fairness conditions given:

- the maximal traces $\mathcal{M}_\infty$ of a program
- a property X to prove (as a set of traces)
- the set F of all (weakly or strongly) fair and of finite traces

$\implies$ prove $\mathcal{M}_\infty \cap F \subseteq X$ instead of $\mathcal{M}_\infty \subseteq X$

Fairness (cont.)

Example: `while x ≥ 0 do x:=x+1 done || x:=-1`

- may not terminate without fairness
- always terminates under weak and strong fairness

Finite prefix trace abstraction

$\mathcal{M}_\infty \cap F \subseteq X$ is abstracted into testing $\alpha_{*\preceq}(\mathcal{M}_\infty \cap F) \subseteq \alpha_{*\preceq}(X)$

for all fairness conditions F, $\alpha_{*\preceq}(\mathcal{M}_\infty \cap F) = \alpha_{*\preceq}(\mathcal{M}_\infty) = \mathcal{T}_p$

$\Longrightarrow$ fairness-dependent properties cannot be proved with finite prefixes only

In the following, we ignore fairness conditions.

(see [Cous85])

Equational state semantics

State abstraction $\mathcal{R}$: as before

- $\mathcal{R} \stackrel{\text{def}}{=} \{\, \sigma \mid \exists n \geq 0, \sigma_0 \stackrel{t_0}{\rightarrow} \cdots \sigma_n \colon \sigma_0 \in \mathcal{I}\ \forall i < n \colon \sigma_i \stackrel{t_i}{\rightarrow}_\tau \sigma_{i+1} \wedge \sigma = \sigma_n \,\}$
- $\mathcal{R} = \alpha_p(\mathcal{T}_p)$ where $\alpha_p(X) \stackrel{\text{def}}{=} \{\, \sigma \mid \exists n \geq 0, \sigma_0 \stackrel{t_0}{\rightarrow} \cdots \sigma_n \in X \colon \sigma = \sigma_n \,\}$
- $\mathcal{R} = \mathrm{lfp}\, F_\mathcal{R}$ where $F_\mathcal{R}(X) = \mathcal{I} \cup \{\, \sigma \mid \exists \sigma' \in X, t \in \mathbb{T} \colon \sigma' \stackrel{t}{\rightarrow}_\tau \sigma \,\}$

Equational form: (without handling errors in Ω)

- for each $L \in \mathbb{T} \rightarrow \mathcal{L}$, a variable $\mathcal{X}_L$ with value in $\mathcal{E}$
- equations are derived from thread equations $eq(\mathtt{prog}_t)$ as:

$$\mathcal{X}_{L_1} = \bigcup_{t \in \mathbb{T}} \{\, F(\mathcal{X}_{L_2}, \ldots, X_{L_N}) \mid \exists (\mathcal{X}_{\ell_1} = F(\mathcal{X}_{\ell_2}, \ldots, \mathcal{X}_{\ell_N})) \in eq(\mathtt{prog}_t) \colon \forall i \leq N \colon L_i(t) = \ell_i,\ \forall u \neq t \colon L_i(u) = L_1(u) \,\}$$

Join with $\cup$ equations from $eq(\mathtt{prog}_t)$ updating a single thread $t \in \mathbb{T}$.

(See course 3 for the full definition of $eq(\mathtt{prog})$.)

Equational state semantics (example)

Example: inferring $0 \leq x \leq y \leq 102$

t_1	t_2
$\ell 1$ `while random do`	$\ell 4$ `while random do`
$\ell 2$ `if x < y then`	$\ell 5$ `if y < 100 then`
$\ell 3$ `x := x + 1`	$\ell 6$ `y := y + [1,3]`

Equation system:

$$
\begin{aligned}
\mathcal{X}_{1,4} &= \mathcal{I} \\
\mathcal{X}_{2,4} &= \mathcal{X}_{1,4} \cup \mathsf{C}[\![x \geq y]\!]\, \mathcal{X}_{2,4} \cup \mathsf{C}[\![x := x + 1]\!]\, \mathcal{X}_{3,4} \\
\mathcal{X}_{3,4} &= \mathsf{C}[\![x < y]\!]\, \mathcal{X}_{2,4} \\
\mathcal{X}_{1,5} &= \mathcal{X}_{1,4} \cup \mathsf{C}[\![y \geq 100]\!]\, \mathcal{X}_{1,5} \cup \mathsf{C}[\![y := y + [1,3]]\!]\, \mathcal{X}_{1,6} \\
\mathcal{X}_{2,5} &= \mathcal{X}_{1,5} \cup \mathsf{C}[\![x \geq y]\!]\, \mathcal{X}_{2,5} \cup \mathsf{C}[\![x := x + 1]\!]\, \mathcal{X}_{3,5} \cup \\
&\qquad \mathcal{X}_{2,4} \cup \mathsf{C}[\![y \geq 100]\!]\, \mathcal{X}_{2,5} \cup \mathsf{C}[\![y := y + [1,3]]\!]\, \mathcal{X}_{2,6} \\
\mathcal{X}_{3,5} &= \mathsf{C}[\![x < y]\!]\, \mathcal{X}_{2,5} \cup \mathcal{X}_{3,4} \cup \mathsf{C}[\![y \geq 100]\!]\, \mathcal{X}_{3,5} \cup \mathsf{C}[\![y := y + [1,3]]\!]\, \mathcal{X}_{3,6} \\
\mathcal{X}_{1,6} &= \mathsf{C}[\![y < 100]\!]\, \mathcal{X}_{1,5} \\
\mathcal{X}_{2,6} &= \mathcal{X}_{1,6} \cup \mathsf{C}[\![x \geq y]\!]\, \mathcal{X}_{2,6} \cup \mathsf{C}[\![x := x + 1]\!]\, \mathcal{X}_{3,6} \cup \mathsf{C}[\![y < 100]\!]\, \mathcal{X}_{2,5} \\
\mathcal{X}_{3,6} &= \mathsf{C}[\![x < y]\!]\, \mathcal{X}_{2,6} \cup \mathsf{C}[\![y < 100]\!]\, \mathcal{X}_{3,5}
\end{aligned}
$$

Equational state semantics (example)

Example: inferring $0 \leq x \leq y \leq 102$

t_1	t_2
$\ell 1$ `while random do`	$\ell 4$ `while random do`
$\ell 2$ `if x < y then`	$\ell 5$ `if y < 100 then`
$\ell 3$ `x := x + 1`	$\ell 6$ `y := y + [1,3]`

Pros:

- easy to construct
- easy to further abstract in an abstract domain $\mathcal{E}^\sharp$

Cons:

- explosion of the number of variables and equations
- explosion of the size of equations
  $\implies$ efficiency issues
- the equation system does *not* reflect the program structure
  (not defined by induction on the concurrent program)

Wish-list

We would like to:

- keep information attached to syntactic program locations
  (control points in $\mathcal{L}$, not control point tuples in $\mathbb{T} \to \mathcal{L}$)
- be able to abstract away control information
  (precision/cost trade-off control)
- avoid duplicating thread instructions
- have a computation structure based on the program syntax
  (denotational style)

Ideally: thread-modular denotational-style semantics

(analyze each thread independently by induction on its syntax)

Simple interference semantics

Intuition and example

t_1

```
ℓ1 while random do
  ℓ2 if x < y then
    ℓ3 x := x + 1
```

t_2

```
ℓ4 while random do
  ℓ5 if y < 100 then
    ℓ6 y := y + [1,3]
```

Principle:

- analyze each thread in isolation

  but also gather interferences

  (abstraction of) the values stored into each variable by each thread

- re-analyze the threads taking interferences into account

  (variable read returns the last value written, or an interference)

  gather new sets of interferences

- iterate until stabilization

  $\Longrightarrow$ one more level of fixpoint iteration

Intuition and example

t_1

```
ℓ1 while random do
  ℓ2 if x < y then
    ℓ3 x := x + 1
```

t_2

```
ℓ4 while random do
  ℓ5 if y < 100 then
    ℓ6 y := y + [1,3]
```

Analysis of t_1 in isolation

(1): $x = y = 0$ $\qquad \mathcal{X}_1 = I$

(2): $x = y = 0$ $\qquad \mathcal{X}_2 = \mathcal{X}_1 \cup \mathsf{C}[\![x \leftarrow x + 1]\!] \mathcal{X}_3 \cup \mathsf{C}[\![x \geq y]\!] \mathcal{X}_2$

(3): $\perp$ $\qquad \mathcal{X}_3 = \mathsf{C}[\![x < y]\!] \mathcal{X}_2$

Intuition and example

t_1

```
ℓ1 while random do
  ℓ2 if x < y then
    ℓ3 x := x + 1
```

t_2

```
ℓ4 while random do
  ℓ5 if y < 100 then
    ℓ6 y := y + [1,3]
```

Analysis of t_2 in isolation

(4): $x = y = 0$ $\qquad \mathcal{X}_4 = I$

(5): $x = 0,\ y \in [0, 102]$ $\qquad \mathcal{X}_5 = \mathcal{X}_4 \cup \mathsf{C}[\![\, y \leftarrow y + [1,3] \,]\!]\, \mathcal{X}_6 \cup \mathsf{C}[\![\, y \geq 100 \,]\!]\, \mathcal{X}_5$

(6): $x = 0,\ y \in [0, 99]$ $\qquad \mathcal{X}_6 = \mathsf{C}[\![\, y < 100 \,]\!]\, \mathcal{X}_5$

output interferences: $y \leftarrow [1, 102]$

Intuition and example

t_1

```
ℓ1 while random do
  ℓ2 if x < y then
    ℓ3 x := x + 1
```

t_2

```
ℓ4 while random do
  ℓ5 if y < 100 then
    ℓ6 y := y + [1,3]
```

Re-analysis of t_1 with interferences from t_2

input interferences: $y \leftarrow [1, 102]$

(1): $x = y = 0$ $\qquad \mathcal{X}_1 = I$

(2): $x \in [0, 102],\ y = 0$ $\qquad \mathcal{X}_2 = \mathcal{X}_{1a} \cup \mathsf{C}[\![\, x \leftarrow x + 1 \,]\!]\, \mathcal{X}_3 \cup \mathsf{C}[\![\, x \geq (y \mid [1, 102]) \,]\!]\, \mathcal{X}_2$

(3): $x \in [0, 102],\ y = 0$ $\qquad \mathcal{X}_3 = \mathsf{C}[\![\, x < (y \mid [1, 102]) \,]\!]\, \mathcal{X}_2$

output interferences: $x \leftarrow [1, 102]$

subsequent re-analyses are identical (fixpoint reached)

Intuition and example

t_1
```
ℓ1 while random do
  ℓ2 if x < y then
    ℓ3 x := x + 1
```

t_2
```
ℓ4 while random do
  ℓ5 if y < 100 then
    ℓ6 y := y + [1,3]
```

Derived abstract analysis:

- similar to a sequential program analysis, but iterated
  (can be parameterized by arbitrary abstract domains)
- efficient (few reanalyses are required in practice)
- interferences are non-relational and flow-insensitive
  (limit inherited from the concrete semantics)

Limitation:

we get $x, y \in [0, 102]$; we don't get that $x \leq y$

simplistic view of thread interferences (volatile variables)

based on an incomplete concrete semantics!

Denotational semantics with interferences

Interferences in $\mathbb{I} \stackrel{\text{def}}{=} \mathbb{T} \times \mathbb{V} \times \mathbb{R}$

$\langle\, t,\, \mathtt{X},\, v \,\rangle$ means: t can store the value v into the variable $\mathtt{X}$

We define the analysis of a thread t
with respect to a set of interferences $I \subseteq \mathbb{I}$.

Expressions with interference: for thread t

$$\mathrm{E_t}[\![\, \mathrm{exp} \,]\!] : (\mathcal{E} \times \mathcal{P}(\mathbb{I})) \to (\mathcal{P}(\mathbb{R}) \times \mathcal{P}(\Omega))$$

- Apply interferences to read variables:
  $\mathrm{E_t}[\![\, \mathtt{X} \,]\!]\langle\, \rho,\, I \,\rangle \stackrel{\text{def}}{=} \langle\, \{\, \rho(\mathtt{X}) \,\} \cup \{\, v \mid \exists u \neq t \colon \langle\, u,\, \mathtt{X},\, v \,\rangle \in I \,\},\, \emptyset \,\rangle$
- Pass recursively I down to sub-expressions:
  $\mathrm{E_t}[\![\, -e \,]\!]\langle\, \rho,\, I \,\rangle \stackrel{\text{def}}{=}$
  $\quad \mathbf{let}\ \langle\, V,\, O \,\rangle = \mathrm{E_t}[\![\, e \,]\!]\langle\, \rho,\, I \,\rangle\ \mathbf{in}\ \langle\, \{\, -v \mid v \in V \,\},\, O \,\rangle$
- ...

Denotational semantics with interferences (cont.)

Statements with interference: for thread t

$$\mathsf{C_t}[\![\,\texttt{prog}\,]\!] : (\mathcal{P}(\mathcal{E}) \times \mathcal{P}(\Omega) \times \mathcal{P}(\mathbb{I})) \to (\mathcal{P}(\mathcal{E}) \times \mathcal{P}(\Omega) \times \mathcal{P}(\mathbb{I}))$$

- pass interferences to expressions
- collect new interferences due to assignments
- accumulate interferences from inner statements

$\mathsf{C_t}[\![\, \texttt{X} := \texttt{e} \,]\!] \langle R, O, I \rangle \stackrel{\text{def}}{=}$
$\langle \emptyset, O, I \rangle \sqcup \bigsqcup_{\rho \in R} \langle \{ \rho[\texttt{X} \mapsto v] \mid v \in V_\rho \}, O_\rho, \{ \langle t, \texttt{X}, v \rangle \mid v \in V_\rho \} \rangle$

$\mathsf{C_t}[\![\, s_1;\, s_2 \,]\!] \stackrel{\text{def}}{=} \mathsf{C_t}[\![\, s_2 \,]\!] \circ \mathsf{C_t}[\![\, s_1 \,]\!]$

...

noting $\langle V_\rho, O_\rho \rangle \stackrel{\text{def}}{=} \mathsf{E_t}[\![\, e \,]\!] \langle \rho, I \rangle$
$\sqcup$ is now the element-wise $\cup$ in $\mathcal{P}(\mathcal{E}) \times \mathcal{P}(\Omega) \times \mathcal{P}(\mathbb{I})$

Denotational semantics with interferences (cont.)

Program semantics: $\mathsf{P}[\![\,\mathtt{parprog}\,]\!] \subseteq \Omega$

Given $\mathtt{parprog} ::= \mathtt{prog}_1 \,||\, \cdots \,||\, \mathtt{prog}_n$, we compute:

$$\mathsf{P}[\![\,\mathtt{parprog}\,]\!] \stackrel{\text{def}}{=} \left[\mathrm{lfp}\, \lambda\langle\, O,\, I\,\rangle. \bigsqcup_{t\in\mathbb{T}} \, [\mathsf{C}_t[\![\,\mathtt{prog}_t\,]\!]\,\langle\, \mathcal{E}_0,\, \emptyset,\, I\,\rangle]_{\Omega,\mathbb{I}}\right]_\Omega$$

- each thread analysis starts in an initial environment set $\mathcal{E}_0 \stackrel{\text{def}}{=} \{\, \lambda \mathtt{V}.0 \,\}$
- $[\mathtt{X}]_{\Omega,\mathbb{I}}$ projects $\mathtt{X} \in \mathcal{P}(\mathcal{E}) \times \mathcal{P}(\Omega) \times \mathcal{P}(\mathbb{I})$ on $\mathcal{P}(\Omega) \times \mathcal{P}(\mathbb{I})$ and interferences and errors from all threads are joined

  (the output environments are ignored)
- $\mathsf{P}[\![\,\mathtt{parprog}\,]\!]$ only outputs the set of possible run-time errors

Interference abstraction

Abstract interferences $\mathbb{I}^\sharp$

$\mathcal{P}(\mathbb{I}) \stackrel{\text{def}}{=} \mathcal{P}(\mathbb{T} \times \mathbb{V} \times \mathbb{R})$ is abstracted as $\mathbb{I}^\sharp \stackrel{\text{def}}{=} (\mathbb{T} \times \mathbb{V}) \to \mathcal{R}^\sharp$
where $\mathcal{R}^\sharp$ abstracts $\mathcal{P}(\mathbb{R})$ (e.g. intervals)

Abstract semantics with interferences $\mathsf{C}^\sharp_t[\![\, s \,]\!]$

derived from $\mathsf{C}^\sharp[\![\, s \,]\!]$ in a generic way:

Example: $\mathsf{C}^\sharp_t[\![\, \mathtt{X} := \mathsf{e} \,]\!]\langle\, R^\sharp,\, \Omega,\, I^\sharp \,\rangle$

- for each $\mathtt{Y}$ in e, get its interference $\mathtt{Y}^\sharp_{\mathcal{R}} = \bigsqcup^\sharp_{\mathcal{R}} \{\, I^\sharp\langle\, u,\, \mathtt{Y} \,\rangle \mid u \neq t \,\}$
- if $\mathtt{Y}^\sharp_{\mathcal{R}} \neq \bot^\sharp_{\mathcal{R}}$, replace $\mathtt{Y}$ in e with $get\langle\, \mathtt{Y},\, R^\sharp \,\rangle \sqcup^\sharp_{\mathcal{R}} \mathtt{Y}^\sharp_{\mathcal{R}}$
  (where $get(\mathtt{Y}, R^\sharp)$ extracts the abstract values in $\mathcal{R}^\sharp$ of a variable $\mathtt{Y}$ from $R^\sharp \in \mathcal{E}^\sharp$)
- compute $\langle\, R^{\sharp\prime},\, O' \,\rangle = \mathsf{C}^\sharp[\![\, e \,]\!]\langle\, R^\sharp,\, O \,\rangle$
- enrich $I^\sharp\langle\, t,\, \mathtt{X} \,\rangle$ with $get(\mathtt{X}, R^{\sharp\prime})$

Static analysis with interferences

Abstract analysis

$$\mathsf{P}^\sharp[\![\,\texttt{parprog}\,]\!] \stackrel{\text{def}}{=} \left[\lim \lambda\langle\, O,\, I^\sharp \,\rangle.\langle\, O,\, I^\sharp \,\rangle \mathbin{\triangledown} \bigsqcup^\sharp_{t\in\mathbb{T}} \left[\mathsf{C}^\sharp_{\mathsf{t}}[\![\,\texttt{prog}_t\,]\!]\,\langle\, \mathcal{E}^\sharp_0,\, \emptyset,\, I^\sharp \,\rangle\right]_{\Omega,\mathbb{I}^\sharp}\right]_\Omega$$

- effective analysis by structural induction
- termination ensured by a widening
- parametrized by a choice of abstract domains $\mathcal{R}^\sharp$, $\mathcal{E}^\sharp$

- interferences are flow-insensitive and non-relational in $\mathcal{R}^\sharp$
- thread analysis remains flow-sensitive and relational in $\mathcal{E}^\sharp$

(reminder: $[X]_\Omega$, $[Y]_{\Omega,\mathbb{I}^\sharp}$ keep only X's component in Ω, Y's components in Ω and $\mathbb{I}^\sharp$)

Abstract rely-guarantee

Rely-guarantee proof method

Reminder: Floyd–Hoare logic

Logic to prove properties about sequential programs [Hoar69].

Hoare triples: $\{P\}\,\texttt{prog}\,\{Q\}$

- annotate programs with logic assertions $\{P\}\,\texttt{prog}\,\{Q\}$

  (if P holds before prog, then Q holds after prog)

- check that $\{P\}\texttt{prog}\{Q\}$ is derivable with the following rules

  (the assertions are program invariants)

$$\frac{}{\{P[e/\texttt{X}]\}\,\texttt{X} := \texttt{e}\,\{P\}} \qquad \frac{\{P \wedge e \bowtie 0\}\,s\,\{Q\} \quad P \wedge e \not\bowtie 0 \Rightarrow Q}{\{P\}\,\texttt{if}\ e \bowtie 0\ \texttt{then}\ s\ \texttt{fi}\,\{Q\}}$$

$$\frac{\{P\}\,s_1\,\{Q\} \quad \{Q\}\,s_2\,\{R\}}{\{P\}\,s_1; s_2\,\{R\}} \qquad \frac{\{P \wedge e \bowtie 0\}\,s\,\{P\}}{\{P\}\,\texttt{while}\ e \bowtie 0\ \texttt{do}\ s\ \texttt{done}\,\{P \wedge e \not\bowtie 0\}}$$

$$\frac{\{P'\}\,s\,\{Q'\} \quad P \Rightarrow P' \quad Q' \Rightarrow Q}{\{P\}\,s\,\{Q\}}$$

Floyd–Hoare logic as abstract interpretation

Link with the equational state semantics: $(\mathcal{X}_\ell)_{\ell \in \mathcal{L}}$

Correspondence between ${}^{\ell}\texttt{prog}^{\ell'}$ and $\{P\}\,\texttt{prog}\,\{Q\}$:

- if P (resp. Q) models exactly the points in $\mathcal{X}_\ell$ (resp. $\mathcal{X}_{\ell'}$) then $\{P\}\,\texttt{prog}\,\{Q\}$ is a derivable Hoare triple
- if $\{P\}\,\texttt{prog}\,\{Q\}$ is derivable, then $\mathcal{X}_\ell \models P$ and $\mathcal{X}_{\ell'} \models Q$
  (all the points in $\mathcal{X}_\ell$ (resp. $\mathcal{X}_{\ell'}$) satisfy P (resp. Q))

$\Longrightarrow \mathcal{X}_\ell$ provides the most precise Hoare assertions in a constructive form

- $\gamma(\mathcal{X}_\ell^\sharp)$ provides (less precise) Hoare assertions in a computable form

Owicki–Gries proof method

Extension of Floyd–Hoare to concurrent programs [Owic76].

Principle: add a new rule, for $||$

$$\frac{\{P_1\}\, s_1\, \{Q_1\} \quad \{P_2\}\, s_2\, \{Q_2\}}{\{P_1 \wedge P_2\}\, s_1 \,||\, s_2\, \{Q_1 \wedge Q_2\}}$$

Owicki–Gries proof method

Extension of Floyd–Hoare to concurrent programs [Owic76].

Principle: add a new rule, for $||$

$$\frac{\{P_1\}\,s_1\,\{Q_1\}\quad\{P_2\}\,s_2\,\{Q_2\}}{\{P_1\wedge P_2\}\,s_1\,||\,s_2\,\{Q_1\wedge Q_2\}}$$

This rule is not always sound!

e.g., we have $\{\mathtt{X}=0,\mathtt{Y}=0\}\ \mathtt{X}:=1\ \{\mathtt{X}=1,\mathtt{Y}=0\}$

and $\{\mathtt{X}=0,\mathtt{Y}=0\}\ \mathtt{if}\ \mathtt{X}=0\ \mathtt{then}\ \mathtt{Y}:=1\ \mathtt{fi}\ \{\mathtt{X}=0,\mathtt{Y}=1\}$

but not $\{\mathtt{X}=0,\mathtt{Y}=0\}\ \mathtt{X}:=1\ ||\ \mathtt{if}\ \mathtt{X}=0\ \mathtt{then}\ \mathtt{Y}:=1\ \mathtt{fi}\ \{\mathit{false}\}$

$\Longrightarrow$ we need a side-condition to the rule:
$\{P_1\}\,s_1\,\{Q_1\}$ and $\{P_2\}\,s_2\,\{Q_2\}$ must not interfere

Owicki–Gries proof method (cont.)

interference freedom

given proofs Δ_1 and Δ_2 of $\{P_1\}\, s_1\, \{Q_1\}$ and $\{P_2\}\, s_2\, \{Q_2\}$

Δ_1 does not interfere with Δ_2 if:
for any Φ appearing before a statement in Δ_1
for any $\{P_2'\}\, s_2'\, \{Q_2'\}$ appearing in Δ_2
$\{\Phi \wedge P_2'\}\, s_2'\, \{\Phi\}$ holds
and moreover $\{Q_1 \wedge P_2'\}\, s_2'\, \{Q_1\}$

i.e.: the assertions used to prove $\{P_1\}\, s_1\, \{Q_1\}$ are stable by s_2

e.g., $\{\mathtt{X} = 0, \mathtt{Y} \in [0,1]\}\ \mathtt{X} := 1\ \{\mathtt{X} = 1, \mathtt{Y} \in [0,1]\}$
$\{\mathtt{X} \in [0,1], \mathtt{Y} = 0\}\ \mathtt{if}\ \mathtt{X} = 0\ \mathtt{then}\ \mathtt{Y} := 1\ \mathtt{fi}\ \{\mathtt{X} \in [0,1], \mathtt{Y} \in [0,1]\}$
$\Longrightarrow$ $\{\mathtt{X} = 0, \mathtt{Y} = 0\}\ \mathtt{X} := 1 \mid\mid \mathtt{if}\ \mathtt{X} = 0\ \mathtt{then}\ \mathtt{Y} := 1\ \mathtt{fi}\ \{\mathtt{X} = 1, \mathtt{Y} \in [0,1]\}$

Summary:

- pros: the invariants are local to threads
- cons: the proof is not compositional

  (proving one thread requires delving into the proof of other threads)

$\Longrightarrow$ not satisfactory

Jones' rely-guarantee proof method

Idea: explicit interferences with (more) annotations [Jone81].

Rely-guarantee "quintuples": $R, G \vdash \{P\}\,\texttt{prog}\,\{Q\}$

- if P is true before `prog` is executed
- and the effect of other threads is included in R (rely)
- then Q is true after `prog`
- and the effect of `prog` is included in G (guarantee)

where:

- P and Q are assertions on states (in $\mathcal{P}(\Sigma)$)
- R and G are assertions on transitions (in $\mathcal{P}(\Sigma \times \mathcal{A} \times \Sigma)$)

The parallel composition rule becomes:

$$\frac{R \vee G_2, G_1 \vdash \{P_1\}\, s_1\, \{Q_1\} \quad R \vee G_1, G_2 \vdash \{P_2\}\, s_2\, \{Q_2\}}{R, G_1 \vee G_2 \vdash \{P_1 \wedge P_2\}\, s_1 \,||\, s_2\, \{Q_1 \wedge Q_2\}}$$

Rely-guarantee example

checking t_1

```
ℓ1 while random do
  ℓ2 if x < y then
    ℓ3 x := x+1
  fi
done
```

$\ell 1$: $x = y = 0$
$\ell 2$: $x, y \in [0, 102]$, $x \leq y$
$\ell 3$: $x \in [0, 101]$, $y \in [1, 102]$, $x < y$

checking t_2

```
ℓ4 while random do
  ℓ5 if y < 100 then
    ℓ6 y := y + [1,3]
  fi
done
```

at $\ell 4$: $x = y = 0$
at $\ell 5$: $x, y \in [0, 102]$, $x \leq y$
at $\ell 6$: $x \in [0, 99]$, $y \in [0, 99]$, $x \leq y$

Rely-guarantee example

checking t_1

```
ℓ1 while random do
    ℓ2 if x < y then
        ℓ3 x := x+1
   fi
done
```

x unchanged
y incremented
$0 \le y \le 102$

$\ell 1: \; x = y = 0$
$\ell 2: \; x, y \in [0, 102], \; x \le y$
$\ell 3: \; x \in [0, 101], \; y \in [1, 102], \; x < y$

checking t_2

y unchanged
$0 \le x \le y$

```
ℓ4 while random do
    ℓ5 if y < 100 then
        ℓ6 y := y + [1,3]
   fi
done
```

at $\ell 4: \; x = y = 0$
at $\ell 5: \; x, y \in [0, 102], \; x \le y$
at $\ell 6: \; x \in [0, 99], \; y \in [0, 99], \; x \le y$

In this example:

- guarantee exactly what is relied on $(R_1 = G_1$ and $R_2 = G_2)$
- rely and guarantee are global assertions

Benefits of rely-guarantee:

- invariants are still local to threads
- checking a thread does not require looking at other threads, only at an abstraction of their semantics

Auxiliary variables

Example

t_1	t_2
$^{\ell 1}$ $\mathtt{x} := \mathtt{x} + 1$ $^{\ell 2}$	$^{\ell 3}$ $\mathtt{x} := \mathtt{x} + 1$ $^{\ell 4}$

Goal: prove $\{\mathtt{x} = 0\}\ t_1 \mid\mid t_2\ \{\mathtt{x} = 2\}$.

Auxiliary variables

Example

t_1	t_2
$^{\ell 1}$ x := x + 1 $^{\ell 2}$	$^{\ell 3}$ x := x + 1 $^{\ell 4}$

Goal: prove $\{x = 0\}\ t_1 \mid\mid t_2\ \{x = 2\}$.

we must rely on and guarantee that
each thread increments x exactly once!

Solution: auxiliary variables
do not change the semantics but store extra information:

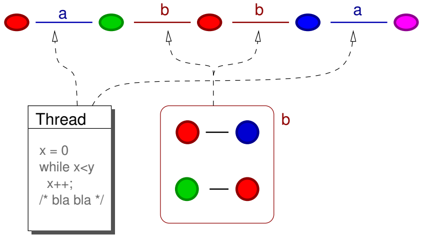
- past values of variables (history of the computation)
- program counter of other threads (pc_t)

Example: for t_1:

$$\{(pc_2 = \ell 3 \wedge x = 0) \vee (pc_2 = \ell 4 \wedge x = 1)\}$$
$$x := x + 1$$
$$\{(pc_2 = \ell 3 \wedge x = 1) \vee (pc_2 = \ell 4 \wedge x = 2)\}$$

Rely-guarantee as abstract interpretation

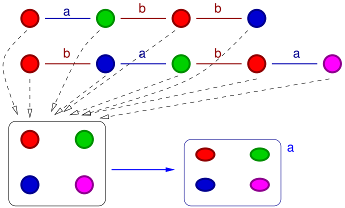
Modularity: main idea



Main idea: separate execution steps

- from the current thread *a*
  - found by analysis by induction on the syntax of *a*
- from other threads *b*
  - given as parameter in the analysis of *a*
  - inferred during the analysis of *b*

Trace decomposition



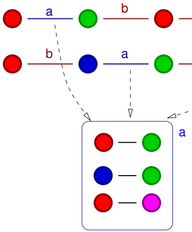
Reachable states projected on thread t: $\mathcal{RI}(t)$

- attached to thread control point in $\mathcal{L}$, not control state in $\mathbb{T} \to \mathcal{L}$
- remember other thread's control point as "auxiliary variables"
  (required for completeness)

$$\mathcal{RI}(t) \stackrel{\text{def}}{=} \pi_t(\mathcal{R}) \subseteq \mathcal{L} \times (\mathbb{V} \cup \{ pc_{t'} \mid t \neq t' \in \mathbb{T} \}) \to \mathbb{R}$$

where $\pi_t(R) \stackrel{\text{def}}{=} \{ \langle L(t), \rho[\forall t' \neq t : pc_{t'} \mapsto L(t')] \rangle \mid \langle L, \rho \rangle \in R \}$

Trace decomposition



Interferences generated by t: $A(t)$ ($\simeq$ guarantees on transitions)

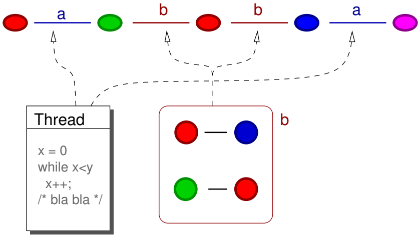
Extract the transitions with action t observed in $\mathcal{T}_p$

(subset of the transition system, containing only transitions actually used in reachability)

$A(t) \stackrel{\text{def}}{=} \alpha^{\mathbb{I}}(\mathcal{T}_p)(t)$

where $\alpha^{\mathbb{I}}(X)(t) \stackrel{\text{def}}{=} \{\, \langle\, \sigma_i,\, \sigma_{i+1} \,\rangle \,|\, \exists \sigma_0 \xrightarrow{a_1} \sigma_1 \cdots \xrightarrow{a_n} \sigma_n \in X \colon a_{i+1} = t \,\}$

Thread-modular concrete semantics



Principle: express $\mathcal{R}I(t)$ and $A(t)$ directly, without computing $\mathcal{T}_p$

States: $\mathcal{R}I$

Interleave:

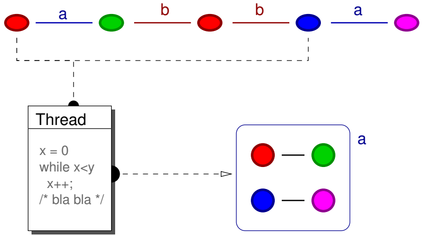
- transitions from the current thread t
- transitions from interferences A by other threads

$\mathcal{R}I(t) = \text{lfp } R_t(A)$, where

$$R_t(Y)(X) \stackrel{\text{def}}{=} \pi_t(I) \cup \{\, \pi_t(\sigma') \mid \exists \pi_t(\sigma) \in X \colon \sigma \xrightarrow{t}_\tau \sigma' \,\} \cup \{\, \pi_t(\sigma') \mid \exists \pi_t(\sigma) \in X \colon \exists t' \neq t \colon \langle\, \sigma, \sigma' \,\rangle \in Y(t') \,\}$$

$\Longrightarrow$ similar to reachability for a sequential program, up to A

Thread-modular concrete semantics



Principle: express $\mathcal{RI}(t)$ and $A(t)$ directly, without computing $\mathcal{T}_p$

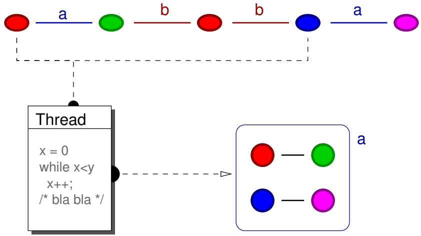
Interferences: A

Collect transitions from a thread t and reachable states $\mathcal{R}$:

$A(t) = B(\mathcal{RI})(t)$, where

$B(Z)(t) \stackrel{\text{def}}{=} \{ \langle \sigma, \sigma' \rangle \mid \pi_t(\sigma) \in Z(t) \wedge \sigma \xrightarrow{t}_{\tau} \sigma' \}$

Thread-modular concrete semantics



Principle: express $\mathcal{RI}(t)$ and $A(t)$ directly, without computing $\mathcal{T}_p$

Recursive definition:

- $\mathcal{RI}(t) = \operatorname{lfp} R_t(A)$
- $A(t) = B(\mathcal{RI})(t)$

$\Longrightarrow$ express the most precise solution as nested fixpoints:

$$\mathcal{RI} = \operatorname{lfp} \lambda Z.\lambda t.\operatorname{lfp} R_t(B(Z))$$

Completeness: $\forall t$: $\mathcal{RI}(t) \simeq \mathcal{R}$ (π_t is bijective thanks to auxiliary variables)

Fixpoint form

Constructive fixpoint form:

Use Kleene's iteration to construct fixpoints:

- $\mathcal{R}I = \text{lfp } H = \bigsqcup_{n \in \mathbb{N}} H^n(\lambda t.\emptyset)$
  in the pointwise powerset lattice $\prod_{t \in \mathbb{T}} \{t\} \to \mathcal{P}(\Sigma_t)$
- $H(Z)(t) = \text{lfp } R_t(B(Z)) = \bigcup_{n \in \mathbb{N}} (R_t(B(Z)))^n(\emptyset)$
  in the powerset lattice $\mathcal{P}(\Sigma_t)$

  (similar to the sequential semantics of thread t in isolation)

$\Longrightarrow$ nested iterations

Abstract rely-guarantee

Suggested algorithm: nested iterations with acceleration

once abstract domains for states and interferences are chosen

- start from $\mathcal{RI}_0^\sharp \stackrel{\text{def}}{=} A_0^\sharp \stackrel{\text{def}}{=} \lambda t.\bot^\sharp$
- while $A_n^\sharp$ is not stable
  - compute $\forall t \in \mathbb{T}: \mathcal{RI}_{n+1}^\sharp(t) \stackrel{\text{def}}{=} \text{lfp } R_t^\sharp(A_n^\sharp)$
    by iteration with widening $\triangledown$

    ($\simeq$ separate analysis of each thread)
  - compute $A_{n+1}^\sharp \stackrel{\text{def}}{=} A_n^\sharp \triangledown B^\sharp(\mathcal{RI}_{n+1}^\sharp)$
- when $A_n^\sharp = A_{n+1}^\sharp$, return $\mathcal{RI}_n^\sharp$

$\Longrightarrow$ thread-modular analysis
parameterized by abstract domains
able to easily reuse existing sequential analyses

Thread-modular abstractions

Flow-insensitive abstraction

Flow-insensitive abstraction:

- reduce as much control information as possible
- but keep flow-sensitivity on each thread's control location

Local state abstraction: remove auxiliary variables

$$\alpha_{\mathcal{R}}^{nf}(X) \stackrel{\text{def}}{=} \{ (\ell, \rho_{|\mathbb{V}}) \mid (\ell, \rho) \in X \} \cup (X \cap \Omega)$$

Interference abstraction: remove all control state

$$\alpha_{A}^{nf}(Y) \stackrel{\text{def}}{=} \{ (\rho, \rho') \mid \exists L, L' \in \mathbb{T} \to \mathcal{L}: ((L, \rho), (L', \rho')) \in Y \}$$

Flow-insensitive abstraction (cont.)

Flow-insensitive fixpoint semantics: (omitting errors Ω)

We apply $\alpha^{nf}_{\mathcal{R}}$ and α^{nf}_{A} to the nested fixpoint semantics.

$\mathcal{RI}^{nf} \stackrel{\text{def}}{=} \text{lfp}\, \lambda Z.\lambda t.\, \text{lfp}\, R^{nf}{}_t(B^{nf}(Z))$, where

$B^{nf}(Z)(t) \stackrel{\text{def}}{=} \{\, (\rho, \rho') \mid \exists \ell, \ell' \in \mathcal{L}: (\ell, \rho) \in Z(t) \wedge (\ell, \rho) \to_t (\ell', \rho') \,\}$
(extract interferences from reachable states)

$R^{nf}_t(Y)(X) \stackrel{\text{def}}{=} R^{loc}_t(X) \cup A^{nf}_t(Y)(X)$ (interleave steps)

$R^{loc}_t(X) \stackrel{\text{def}}{=} \{(\ell^i_t, \lambda \mathtt{V}.0)\} \cup \{\, (\ell', \rho') \mid \exists (\ell, \rho) \in X: (\ell, \rho) \to_t (\ell', \rho') \,\}$ (thread step)

$A^{nf}_t(Y)(X) \stackrel{\text{def}}{=} \{\, (\ell, \rho') \mid \exists \rho,\, u \neq t: (\ell, \rho) \in X \wedge (\rho, \rho') \in Y(u) \,\}$ (interference step)

where $\to_t$ is the transition relation for thread t alone: $\tau[\mathtt{prog}_t]$

Cost/precision trade-off:

- less variables
  $\Longrightarrow$ subsequent numeric abstractions are more efficient
- sufficient to analyze our first example (slide 26)
- insufficient to analyze $\mathtt{x} := \mathtt{x} + 1 \,||\, \mathtt{x} := \mathtt{x} + 1$ (slide 35)

Retrieving the simple interference-based analysis

Cartesian abstraction: on interferences

- forget the relations between variables
- forget the relations between values before and after transitions (input-output relationship)
- only remember which variables are modified, and their value:

$$\alpha_A^{nr}(Y) \stackrel{\text{def}}{=} \lambda \mathtt{V}.\{\, x \in \mathbb{V} \mid \exists (\rho, \rho') \in Y \colon \rho(\mathtt{V}) \neq x \wedge \rho'(\mathtt{V}) = x \,\}$$

- to apply interferences, we get, in the nested fixpoint form:

$$A_t^{nr}(Y)(X) \stackrel{\text{def}}{=} \{\, (\ell, \rho[\mathtt{V} \mapsto v]) \mid (\ell, \rho) \in X, \mathtt{V} \in \mathbb{V}, \exists u \neq t \colon v \in Y(u)(\mathtt{V}) \,\}$$

- no modification on the state (the analysis of each thread can still be relational)

$\Longrightarrow$ we get back our simple interference analysis!

Finally, use a numeric abstract domain $\alpha : \mathcal{P}(\mathbb{V} \to \mathbb{R}) \to \mathcal{D}^\sharp$
(for interferences, $\mathbb{V} \to \mathcal{P}(\mathbb{R})$ is abstracted as $\mathbb{V} \to \mathcal{D}^\sharp$)

A note on unbounded threads

Extension: relax the finiteness constraint on $\mathbb{T}$

- there is still a finite syntactic set of threads $\mathbb{T}_s$
- some threads $\mathbb{T}_\infty \subseteq \mathbb{T}_s$ can have several instances

  (possibly an unbounded number)

Flow-insensitive analysis:

- local state and interference domains have finite dimensions

  ($\mathcal{E}_t$ and $(\mathcal{L} \times \mathcal{E}) \times (\mathcal{L} \times \mathcal{E})$, as opposed to $\mathcal{E}$ and $\mathcal{E} \times \mathcal{E}$)

- all instances of a thread $t \in \mathbb{T}_s$ are isomorphic

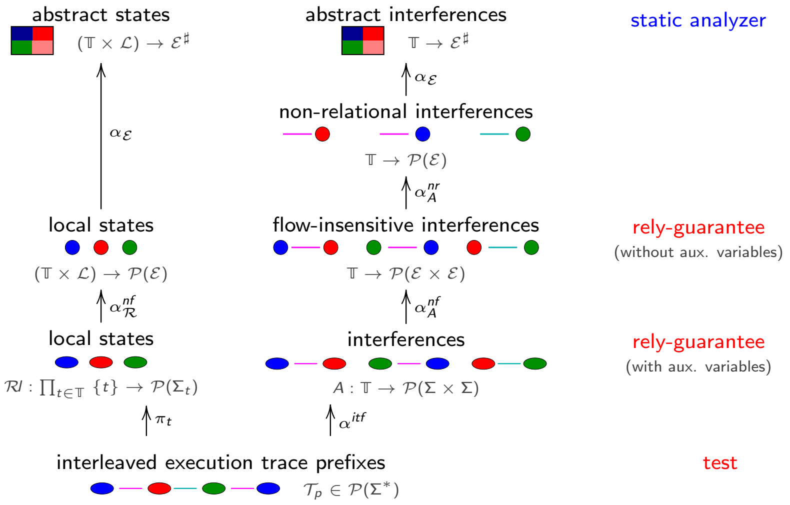
  $\Longrightarrow$ iterate the analysis on the finite set $\mathbb{T}_s$ (instead of $\mathbb{T}$)

- we must handle self-interferences for threads in $\mathbb{T}_\infty$:

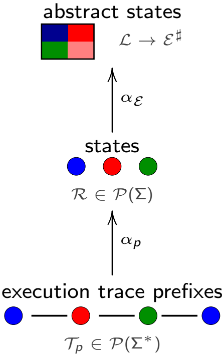
  $A_t^{nf}(Y)(X) \stackrel{\text{def}}{=}$

  $\{\,(\ell,\rho') \mid \exists \rho,\, u{:}\, (u \neq t \vee t \in \mathbb{T}_\infty) \wedge (\ell,\rho) \in X \wedge (\rho,\rho') \in Y(u)\,\}$

From traces to thread-modular analyses



Compare with sequential analyses



static analyzer

reachability

test

Beyond simple interferences

Scheduling

Synchronization primitives

$$\begin{array}{lcl} \texttt{prog} & ::= & \texttt{lock}(m) \\ & | & \texttt{unlock}(m) \end{array}$$

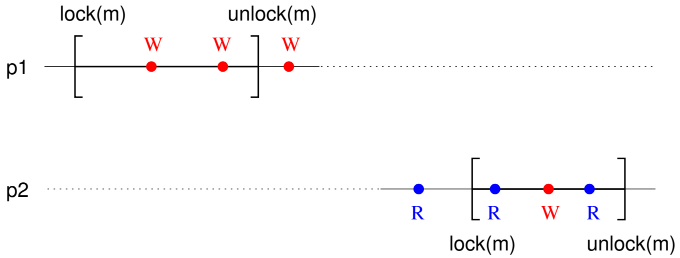
$m \in \mathbb{M}$: finite set of non-recursive mutexes

Scheduling

mutexes ensure mutual exclusion

at each time, each mutex can be locked by a single thread

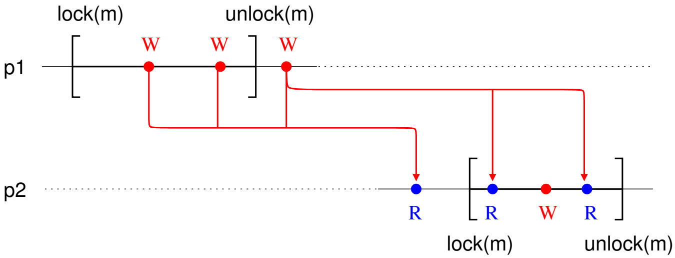
Mutual exclusion



We use a refinement of the simple interference semantics by partitioning wrt. an abstract local view of the scheduler $\mathbb{C}$

- $\mathcal{E} \rightsquigarrow \mathcal{E} \times \mathbb{C}$, $\quad \mathcal{E}^\sharp \rightsquigarrow \mathbb{C} \to \mathcal{E}^\sharp$
- $\mathbb{I} \stackrel{\text{def}}{=} \mathbb{T} \times \mathbb{V} \times \mathbb{R} \rightsquigarrow \mathbb{I} \stackrel{\text{def}}{=} \mathbb{T} \times \mathbb{C} \times \mathbb{V} \times \mathbb{R}$,
  $\mathbb{I}^\sharp \stackrel{\text{def}}{=} (\mathbb{T} \times \mathbb{V}) \to \mathcal{R}^\sharp \rightsquigarrow \mathbb{I}^\sharp \stackrel{\text{def}}{=} (\mathbb{T} \times \mathbb{C} \times \mathbb{V}) \to \mathcal{R}^\sharp$

Mutual exclusion

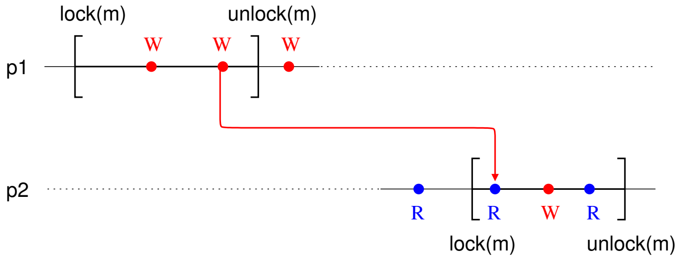


Data-race effects

Across read / write not protected by a mutex.
Partition wrt. mutexes $M \subseteq \mathbb{M}$ held by the current thread t.

- $\mathsf{C_t}[\![\, \mathtt{X} := e \,]\!]\langle \rho, M, I \rangle$ adds $\{ \langle t, M, \mathtt{X}, v \rangle \mid v \in \mathsf{E_t}[\![\, X \,]\!]\langle \rho, M, I \rangle \}$ to I
- $\mathsf{E_t}[\![\, \mathtt{X} \,]\!]\langle \rho, M, I \rangle =$ $\{ \rho(\mathtt{X}) \} \cup \{ v \mid \langle t', M', \mathtt{X}, v \rangle \in I,\ t \neq t',\ M \cap M' = \emptyset \}$

Mutual exclusion



Well-synchronized effects

- last write before unlock affects first read after lock
- partition interferences wrt. a protecting mutex m (and M)
- $C_t[\![\texttt{unlock}(m)]\!] \langle \rho, M, I \rangle$ stores $\rho(\texttt{X})$ into I
- $C_t[\![\texttt{lock}(m)]\!] \langle \rho, M, I \rangle$ imports values form I into ρ
- imprecision: non-relational, largely flow-insensitive

Example analysis

abstract consumer/producer

N consumers	N producers

`while 0=0 do`
`lock(m);` $^{\ell 1}$
`if X>0 then` $^{\ell 2}$`X:=X-1 fi;`
`unlock(m);`
$^{\ell 3}$`Y:=X`
`done`
	`while 0=0 do`
	`lock(m);`
	`X:=X+1;`
	`if X>100 then X:=100 fi;`
	`unlock(m)`
	`done`

Assuming we have several (N) producers and consumers:

- no data-race interference (proof of the absence of data-race)
- well-synchronized interferences:

  consumer: $x \leftarrow [0, 99]$

  producer: $x \leftarrow [1, 100]$
- $\implies$ we get that $x \in [0, 100]$

(without locks, if $N > 1$, our concrete semantics cannot bound x!)

Locks and priorities

priority-based critical sections

high thread	low thread
`L := isLocked(m);`	`lock(m);`
`if L = 0 then`	`Z := Y;`
`  Y := Y+1;`	`Y := 0;`
`  yield()`	`unlock(m)`

Real-time scheduling

- only the highest priority unblocked thread can run
- `lock` and `yield` may block
- `yielding` threads wake up non-deterministically (preempting lower-priority threads)
- explicit synchronisation enforces memory consistency

Locks and priorities

priority-based critical sections

high thread	low thread
`L := isLocked(m);`	`lock(m);`
`if L = 0 then`	`Z := Y;`
`  Y := Y+1;`	`Y := 0;`
`  yield()`	`unlock(m)`

Partition interferences and environments wrt. scheduling state

- partition wrt. mutexes tested with `isLocked`
- `X := isLocked(m)` creates two partitions
  - P_0 where $\texttt{X} = 0$ and m is free
  - P_1 where $\texttt{X} = 1$ and m is locked
- P_0 handled as if m where locked
- blocking primitives merge P_0 and P_1 (`lock`, `yield`)

Weakly relational interferences

Clock thread

```
while Clock < 10^6 do
  Clock := Clock + 1;
  ...
done
```

Accumulator thread

```
while random do
  Prec := Clock;
  ...
  delta := Clock - Prec;
  if random then x := x + delta endif;
  ...
done
```

- `lock` is a global, increasing clock
- `x` accumulates periods of time
- no overflow on `Clock - Prec`, nor `x := x + delta`

To prove this we need relational abstractions of interferences
(keep input-output relationships)

Monotonicity abstraction

Abstraction:

map variables to $\nearrow$ monotonic or $\top$ don't know

$$\alpha_A^{\text{mono}}(Y) \stackrel{\text{def}}{=} \lambda V.\text{if } \forall\langle\, \rho,\, \rho' \,\rangle \in Y \colon \rho(V) \le \rho'(V) \text{ then } \nearrow \text{ else } \top$$

- keep some input-output relationships
- forgets all relations between variables
- flow-insensitive

Inference and use

- **gather:**
  $A^{\text{mono}}(t)(V) = \nearrow \iff$
  all assignments to V in t have the form $V \leftarrow V + e$, with $e \ge 0$
- **use:** combined with non-relational interferences
  if $\forall t \colon A^{\text{mono}}(t)(V) = \nearrow$
  then any test with non-relational interference $\mathsf{C}[\![\, X \le (V \mid [a, b]) \,]\!]$
  can be strengthened into $\mathsf{C}[\![\, X \le V \,]\!]$

Relational invariant interferences

Abstraction: keep relations maintained by interferences

- remove control state in interferences (α_A^{nf})
- keep mutex state M (set of mutexes held)
- forget input-output relationships
- keep relationships between variables

$$\alpha_A^{\text{inv}}(Y) \stackrel{\text{def}}{=} \{\, \langle\, M,\, \rho \,\rangle \mid \exists \rho' \colon \langle\, \langle\, M,\, \rho \,\rangle,\, \langle\, M,\, \rho' \,\rangle\, \rangle \in Y \lor \langle\, \langle\, M,\, \rho' \,\rangle,\, \langle\, M,\, \rho \,\rangle\, \rangle \in Y \,\}$$

$\langle\, M,\, \rho \,\rangle \in \alpha_A^{\text{inv}}(Y) \implies \langle\, M,\, \rho \,\rangle \in \alpha_A^{\text{inv}}(Y)$ after any sequence of interferences from Y

Lock invariant:

$$\{\, \rho \mid \exists t \in \mathcal{T}, M \colon \langle\, M,\, \rho \,\rangle \in \alpha_A^{\text{inv}}(\mathbb{I}(t)),\ m \notin M \,\}$$

- property maintained outside code protected by m
- possibly broken while m is locked
- restored before unlocking m

Weakly relational interference example

analyzing t_1

t_1	t_2

`while random do`
`lock(m);`
`if x < y then`
`x := x + 1;`
`unlock(m)`
	`x unchanged`
	`y incremented`
	$0 \leq y \leq 102$

analyzing t_2

t_1	t_2

`y unchanged`
$0 \leq x$, $x \leq y$
	`while random do`
	`lock(m);`
	`if y < 100 then`
	`y := y + [1,3];`
	`unlock(m)`

Using all three interference abstractions:

- non-relational interferences $(0 \leq y \leq 102, 0 \leq x)$
- lock invariants, with the octagon domain $(x \leq y)$
- monotonic interferences (y monotonic)

we can prove automatically that $x \leq y$ holds

Subsequence interference

t_1: clock in H	t_2: sample H into C	t_3: accumulate time in T

`while random do`
`  if H < 10,000 then`
`    H := H+1`
	`while random do`
	`  C := H`
		`while random do`
		`  if random then T := 0`
		`  else T := T + (C-L)`
		`  L := C`

Problem: we wish to prove that $T \le L \le C \le H$

it is sufficient to prove the monotony of H, C, and L

but monotony is not transitive

X is only assigned monotonic variables $\not\Longrightarrow$ X is monotonic

$\Longrightarrow$ we infer an additional property implying monotony

Abstraction: subsequence

- $A^{\text{sseq}}(t)(V) = \{ W \in \mathbb{V} \mid V\text{'s values are a subsequence of } W\text{'s values} \}$
- $\alpha_{\mathcal{R}}^{\text{sseq}}(X)(V) \stackrel{\text{def}}{=} \{ W \mid$
  $\forall \langle\langle \ell_0, \rho_0 \rangle, \ldots, \langle \ell_n, \rho_n \rangle\rangle \in X: \exists i_0, \ldots, i_n:$
  $\forall k: i_k \le k \wedge i_k \le i_{k+1} \wedge \forall j: \rho_j(V) = \rho_{i_j}(W) \}$

based on a **trace version** of the modular semantics

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