Shape analysis based on separation logic

MPRI — Cours "Interprétation abstraite :
application à la vérification et à l'analyse statique"

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Overview of the lecture

How to reason about memory properties

Last lecture:

- concrete and abstract memory models
- abstractions for pointers and arrays
- issues specific to the precise analysis of updates
- an introduction to shape analysis with TVLA

Today: systematically avoid weak updates

- a logic to describe properties of memory states
- abstract domain
- static analysis algorithms
- combination with numerical domains
- widening operators...

Weak update problems

$$x \in [-10, -5];\ y \in [5, 10]$$

```
1 :  int * p;
2 :  if(?){
3 :      p = &x;
4 :  } else {
5 :      p = &y;
6 :  }
7 :  *p = 0;
8 :  ...
```

	&x	&y	&p
1	$[-10, -5]$	$[5, 10]$	$\top$
2	$[-10, -5]$	$[5, 10]$	$\top$
3	$[-10, -5]$	$[5, 10]$	$\top$
4	$[-10, -5]$	$[5, 10]$	{&x}
5	$[-10, -5]$	$[5, 10]$	$\top$
6	$[-10, -5]$	$[5, 10]$	{&y}
7	$[-10, -5]$	$[5, 10]$	{&x, &y}
8	$[-10, 0]$	$[0, 10]$	{&x, &y}

- What is the final range for x ?
- What is the final range for y ?

Abstract locations: {&x, &y, &p}

Imprecise results

- The abstract information about both x and y are weakened
- The fact that $x \neq y$ is lost

Outline

Our model

Not all memory cell corresponds to a variable

- a variable may correspond to **several cells** (structures...)
- **dynamically allocated cells** correspond to no variable at all...

Environment + Heap

- **Addresses** are values: $\mathbb{V}_{\mathrm{addr}} \subseteq \mathbb{V}$
- **Environments** $e \in \mathbb{E}$ map variables into their addresses
- **Heaps** ($h \in \mathbb{H}$) map addresses into values

$$\begin{array}{rcl} \mathbb{E} & = & \mathbb{X} \rightarrow \mathbb{V}_{\mathrm{addr}} \\ \mathbb{H} & = & \mathbb{V}_{\mathrm{addr}} \rightarrow \mathbb{V} \end{array}$$

  h is actually only a partial function
- **Memory states** (or **memories**): $\mathbb{M} = \mathbb{E} \times \mathbb{H}$

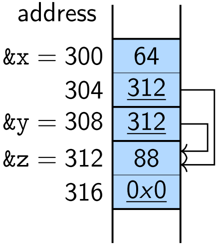
Avoid confusion between heap (function from addresses to values) and dynamic allocation space (often referred to as “heap”)

Example of a concrete memory state (variables)

- x and z are two list elements containing values 64 and 88, and where the former points to the latter
- y stores a pointer to z

Memory layout

(pointer values underlined)

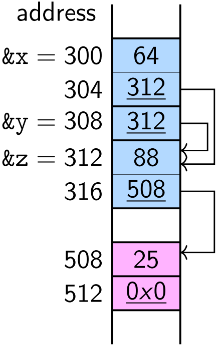


$$
\begin{array}{llll}
e: & \mathrm{x} & \mapsto & 300 \\
 & \mathrm{y} & \mapsto & 308 \\
 & \mathrm{z} & \mapsto & 312 \\
 & & & \\
h: & 300 & \mapsto & 64 \\
 & 304 & \mapsto & 312 \\
 & 308 & \mapsto & 312 \\
 & 312 & \mapsto & 88 \\
 & 316 & \mapsto & 0
\end{array}
$$

Example of a concrete memory state (variables + dyn. cell)

- same configuration
- + z points to a heap allocated list element (in purple)

Memory layout



$$
\begin{array}{llll}
e: & \mathtt{x} & \mapsto & 300 \\
 & \mathtt{y} & \mapsto & 308 \\
 & \mathtt{z} & \mapsto & 312 \\
 & & & \\
h: & 300 & \mapsto & 64 \\
 & 304 & \mapsto & 312 \\
 & 308 & \mapsto & 312 \\
 & 312 & \mapsto & 88 \\
 & 316 & \mapsto & 508 \\
 & 508 & \mapsto & 25 \\
 & 512 & \mapsto & 0
\end{array}
$$

Separation logic principle: avoid weak updates

How to deal with weak updates ?

Avoid them !

- Always materialize exactly the cell that needs be modified
- Can be very costly to achieve, and not always feasible

- Notion of property that holds **over a memory region**:
  special separating conjunction operator $*$
- **Local reasoning**:
  powerful principle, which allows to consider only part of the memory
- Separation logic has been used in **many contexts**, including **manual verification**, **static analysis**, etc...

Separation logic

Several kinds of **formulas**:

- **pure formulas** behave like formulas in first-order logic
  i.e., are not attached to a memory region
- **spatial formulas** describe properties attached to a memory region

Pure formulas denote value properties

$$
\begin{array}{rcll}
\mathtt{e} & ::= & n \qquad (n \in \mathbb{N}) & \text{constants} \\
 & | & \mathtt{l} & \text{l-value} \\
 & | & \mathtt{e}_0 + \mathtt{e}_1 & \text{binary operations} \\
 & | & \ldots & \\
\mathtt{P} & ::= & \mathtt{e}_0 = \mathtt{e}_1 \mid \mathtt{P}' \vee \mathtt{P}'' \mid \mathtt{P}' \wedge \mathtt{P}'' \ldots & \text{pure predicates}
\end{array}
$$

Pure formulas semantics: $\gamma(\mathtt{P}) \subseteq \mathbb{E} \times \mathbb{M}$

Separation logic: points-to predicates

The next slides introduce the main **separation logic formulas** $\mathtt{F} ::= \ldots$

We start with the most basic predicate, that **describes a single cell**:

Points-to predicate

- **Predicate:**

$$\mathtt{F} ::= \ldots \mid \mathtt{l} \mapsto \mathtt{v}$$

- **Concretization:**

$$(e, h) \in \gamma(\mathtt{l} \mapsto \mathtt{v}) \quad \text{if and only if} \quad h = [[\![\mathtt{l}]\!](e, h) \mapsto \mathtt{v}]$$

- **Example:**

$$\mathtt{F} = \mathtt{x} \mapsto 18 \qquad\qquad \&\mathtt{x} = 308 \ \boxed{18}$$

- We also note $\mathtt{l} \mapsto \mathtt{e}$

Separation logic: separating conjunction

Merge of concrete heaps: let $h_0, h_1 \in (\mathbb{V}_{\mathrm{addr}} \to \mathbb{V})$, such that $\mathbf{dom}(h_0) \cap \mathbf{dom}(h_1) = \emptyset$; then, we let $h_0 \circledast h_1$ be defined by:

$$\begin{array}{rlcl} h_0 \circledast h_1 : & \mathbf{dom}(h_0) \cup \mathbf{dom}(h_1) & \longrightarrow & \mathbb{V} \\ & x \in \mathbf{dom}(h_0) & \longmapsto & h_0(x) \\ & x \in \mathbf{dom}(h_1) & \longmapsto & h_1(x) \end{array}$$

Separating conjunction

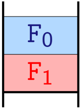
- **Predicate:**

$$\mathtt{F} ::= \ldots \mid \mathtt{F}_0 * \mathtt{F}_1$$

- **Concretization:**

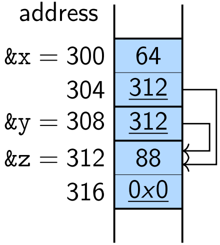
$$\gamma(\mathtt{F}_0 * \mathtt{F}_1) = \{(e, h_0 \circledast h_1) \mid (e, h_0) \in \gamma(\mathtt{F}_0) \wedge (e, h_1) \in \gamma(\mathtt{F}_1)\}$$

$\mathtt{F}_0 * \mathtt{F}_1$



An example

Concrete memory layout
(pointer values underlined)



$$
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 & & & \\
h: & 300 & \mapsto & 64 \\
 & 304 & \mapsto & 312 \\
 & 308 & \mapsto & 312 \\
 & 312 & \mapsto & 88 \\
 & 316 & \mapsto & 0
\end{array}
$$

A formula that abstracts away the addresses:

$$x \mapsto \langle 64, \&z \rangle * y \mapsto \&z * z \mapsto \langle 88, 0 \rangle$$

Separation logic: non separating conjunction

We can also add the **conventional conjunction operator**, with its **usual concretization**:

Non separating conjunction

- **Predicate:**

$$\mathtt{F} ::= \ldots \mid \mathtt{F}_0 \wedge \mathtt{F}_1$$

- **Concretization:**

$$\gamma(\mathtt{F}_0 \wedge \mathtt{F}_1) = \gamma(\mathtt{F}_0) \cap \gamma(\mathtt{F}_1)$$

Exercise: **describe** and **compare** the concretizations of

- $\mathtt{a} \mapsto \&\mathtt{b} \wedge \mathtt{b} \mapsto \&\mathtt{a}$
- $\mathtt{a} \mapsto \&\mathtt{b} * \mathtt{b} \mapsto \&\mathtt{a}$

Separating conjunction vs non separating conjunction

- **Classical conjunction**: properties for the same memory region
- **Separating conjunction**: properties for **disjoint** memory regions

$\mathtt{a} \mapsto \&\mathtt{b} \wedge \mathtt{b} \mapsto \&\mathtt{a}$

- the same heap verifies $\mathtt{a} \mapsto \&\mathtt{b}$ and $\mathtt{b} \mapsto \&\mathtt{a}$
- there can be only **one cell**
- thus $\mathtt{a} = \mathtt{b}$

$\mathtt{a} \mapsto \&\mathtt{b} * \mathtt{b} \mapsto \&\mathtt{a}$

- two **separate** sub-heaps respectively satisfy $\mathtt{a} \mapsto \&\mathtt{b}$ and $\mathtt{b} \mapsto \&\mathtt{a}$
- thus $\mathtt{a} \neq \mathtt{b}$

- Separating conjunction and non-separating conjunction have **very different properties**
- Both **express very different properties**
  e.g., no ambiguity on weak / strong updates

Separating and non separating conjunction

Logic rules of the two conjunction operators of SL:

- **Separating conjunction:**

$$\frac{(e, h_0) \in \gamma(\mathtt{F}_0) \qquad (e, h_1) \in \gamma(\mathtt{F}_1)}{(e, h_0 \circledast h_1) \in \gamma(\mathtt{F}_0 * \mathtt{F}_1)}$$

- **Non separating conjunction:**

$$\frac{(e, h) \in \gamma(\mathtt{F}_0) \qquad (e, h) \in \gamma(\mathtt{F}_1)}{(e, h) \in \gamma(\mathtt{F}_0 \wedge \mathtt{F}_1)}$$

Reminiscent of Linear Logic [Girard87]:
resource aware / non resource aware conjunction operators

Separation logic: empty store

Empty store

- **Predicate:**

$$\mathtt{F} ::= \ldots \mid \mathbf{emp}$$

- **Concretization:**

$$\gamma(\mathbf{emp}) = \{(e, []) \mid e \in \mathbb{E}\} = \mathbb{E} \times \{[]\}$$

where [] denotes the empty store

- **emp** is the **neutral element for** $*$
- by contrast the **neutral element for** $\wedge$ is TRUE, with concretization:

$$\gamma(\mathtt{TRUE}) = \mathbb{E} \times \mathbb{H}$$

Separation logic: other connectors

Disjunction:

- $\mathtt{F} ::= \ldots \mid \mathtt{F}_0 \vee \mathtt{F}_1$
- concretization:

$$\gamma(\mathtt{F}_0 \vee \mathtt{F}_1) = \gamma(\mathtt{F}_0) \cup \gamma(\mathtt{F}_1)$$

Spatial implication (*aka*, magic wand):

- $\mathtt{F} ::= \ldots \mid \mathtt{F}_0 \mathrel{-\!\!*} \mathtt{F}_1$
- concretization:

$$\gamma(\mathtt{F}_0 \mathrel{-\!\!*} \mathtt{F}_1) = \\ \{(e, \hbar) \mid \forall \hbar_0 \in \mathbb{H},\ (e, \hbar_0) \in \gamma(\mathtt{F}_0) \Longrightarrow (e, \hbar \circledast \hbar_0) \in \gamma(\mathtt{F}_1)\}$$

- very powerful connector to describe **structure segments**, used in complex SL proofs

Separation logic

Summary of the main separation logic constructions seen so far:

Separation logic main connectors

$$
\begin{array}{rcl}
\gamma(\mathbf{emp}) & = & \mathbb{E} \times \{[]\} \\
\gamma(\mathtt{TRUE}) & = & \mathbb{E} \times \mathbb{H} \\
\gamma(\mathtt{l} \mapsto \mathtt{v}) & = & \{(e, [[\![\mathtt{l}]\!](e, h) \mapsto \mathtt{v}]) \mid e \in \mathbb{E}\} \\
\gamma(\mathtt{F}_0 * \mathtt{F}_1) & = & \{(e, h_0 \circledast h_1) \mid (e, h_0) \in \gamma(\mathtt{F}_0) \wedge (e, h_1) \in \gamma(\mathtt{F}_1)\} \\
\gamma(\mathtt{F}_0 \wedge \mathtt{F}_1) & = & \gamma(\mathtt{F}_0) \cap \gamma(\mathtt{F}_1)
\end{array}
$$

Concretization of pure formulas is standard

How does this help for program reasoning ?

Programs with pointers: syntax

Syntax extension: quite a few additional constructions

$\mathtt{l}$	$::=$	**l-values**	
	$\mid$	$\mathtt{x}$	$(\mathtt{x} \in \mathbb{X})$
	$\mid$	$\ldots$	
	$\mid$	$*\mathtt{e}$	pointer dereference
	$\mid$	$\mathtt{l} \cdot \mathtt{f}$	field read
$\mathtt{e}$	$::=$	**expressions**	
	$\mid$	$\mathtt{l}$	
	$\mid$	$\ldots$	
	$\mid$	$\&\mathtt{l}$	"address of" operator
$\mathtt{s}$	$::=$	**statements**	
	$\mid$	$\ldots$	
	$\mid$	$\mathtt{x} = \mathbf{malloc}(c)$	allocation of c bytes
	$\mid$	$\mathbf{free}(\mathtt{x})$	deallocation of the block pointed to by $\mathtt{x}$

We do not consider **pointer arithmetics here**

Programs with pointers: semantics

Case of l-values:

$$
\begin{array}{rcl}
[\![\mathtt{x}]\!](e, h) & = & e(\mathtt{x}) \\
[\![*\mathtt{e}]\!](e, h) & = & \left\{ \begin{array}{ll} h([\![\mathtt{e}]\!](e, h)) & \text{if } [\![\mathtt{e}]\!](e, h) \neq 0 \wedge [\![\mathtt{e}]\!](e, h) \in \mathbf{Dom}(h) \\ \Omega & \text{otherwise} \end{array} \right. \\
[\![\mathtt{l} \cdot \mathtt{f}]\!](e, h) & = & [\![\mathtt{l}]\!](e, h) + \mathbf{offset}(\mathtt{f}) \text{ (numeric offset)}
\end{array}
$$

Case of expressions:

$$
[\![\mathtt{l}]\!](e, h) = h([\![\mathtt{l}]\!](e, h)) \qquad\qquad [\![\&\mathtt{l}]\!](e, h) = [\![\mathtt{l}]\!](e, h)
$$

Case of statements:

- **memory allocation** $\mathtt{x} = \mathbf{malloc}(c)$: $(e, h) \rightarrow (e, h')$ where $h' = h[e(\mathtt{x}) \leftarrow k] \uplus \{k \mapsto v_k, k+1 \mapsto v_{k+1}, \ldots, k+c-1 \mapsto v_{k+c-1}\}$ and $k, \ldots, k+c-1$ are fresh in h
- **memory deallocation** $\mathbf{free}(\mathtt{x})$: $(e, h) \rightarrow (e, h')$ where $k = e(\mathtt{x})$ and $h = h' \uplus \{k \mapsto v_k, k+1 \mapsto v_{k+1}, \ldots, k+c-1 \mapsto v_{k+c-1}\}$

Separation logic triple

Program proofs based on triples

- **Notation**: $\{\mathtt{F}\}p\{\mathtt{F}'\}$ if and only if:
$$\forall s, s' \in \mathbb{S},\ s \in \gamma(\mathtt{F}) \wedge s' \in [\![p]\!](s) \Longrightarrow s' \in \gamma(\mathtt{F}')$$
Hoare triples
- Application: **formalize proofs of programs**

A few rules (straightforward proofs):

$$\frac{\mathtt{F}_0 \Longrightarrow \mathtt{F}_0' \qquad \{\mathtt{F}_0'\}\mathtt{b}\{\mathtt{F}_1'\} \qquad \mathtt{F}_1' \Longrightarrow \mathtt{F}_1}{\{\mathtt{F}_0\}\mathtt{b}\{\mathtt{F}_1\}}\ \textit{consequence}$$

$$\frac{}{\{\mathtt{x} \mapsto ?\}\mathtt{x} := \mathtt{e}\{\mathtt{x} \mapsto \mathtt{e}\}}\ \textit{mutation}$$

$$\frac{\mathtt{x} \text{ does not appear in } \mathtt{F}}{\{\mathtt{x} \mapsto ? * \mathtt{F}\}\mathtt{x} := \mathtt{e}\{\mathtt{x} \mapsto \mathtt{e} * \mathtt{F}\}}\ \textit{mutation} - 2$$

(we assume that e does not allocate memory)

The frame rule

What about the resemblance between rules "mutation" and "mutation-2" ?

Theorem: the frame rule

$$\frac{\{\mathtt{F}_0\}\mathtt{b}\{\mathtt{F}_1\} \qquad \mathtt{freevar}(\mathtt{F}) \cap \mathtt{write}(\mathtt{b})}{\{\mathtt{F}_0 * \mathtt{F}\}\mathtt{b}\{\mathtt{F}_1 * \mathtt{F}\}} \ \textit{frame}$$

- Proof by induction on the logical rules on program statements, *i.e.*, essentially a large case analysis
  (see biblio for a more complete set of rules)
- Rules are proved by case analysis on the program syntax

The frame rule allows to reason locally about programs

Application of the frame rule

A program with intermittent invariants, derived using **the frame rule**, since each step **impacts a disjoint region**:

$$
\begin{array}{l}
\quad \textbf{int}\ \mathtt{i};\\
\quad \textbf{int} * \mathtt{x};\\
\quad \textbf{int} * \mathtt{y};\\
\{\mathtt{i} \mapsto ? * \mathtt{x} \mapsto ? * \mathtt{y} \mapsto ?\}\\
\quad \mathtt{x} = \&\mathtt{i};\\
\{\mathtt{i} \mapsto ? * \mathtt{x} \mapsto \&\mathtt{i} * \mathtt{y} \mapsto ?\}\\
\quad \mathtt{y} = \&\mathtt{i};\\
\{\mathtt{i} \mapsto ? * \mathtt{x} \mapsto \&\mathtt{i} * \mathtt{y} \mapsto \&\mathtt{i}\}\\
\quad * \mathtt{x} = 42;\\
\{\mathtt{i} \mapsto 42 * \mathtt{x} \mapsto \&\mathtt{i} * \mathtt{y} \mapsto \&\mathtt{i}\}
\end{array}
$$

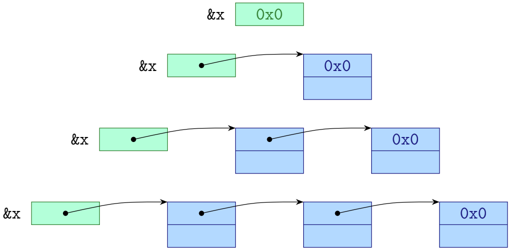
Many other program proofs done using separation logic
e.g., verification of the Deutsch-Shorr-Waite algorithm (biblio)

Summarization and inductive definitions

What do we still miss ?

So far, formulas denote **fixed sets of cells**
Thus, no summarization of unbounded regions...

- **Example** all lists pointed to by `x`, such as:



- How to precisely abstract these stores with **a single formula** *i.e.*, no infinite disjunction ?

Inductive definitions in separation logic

List definition

$$\begin{array}{rcl} \alpha \cdot \mathbf{list} & := & \alpha = 0 \wedge \mathbf{emp} \\ & \vee & \alpha \neq 0 \wedge \alpha \cdot \mathtt{next} \mapsto \delta * \alpha \cdot \mathtt{data} \mapsto \beta * \delta \cdot \mathbf{list} \end{array}$$

- Formula abstracting our set of structures:

$$\&\mathtt{x} \mapsto \alpha * \alpha \cdot \mathbf{list}$$

- **Summarization**:
  this formula is finite and describe infinitely many heaps
- **Concretization**: next slide...

Practical implementation in verification/analysis tools

- **Verification:** hand-written definitions
- **Analysis:** either built-in or user-supplied, or partly inferred

Concretization by unfolding

Intuitive semantics of inductive predicates

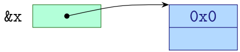
- Inductive predicates can be **unfolded**, by **unrolling their definitions**
  Syntactic unfolding is noted $\xrightarrow{\mathcal{U}}$
- A formula $\mathtt{F}$ with inductive predicates describes all stores described by all formulas $\mathtt{F}'$ such that $\mathtt{F} \xrightarrow{\mathcal{U}} \mathtt{F}'$

Example:

- Let us start with $\mathtt{x} \mapsto \alpha_0 * \alpha_0 \cdot \textbf{list}$; we can unfold it as follows:

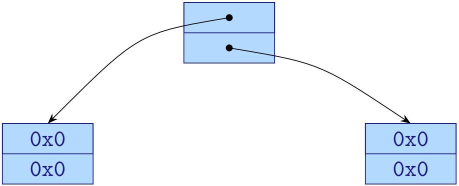
$$\begin{array}{ll} & \&\mathtt{x} \mapsto \alpha_0 * \alpha_0 \cdot \textbf{list} \\ \xrightarrow{\mathcal{U}} & \&\mathtt{x} \mapsto \alpha_0 * \alpha_0 \cdot \mathtt{next} \mapsto \alpha_1 * \alpha_0 \cdot \mathtt{data} \mapsto \beta_1 * \alpha_1 \cdot \textbf{list} \\ \xrightarrow{\mathcal{U}} & \&\mathtt{x} \mapsto \alpha_0 * \alpha_0 \cdot \mathtt{next} \mapsto \alpha_1 * \alpha_0 \cdot \mathtt{data} \mapsto \beta_1 * \textbf{emp} \wedge \alpha_1 = \textbf{0x0} \end{array}$$

- We get the concrete state below:



Example: tree

- **Example:**



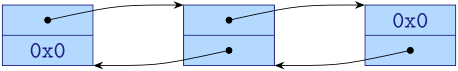
Inductive definition

- **Two recursive calls** instead of one:

$$\begin{array}{rcl} \alpha \cdot \textbf{tree} & := & \quad \alpha = 0 \wedge \textbf{emp} \\ & \vee & \alpha \neq 0 \wedge \alpha \cdot \texttt{left} \mapsto \beta * \alpha \cdot \texttt{right} \mapsto \delta \\ & & * \, \beta \cdot \textbf{tree} * \delta \cdot \textbf{tree} \end{array}$$

Example: doubly linked list

- **Example:**



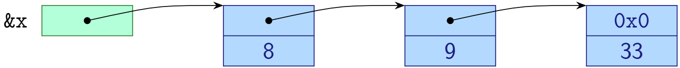
Inductive definition

- We need to propagate the prev pointer as an additional parameter:

$$\begin{array}{rcl} \alpha \cdot \mathbf{dll}(\delta) & := & \alpha = 0 \land \mathbf{emp} \\ & \lor & \alpha \neq 0 \land \alpha \cdot \mathtt{next} \mapsto \beta * \alpha \cdot \mathtt{prev} \mapsto \delta \\ & & * \beta \cdot \mathbf{dll}(\alpha) \end{array}$$

Example: sortedness

- **Example:** sorted list



Inductive definition

- Each element should be greater than the previous one
- The first element simply needs be greater than $-\infty$...
- We need to propagate **the lower bound**, using a scalar parameter

$$
\begin{array}{rcl}
\alpha \cdot \mathbf{lsort}_{\mathrm{aux}}(n) & := & \alpha = 0 \wedge \mathbf{emp} \\
 & \vee & \alpha \neq 0 \wedge \beta \leq n \wedge \alpha \cdot \mathtt{next} \mapsto \delta \\
 & & * \, \alpha \cdot \mathtt{data} \mapsto \beta * \delta \cdot \mathbf{lsort}_{\mathrm{aux}}(\beta) \\
 & & \\
\alpha \cdot \mathbf{lsort}() & := & \alpha \cdot \mathbf{lsort}_{\mathrm{aux}}(-\infty)
\end{array}
$$

A new overview of the remaining part of the lecture

How to apply separation logic to static analysis and design abstract interpretation algorithms based on it ?

In remainder of this lecture, we will:

- choose a **small but expressive set of separation logic formulas**
- combine it **with a numerical abstract domain**
- study algorithms for **local concretization** (equivalent to focus) and **global abstraction** (widening...)

Outline

Design of an abstract domain

A lot of things are missing to turn SL into an abstract domain

Set of logical predicates:

- separation logic formulas are **very expressive**
  e.g., arbitrary **alternations** of $\wedge$ and $*$
- such expressiveness is not necessarily required in static analysis

Representation:

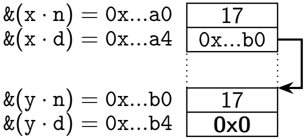
- unstructured formulas can be represented as **ASTs**,
  but this representation is **not easy to manipulate efficiently**
- intuition over memory states typically involves **graphs**

Analysis algorithms:

- inference of "optimal" invariants in SL obviously **not computable**

Basic abstraction: structures and their contents (1/2)

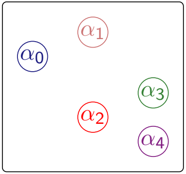
- **Concrete memory states**
  - very **low level** description
  - pointers, numeric values: **raw sequences of bits**

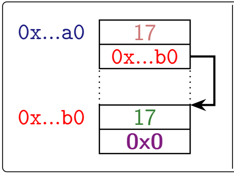


Basic abstraction: structures and their contents (1/2)

- **Concrete memory states**

- **Abstraction of values into symbolic variables** (nodes)



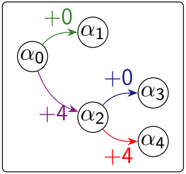


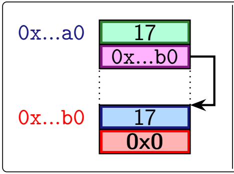
$\nu(\alpha_0) = \texttt{0x...a0}$
$\nu(\alpha_1) = 17$
$\nu(\alpha_2) = \texttt{0x...b0}$
$\nu(\alpha_3) = 17$
$\nu(\alpha_4) = \mathbf{0x0}$

  - characterized by **valuation** ν
  - ν maps **symbolic variables** into **concrete addresses**

Basic abstraction: structures and their contents (1/2)

- **Concrete memory states**
- **Abstraction of values into symbolic variables** / nodes
- **Abstraction of regions into points-to edges**

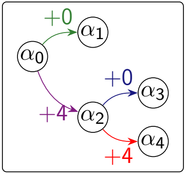


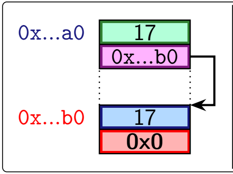


$\nu(\alpha_0) = \texttt{0x...a0}$
$\nu(\alpha_1) = 17$
$\nu(\alpha_2) = \texttt{0x...b0}$
$\nu(\alpha_3) = 17$
$\nu(\alpha_4) = \mathbf{0x0}$

Basic abstraction: structures and their contents (1/2)

- **Concrete memory states**
- **Abstraction of values into symbolic variables** / nodes
- **Abstraction of regions into points-to edges**





$\nu(\alpha_0) = \texttt{0x...a0}$
$\nu(\alpha_1) = 17$
$\nu(\alpha_2) = \texttt{0x...b0}$
$\nu(\alpha_3) = 17$
$\nu(\alpha_4) = \mathbf{0x0}$

- **Shape graph concretization**

$$\gamma_{\mathrm{S}}(G) = \{(\hbar, \nu) \mid \ldots\}$$

valuation ν plays an important role to combine abstraction...

Structure of shape graphs

Valuations bridge the gap between nodes and values

Symbolic variables / **nodes** and intuitively abstract concrete values:

Symbolic variables

We let $\mathbb{V}^\sharp$ denote a countable set of **symbolic variables**; we usually let them be denoted by Greek letters in the following: $\mathbb{V}^\sharp = \{\alpha, \beta, \delta, \ldots\}$

When concretizing a shape graph, we need to **characterize how the concrete instance evaluates each symbolic variable**, which is the purpose of the **valuation functions**:

Valuations

A **valuation** is a function from symbolic variables into concrete values (and is often denoted by ν): $\mathbf{Val} = \mathbb{V}^\sharp \longrightarrow \mathbb{V}$

Note that valuations treat **in the same way** addresses and raw values

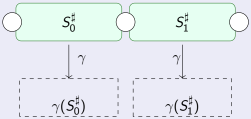
Structure of shape graphs

Distinct edges describe separate regions

In particular, if we **split** a graph into **two parts**:

Separating conjunction

$$\gamma_{\mathrm{S}}(S_0^\sharp * S_1^\sharp) \quad = \quad \{(h_0 \circledast h_1, \nu) \mid (h_0, \nu) \in \gamma_{\mathrm{S}}(S_0^\sharp) \wedge (h_1, \nu) \in \gamma_{\mathrm{S}}(S_1^\sharp)\}$$



Similarly, when considering the **empty set of edges**, we get the empty heap (where $\mathbb{V}^\sharp$ is the set of nodes):

$$\gamma_{\mathrm{S}}(\mathbf{emp}) = \{(\emptyset, \nu) \mid \nu : \mathbb{V}^\sharp \to \mathbb{V}\}$$

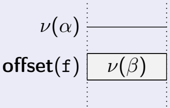
Abstraction of contiguous regions

A single points-to edge represents one heap cell

A **points-to edge** encodes **basic points to predicate in separation logic**:

Points-to edges

- **Syntax**

Graph edge	Separation logic formula	Concrete view
$\alpha \xrightarrow{\mathtt{f}} \beta$	$\alpha \cdot \mathtt{f} \mapsto \beta$	

- **Concretization**:

$$\gamma_{\mathrm{S}}(\alpha \cdot \mathtt{f} \mapsto \beta) = \\ \{([\nu(\alpha) + \textbf{offset}(\mathtt{f}) \mapsto \nu(\beta)], \nu) \mid \nu : \{\alpha, \beta, \ldots\} \to \mathbb{N}\}$$

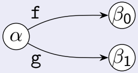
Abstraction of contiguous regions

Contiguous regions are described by adjacent points-to edges

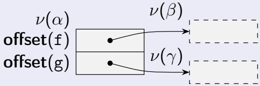
To describe **blocks** containing series of **cells** (*e.g.*, in a **C structure**), shape graphs utilize several outgoing edges from the node representing the base address of the block

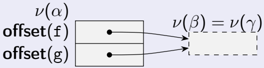
Field splitting model

- Separation impacts edges / fields, *not pointers*
- Shape graph



accounts for both abstract states below:

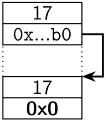


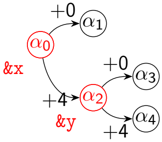


Abstraction of the environment

Environments bind variables to their (concrete / abstract) address

$$\&\mathtt{x} = \&(\mathtt{x}\cdot\mathtt{n}) = \mathtt{0x...a0}$$
$$\&(\mathtt{x}\cdot\mathtt{d}) = \mathtt{0x...a4}$$
$$\&\mathtt{y} = \&(\mathtt{y}\cdot\mathtt{n}) = \mathtt{0x...b0}$$
$$\&(\mathtt{y}\cdot\mathtt{d}) = \mathtt{0x...b4}$$





$$\begin{array}{rlcl} \nu: & \alpha_0 & \mapsto & \mathtt{0x...a0} \\ & \alpha_2 & \mapsto & \mathtt{0x...b0} \\ & \ldots & \mapsto & \ldots \end{array}$$

$$\begin{array}{rll} e^\sharp: & \mathtt{x} \mapsto \alpha_0 & (\overset{\nu}{\mapsto} \mathtt{0x...a0}) \\ & \mathtt{y} \mapsto \alpha_2 & (\overset{\nu}{\mapsto} \mathtt{0x...b0}) \end{array}$$

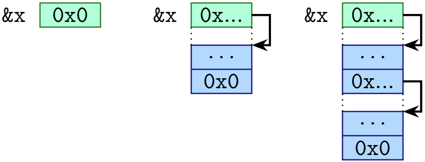
Abstract environments

- An **abstract environment** is a function $e^\sharp$ from **variables** to **symbolic nodes**
- The **concretization** extends as follows:

$$\gamma_{\mathrm{M}}(e^\sharp, S^\sharp) = \{(e, \hbar, \nu) \mid (\hbar, \nu) \in \gamma_{\mathrm{S}}(S^\sharp) \wedge e = \nu \circ e^\sharp\}$$

Basic abstraction: summarization

Set of all lists of any length:

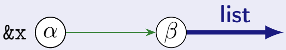


Well-founded list inductive def.

$$\begin{array}{ll} \alpha \cdot \textbf{list} := & \\ & (\textbf{emp} \wedge \alpha = \textbf{0x0}) \\ \vee & (\alpha \cdot \mathtt{d} \mapsto \beta_0 * \alpha \cdot \mathtt{n} \mapsto \beta_1 \\ & \quad * \beta_1 \cdot \textbf{list} \wedge \alpha \neq \textbf{0x0}) \end{array}$$

well-founded predicate

Inductive summary predicates



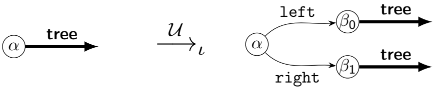
Concretization based on **unfolding** and **least-fixpoint**:

- $\xrightarrow{\mathcal{U}}$ replaces **an $\alpha \cdot$ list predicate** with **one of its premises**
- $\gamma(S^\sharp, \mathtt{F}) = \bigcup \{ \gamma(S^\sharp_u, \mathtt{F}_u) \mid (S^\sharp, \mathtt{F}) \xrightarrow{\mathcal{U}} (S^\sharp_u, \mathtt{F}_u) \}$

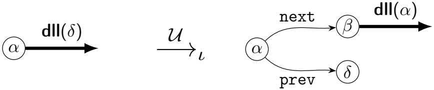
Inductive structures: a few instances

As before, **many interesting inductive predicates** encode nicely into graph inductive definitions:

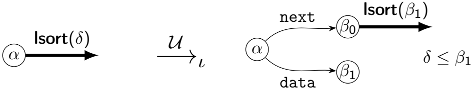
- **More complex shapes**: **trees**



- **Relations among pointers**: **doubly-linked lists**

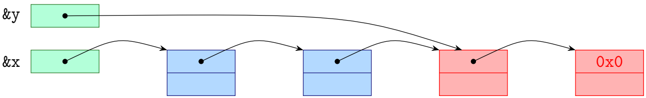


- **Relations between pointers and numerical**: **sorted lists**



Inductive segments

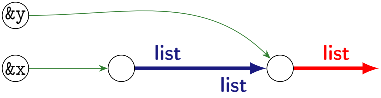
- **A frequent pattern:**



- Could be **expressed directly** as an inductive with a parameter:

$$\begin{array}{rcl} \alpha \cdot \textbf{list\_endp}(\pi) & ::= & (\textbf{emp}, \alpha = \pi) \\ & | & (\alpha \cdot \texttt{next} \mapsto \beta_0 * \alpha \cdot \texttt{data} \mapsto \beta_1 \\ & & \quad * \beta_0 \cdot \textbf{list\_endp}(\pi), \alpha \neq 0) \end{array}$$

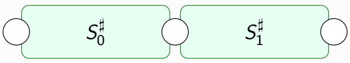
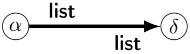
- This definition **straightforwardly derives** from **list**
  Thus, we make **segments** part of the **fundamental predicates of the domain**



- **Multi-segments**: possible, but harder for analysis

Shape graphs and separation logic

Semantic preserving translation Π of graphs into separation logic formulas:

Graph $S^\sharp \in \mathbb{D}^\sharp_{\mathrm{S}}$	Translated formula $\Pi(S^\sharp)$
	$\alpha \cdot \mathtt{f} \mapsto \beta$
	$\Pi(S^\sharp_0) * \Pi(S^\sharp_1)$
	$\alpha \cdot \textbf{list}$
	$\alpha \cdot \textbf{list\_endp}(\delta)$
other inductives and segments	similar

Note that:

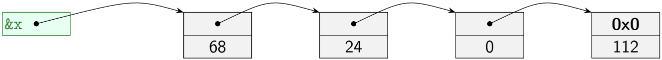
- **shape graphs can be encoded into separation logic formula**
- **the opposite is usually not true**

Outline

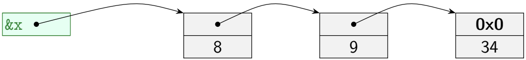
Example

How to express both shape and numerical properties ?

- Hybrid stores: data stored next to structures
- List of even elements:



- Sorted list:



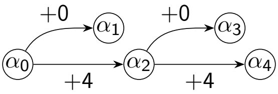
- Many other examples:
  - list of open filed descriptors
  - tries
  - balanced trees: red-black, AVL...
- Note: inductive definitions also talk about data

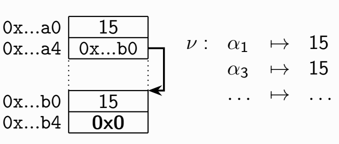
Adding value information (here, numeric)

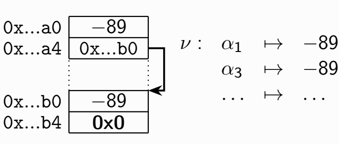
Concrete numeric values appear in the valuation thus the abstracting contents boils down to abstracting ν !

Example: all lists of length 2, with equal data fields **Memory**

abstraction:







Abstraction of valuations: $\nu(\alpha_1) = \nu(\alpha_3)$, **(constraint $\alpha_1 = \alpha_3$)**

A first approach to domain combination

Assumptions:

- Graphs form a **shape domain** $\mathbb{D}^\sharp_{\mathrm{S}}$
  abstract stores together with a **physical mapping** of nodes
  $$\gamma_{\mathrm{S}} : \mathbb{D}^\sharp_{\mathrm{S}} \to \mathcal{P}((\mathbb{D}^\sharp_{\mathrm{S}} \to \mathbb{M}) \times (\mathbb{V}^\sharp \to \mathbb{V}))$$
- Numerical values are taken in a **numerical domain** $\mathbb{D}^\sharp_{\mathrm{num}}$
  abstracts physical mapping of nodes
  $$\gamma_{\mathrm{num}} : \mathbb{D}^\sharp_{\mathrm{num}} \to \mathcal{P}((\mathbb{V}^\sharp \to \mathbb{V}))$$

Combined domain **[CR]**

- **Set of abstract values:** $\mathbb{D}^\sharp = \mathbb{D}^\sharp_{\mathrm{S}} \times \mathbb{D}^\sharp_{\mathrm{num}}$
- **Concretization:**

$$\gamma(S^\sharp, N^\sharp) = \{(\hbar, \nu) \in \mathbb{M} \mid \nu \in \gamma_{\mathrm{num}}(N^\sharp) \wedge (\hbar, \nu) \in \gamma_{\mathrm{S}}(S^\sharp)\}$$

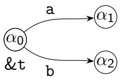
Formalizing the product domain

Can it be described as a reduced product ?

- **Product abstraction:** $\mathbb{D}^\sharp = \mathbb{D}^\sharp_0 \times \mathbb{D}^\sharp_1$
- **Concretization:** $\gamma(x_0, x_1) = \gamma(x_0) \cap \gamma(x_1)$
- **Reduction:** $\mathbb{D}^\sharp_r$ is the quotient of $\mathbb{D}^\sharp$ by the equivalence relation $\equiv$ defined by $(x_0, x_1) \equiv (x'_0, x'_1) \iff \gamma(x_0, x_1) = \gamma(x'_0, x'_1)$
- **Abstract order:** pairwise on reduced elements

Several issues:

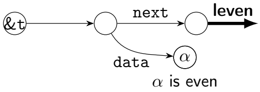
Shape + octagons:



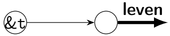
$\alpha_1 \leq \alpha_2$
$\alpha_2 \leq \alpha_1 + \alpha_3$

... **what is α_3 ?**

How to compare the two elements below ?



and



Towards a more adapted combination operator

Why does this fail here ?

- The set of nodes / symbolic variables **is not fixed**
- Variables represented in the numerical domain depend on the shape abstraction

$\Rightarrow$ Thus the product is **not** symmetric

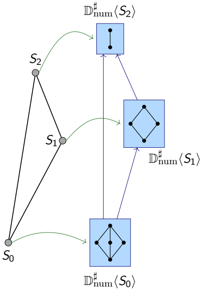
Intuitions

- Graphs form a **shape domain** $\mathbb{D}^\sharp_{\mathrm{S}}$
- For **each** graph $S^\sharp \in \mathbb{D}^\sharp_{\mathrm{S}}$, we have a **numerical lattice** $\mathbb{D}^\sharp_{\mathbf{num}\langle S^\sharp \rangle}$
  - example: if graph $S^\sharp$ contains nodes $\alpha_0, \alpha_1, \alpha_2$, $\mathbb{D}^\sharp_{\mathbf{num}\langle S^\sharp \rangle}$ should abstract $\{\alpha_0, \alpha_1, \alpha_2\} \to \mathbb{V}$
- **An abstract value is a pair $(S^\sharp, N^\sharp)$, such that $N^\sharp \in \mathbb{D}^\sharp_{\mathbf{num}\langle N^\sharp \rangle}$**

Cofibered domain

Definition [AV]

- **Basis:** abstract domain $(\mathbb{D}^\sharp_0, \sqsubseteq^\sharp{}_0)$, with concretization $\gamma_0 : \mathbb{D}^\sharp_0 \to \mathbb{D}$
- **Function:** $\phi : \mathbb{D}^\sharp_0 \to \mathcal{D}_1$, where each element of $\mathcal{D}_1$ is an abstract domain $(\mathbb{D}^\sharp_1, \sqsubseteq^\sharp{}_1)$, with a concretization $\gamma_{\mathbb{D}^\sharp_1} : \mathbb{D}^\sharp_1 \to \mathbb{D}$
- **Domain:** $\mathbb{D}^\sharp$ is the set of **pairs** $(x^\sharp_0, x^\sharp_1)$ **where** $x^\sharp_1 \in \phi(x^\sharp_0)$
- **Lift functions:** $\forall x^\sharp, y^\sharp \in \mathbb{D}^\sharp_0$, such that $x^\sharp \sqsubseteq^\sharp{}_0\, y^\sharp$, there exists a function $\Pi_{x^\sharp, y^\sharp} : \phi(x^\sharp) \to \phi(y^\sharp)$, that is monotone for $\gamma_{x^\sharp}$ and $\gamma_{y^\sharp}$



- **Generic product**, where the second lattice depends on the first
- Provides a generic scheme for **widening**, **comparison**

Domain operations

- **Lift functions** allow to **switch domain when needed**

Comparison of $(x_0^\sharp, x_1^\sharp)$ and $(y_0^\sharp, y_1^\sharp)$

1. First, **compare** $x_0^\sharp$ **and** $y_0^\sharp$ in $\mathbb{D}_0^\sharp$
2. If $x_0^\sharp \sqsubseteq^\sharp{}_0\ y_0^\sharp$, **compare** $\Pi_{x_0^\sharp, y_0^\sharp}(x_1^\sharp)$ **and** $y_1^\sharp$

Widening of $(x_0^\sharp, x_1^\sharp)$ and $(y_0^\sharp, y_1^\sharp)$

1. First, compute the **widening in the basis** $z_0^\sharp = x_0^\sharp \triangledown y_0^\sharp$
2. Then **move to** $\phi(z_0^\sharp)$, by computing $x_2^\sharp = \Pi_{x_0^\sharp, z_0^\sharp}(x_1^\sharp)$ and $y_2^\sharp = \Pi_{y_0^\sharp, z_0^\sharp}(y_1^\sharp)$
3. Last **widen in** $\phi(z_0^\sharp)$: $z_1^\sharp = x_2^\sharp \triangledown_{z_0^\sharp} y_2^\sharp$

$$(x_0^\sharp, x_1^\sharp) \triangledown (y_0^\sharp, y_1^\sharp) = (z_0^\sharp, z_1^\sharp)$$

Domain operations

Transfer functions, *e.g.*, assignment

- Require memory location be **materialized** in the graph
  - *i.e.*, the graph may have to be modified
  - the numerical component should be updated with lift functions
- Require **update** in the graph and the numerical domain
  - *i.e.*, the numerical component should be kept coherent with the graph

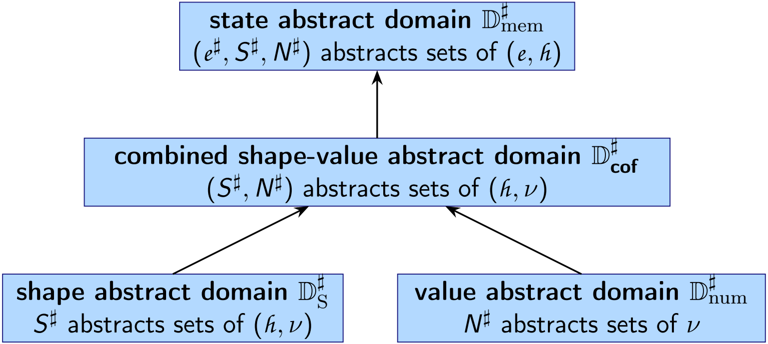
Proofs of soundness of transfer functions rely on:

- the soundness of the lift functions
- the soundness of both domain transfer functions

Overall abstract domain structure

Modular structure

- **Each layer** accounts for one **aspect of the concrete states**
- **Each layer** boils down to a **module or functor in ML**



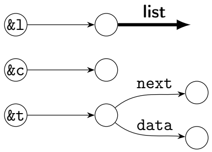
Outline

Static analysis overview

A list insertion function:

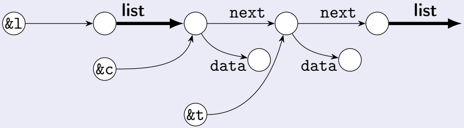
```
list ⋆ l assumed to point to a list
list ⋆ t assumed to point to a list element
list ⋆ c = l;
while(c != NULL && c -> next != NULL && (...)){
    c = c -> next;
}
t -> next = c -> next;
c -> next = t;
```

- **list** inductive structure def.
- Abstract precondition:



Result of the (interprocedural) analysis

- **Over-approximations** of reachable concrete states
  e.g., after the insertion:



Transfer functions

Abstract interpreter design

- **Follows the semantics** of the language under consideration
- The abstract domain should provide **sound transfer functions**

Transfer functions:

- **Assignment:** $\mathrm{x} \rightarrow f = \mathrm{y} \rightarrow g$ or $\mathrm{x} \rightarrow f = e_{\mathrm{arith}}$
- **Test:** analysis of conditions (if, while)
- Variable **creation** and **removal**
- **Memory management:** **malloc**, **free**

Abstract operators:

- **Join** and **widening**: over-approximation
- **Inclusion checking**: check stabilization of abstract iterates

Should be **sound** *i.e.*, not forget any concrete behavior

Abstract operations

Denotational style abstract interpreter

- Concrete **denotational semantics** $[\![\mathtt{b}]\!] : \mathbb{S} \longrightarrow \mathcal{P}(\mathbb{S})$
- **Abstract post-condition** $[\![\mathtt{b}]\!]^\sharp(\mathbf{S})$, computed by the analysis:

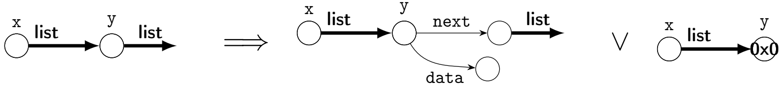
$$s \in \gamma(\mathbf{S}) \Longrightarrow [\![\mathtt{b}]\!](s) \subseteq \gamma([\![\mathtt{b}]\!]^\sharp(\mathbf{S}))$$

Analysis by induction on the syntax using **domain operators**

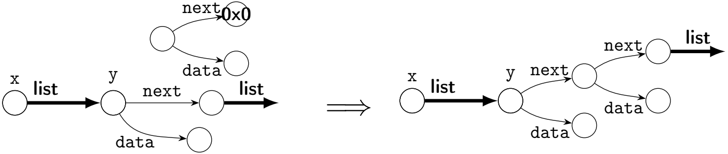
$$\begin{array}{rcl}
[\![\mathtt{b}_0; \mathtt{b}_1]\!]^\sharp(\mathbf{S}) & = & [\![\mathtt{b}_1]\!]^\sharp \circ [\![\mathtt{b}_0]\!]^\sharp(\mathbf{S}) \\
[\![\mathtt{l} = \mathtt{e}]\!]^\sharp(\mathbf{S}) & = & \mathit{assign}(\mathtt{l}, \mathtt{e}, \mathbf{S}) \\
[\![\mathtt{l} = \mathbf{malloc}(n)]\!]^\sharp(\mathbf{S}) & = & \mathit{alloc}(\mathtt{l}, n, \mathbf{S}) \\
[\![\mathbf{free}(\mathtt{l})]\!]^\sharp(\mathbf{S}) & = & \mathit{free}(\mathtt{l}, n, \mathbf{S}) \\
[\![\mathbf{if}(\mathtt{e})\ \mathtt{b}_t\ \mathbf{else}\ \mathtt{b}_f]\!]^\sharp(\mathbf{S}) & = & \left\{ \begin{array}{l} \mathit{join}([\![\mathtt{b}_t]\!]^\sharp(\mathit{test}(\mathtt{e}, \mathbf{S})), \\ \qquad\qquad [\![\mathtt{b}_f]\!]^\sharp(\mathit{test}(\mathtt{e} = \mathbf{false}, \mathbf{S}))) \end{array} \right. \\
[\![\mathbf{while}(\mathtt{e})\mathtt{b}]\!]^\sharp(\mathbf{S}) & = & \mathit{test}(\mathtt{e} = \mathbf{false}, \mathbf{lfp}^\sharp{}_{\mathbf{S}} F^\sharp) \\
\text{where, } F^\sharp : \mathbf{S}_0 \mapsto [\![\mathtt{b}]\!]^\sharp(\mathit{test}(\mathtt{e}, \mathbf{S}_0)) & &
\end{array}$$

The algorithms underlying the transfer functions

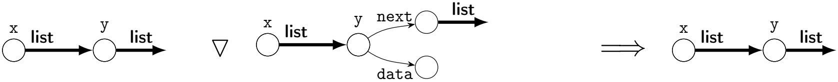
- **Unfolding**: cases analysis on summaries



- **Abstract postconditions**, on "exact" regions, *e.g.* insertion



- **Widening**: builds summaries and ensures termination



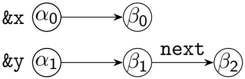
Outline

Analysis of an assignment in the graph domain

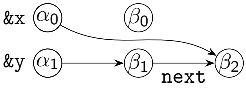
Steps for analyzing `x = y -> next` (local reasoning)

1. Evaluate **l-value** `x` into **points-to edge** $\alpha \mapsto \beta$
2. Evaluate **r-value** `y -> next` into **node** β'
3. Replace points-to edge $\alpha \mapsto \beta$ with **points-to edge** $\alpha \mapsto \beta'$

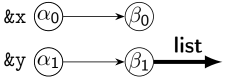
With pre-condition:



- Step 1 produces $\alpha_0 \mapsto \beta_0$
- Step 2 produces β_2
- End result:



With pre-condition:



- Step 1 produces $\alpha_0 \mapsto \beta_0$
- **Step 2 fails**

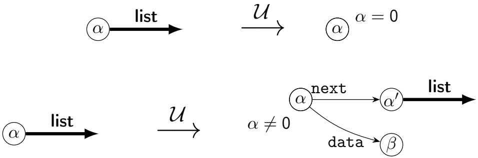
- Abstract state **too abstract**
- We need to **refine it**

Unfolding as a local case analysis

Unfolding principle

- **Case analysis**, based on the inductive definition
- Generates **symbolic disjunctions** (analysis performed in a **disjunction domain**, *e.g.*, trace partitioning)

- Example, for lists:



- **Numeric predicates: approximated in the numerical domain**

Soundness: by definition of the concretization of inductive structures

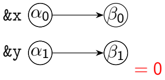
$$\gamma_{\mathrm{S}}(S^\sharp) \subseteq \bigcup \{\gamma_{\mathrm{S}}(S^\sharp_0) \mid S^\sharp \xrightarrow{\mathcal{U}} S^\sharp_0\}$$

Analysis of an assignment, with unfolding

Principle

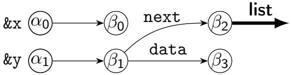
- We have $\gamma_{\mathrm{S}}(\alpha \cdot \iota) = \bigcup\{\gamma_{\mathrm{S}}(S^\sharp) \mid \alpha \cdot \iota \xrightarrow{\mathcal{U}} S^\sharp\}$
- Replace $\alpha \cdot \iota$ with a finite number of disjuncts and continue

Disjunct 1:

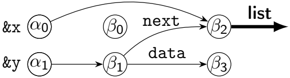


- Step 1 produces $\alpha_0 \mapsto \beta_0$
- **Step 2 fails: Null pointer !**
- In a **correct** program, would be ruled out by a **condition** $\mathtt{y} \neq 0$ *i.e.*, $\beta_1 \neq 0$ in $\mathbb{D}^\sharp_{\mathrm{num}}$

Disjunct 2:



- Step 1 produces $\alpha_0 \mapsto \beta_0$
- Step 2 produces β_2
- **End result**:



Unfold, compute abstract post, and...

Evaluation of a transfer functions (assignment, test...)

1. **evaluate** all **expressions and l-values** that are required
   unfold inductive definitions if needed
2. **compute the effect** of the concrete operation on **fully materialized graph chunks**

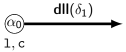
Comparison with the previous lecture:

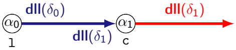
Using separation logic shape graphs	In TVLA
$\begin{tikzcd} \mathcal{S}^\sharp_{\mathrm{pre}} \arrow[d, "\text{unfold (materialize)}"'] & \\ \mathcal{S}^\sharp_{\mathrm{pre,ref}} \arrow[r, "f"] & \mathcal{S}^\sharp_{\mathrm{post}} \end{tikzcd}$	$\begin{tikzcd} \mathcal{S}^\sharp_{\mathrm{pre}} \arrow[d, "\text{partially concretize}"'] & \mathcal{S}^\sharp_{\mathrm{post}} \\ \mathcal{S}_{\mathrm{pre}} \arrow[r, "f"] & \mathcal{S}_{\mathrm{post}} \arrow[u, "\text{abstract}"'] \end{tikzcd}$

When does the abstraction takes place ? More on this a bit later

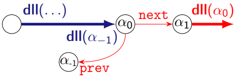
Unfolding and degenerated cases

```
assume(l points to a dll)
c = l;
① while(c ≠ NULL && condition)
      c = c -> next;
② if(c ≠ 0 && c -> prev ≠ 0)
      c = c -> prev → prev;
```

- at ①: 

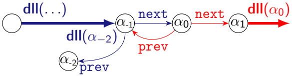
- at ②: 

⇒ **non trivial unfolding**

- **Materialization of** c -> prev: 

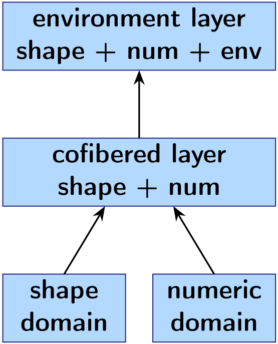
Segment splitting lemma: basis for segment unfolding

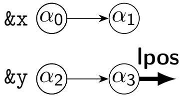
$\alpha_0 \xrightarrow[\iota']{\iota\ \ i+j} \alpha_2$ describes the same set of stores as $\alpha_0 \xrightarrow[\iota'']{\iota\ \ i} \alpha_1 \xrightarrow[\iota']{\iota''\ \ j} \alpha_2$

- **Materialization of** c -> prev -> prev: 

- **Implementation issue**: discover **which inductive edge** to unfold
  very hard !

Analysis of an assignment in the combined domain



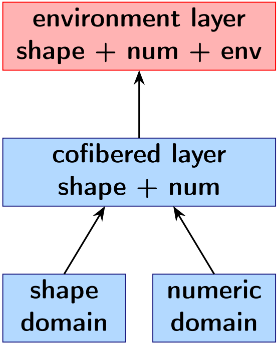


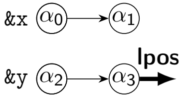
$N^\sharp = \alpha_1 \geq 0 \wedge \alpha_3 \neq \mathbf{0x0}$

$$\mathtt{y} \mathrel{\texttt{->}} \mathtt{d} = \mathtt{x} + 1$$

Abstract post-condition ?

Analysis of an assignment in the combined domain





$N^\sharp = \alpha_1 \geq 0 \wedge \alpha_3 \neq$ **0x0**

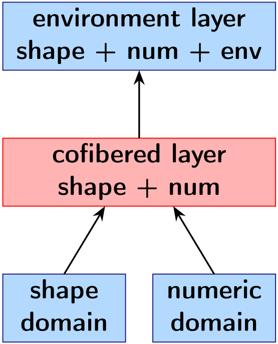
$$\mathtt{y \text{->} d} = \mathtt{x} + 1 \quad \Rightarrow \quad (*\alpha_2) \cdot \mathtt{d} = (*\alpha_0) + 1$$

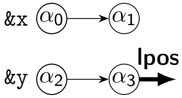
Abstract post-condition ?

Stage 1: environment resolution

- replaces $\mathtt{x}$ with $*e^\sharp(\mathtt{x})$

Analysis of an assignment in the combined domain





$$N^\sharp = \alpha_1 \geq 0 \wedge \alpha_3 \neq \mathbf{0x0}$$

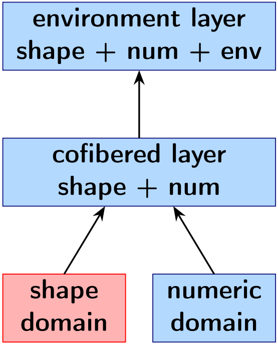
$$(*\alpha_2) \cdot \mathtt{d} = (*\alpha_0) + 1$$

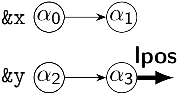
Abstract post-condition ?

Stage 2: propagate into the shape + numerics domain

- only symbolic nodes appear

Analysis of an assignment in the combined domain





$$N^\sharp = \alpha_1 \geq 0 \wedge \alpha_3 \neq \mathbf{0x0}$$

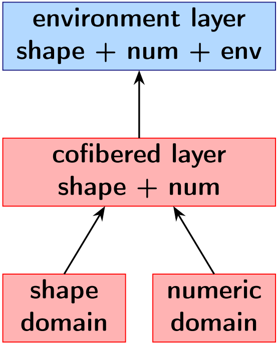
$$(*\alpha_2) \cdot \mathtt{d} = (*\alpha_0) + 1$$

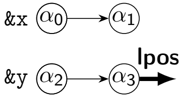
Abstract post-condition ?

Stage 3: resolve cells in the shape graph abstract domain

- $*\alpha_0$ evaluates to α_1; $*\alpha_2$ evaluates to α_3
- $(*\alpha_2) \cdot \mathtt{d}$ **fails to evaluate**: **no points-to out of** α_3

Analysis of an assignment in the combined domain





$$N^\sharp = \alpha_1 \geq 0 \wedge \alpha_3 \neq \mathbf{0x0}$$

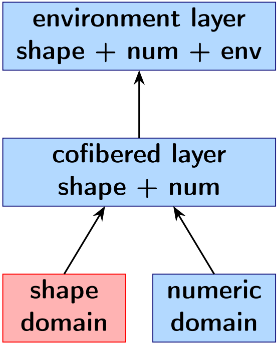
$$(*\alpha_2) \cdot \mathtt{d} = (*\alpha_0) + 1$$

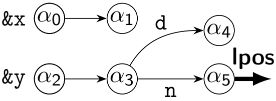
Abstract post-condition ?

Stage 4 (a): unfolding triggered

- the analysis needs to locally materialize $\alpha_3 \cdot$ **lpos**...
- thus, unfolding starts at symbolic variable α_3

Analysis of an assignment in the combined domain





$$N^\sharp = \alpha_1 \geq 0 \wedge \alpha_3 \neq \mathbf{0x0}$$

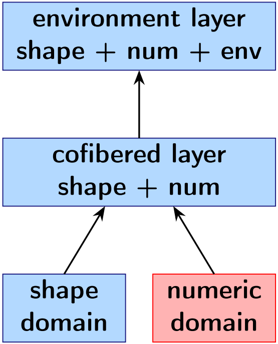
$$(*\alpha_2) \cdot \mathtt{d} = (*\alpha_0) + 1$$

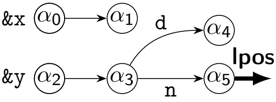
Abstract post-condition ?

Stage 4 (b): unfolding, shape part

- unfolding of the memory predicate part
- numerical predicates still need be taken into account

Analysis of an assignment in the combined domain





$N^\sharp = \alpha_1 \geq 0 \wedge \alpha_3 \neq \mathbf{0x0} \wedge \alpha_4 \geq 0$

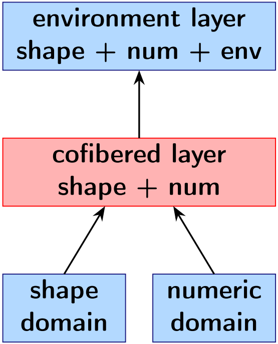
$$(*\alpha_2) \cdot \mathtt{d} = (*\alpha_0) + 1$$

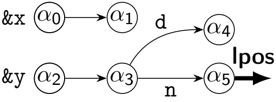
Abstract post-condition ?

Stage 4 (c): unfolding, numeric part

- numerical predicates taken into account
- l-value $\alpha_3 \cdot \mathtt{d}$ **now evaluates into edge** $\alpha_3 \cdot \mathtt{d} \mapsto \alpha_4$

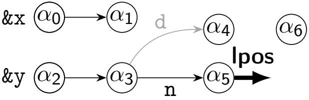
Analysis of an assignment in the combined domain





$$N^\sharp = \alpha_1 \geq 0 \wedge \alpha_3 \neq \mathbf{0x0} \wedge \alpha_4 \geq 0$$

create node α_6

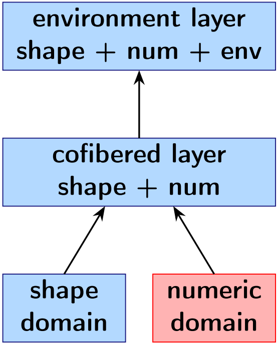


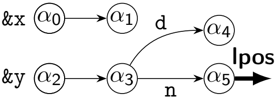
$$N^\sharp = \alpha_1 \geq 0 \wedge \alpha_3 \neq \mathbf{0x0} \wedge \alpha_4 \geq 0$$

Stage 5: create a new node

- new node α_6 denotes a new value
  will store the new value

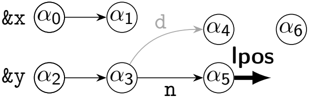
Analysis of an assignment in the combined domain





$$N^\sharp = \alpha_1 \geq 0 \wedge \alpha_3 \neq \mathbf{0x0} \wedge \alpha_4 \geq 0$$

$\alpha_6 \leftarrow \alpha_1 + 1$ in numerics

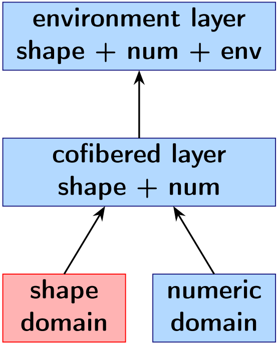


$$N^\sharp = \alpha_1 \geq 0 \wedge \alpha_3 \neq \mathbf{0x0} \wedge \alpha_4 \geq 0 \wedge \alpha_6 \geq 1$$

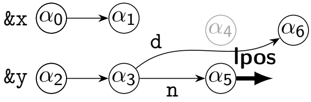
Stage 6: perform numeric assignment

- numeric assignment **completely ignores pointer structures** to the new node

Analysis of an assignment in the combined domain



mutate $(\alpha_3 \cdot \mathtt{d}) \mapsto \alpha_4$ into α_6



$N^\sharp = \alpha_1 \geq 0 \land \alpha_3 \neq \mathbf{0x0} \land \alpha_4 \geq 0 \land \alpha_6 \geq 1$

Stage 7: perform the update in the graph

- classic **strong update** in a pointer aware domain
- symbolic node α_4 becomes redundant and can be removed

Outline

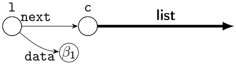
Need for a folding operation

Back to the **list traversal** example:

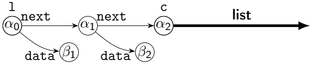
First iterates in the loop:

- at **iteration 0** (before entering the loop):

- at **iteration 1**:



- at **iteration 2**:




```
assume(l points to a list)
c = l;
while(c ≠ NULL){
  c = c → next;
}
```

The analysis **unfolds**, but **never folds**:

$$\begin{tikzcd}
S_0 \arrow[d, "\text{unfold}"'] & & \\
S_{0,u} \arrow[r, "f"] & S_1 \arrow[d, "\text{unfold}"'] & \\
& S_{1,u} \arrow[r, "f"] & S_2 \quad \cdots
\end{tikzcd}$$

- How to guarantee **termination** of the analysis ?
- How to **introduce segment edges** / perform **abstraction** ?

Widening

- The lattice of shape abstract values has **infinite height**
- Thus iteration sequences **may not terminate**

Definition of a widening operator $\triangledown$

- **Over-approximates join:**

$$\begin{cases} \gamma(X^\sharp) & \subseteq & \gamma(X^\sharp \triangledown Y^\sharp) \\ \gamma(Y^\sharp) & \subseteq & \gamma(X^\sharp \triangledown Y^\sharp) \end{cases}$$

- **Enforces termination:** for all sequence $(X_n^\sharp)_{n\in\mathbb{N}}$, the **sequence $(Y_n^\sharp)_{n\in\mathbb{N}}$ defined below is ultimately stationary**

$$\begin{cases} Y_0^\sharp & = & X_0^\sharp \\ \forall n \in \mathbb{N},\ Y_{n+1}^\sharp & = & Y_n^\sharp \triangledown X_{n+1}^\sharp \end{cases}$$

Canonicalization

Upper closure operator

$\rho : \mathbb{D}^\sharp \longrightarrow \mathbb{D}^\sharp_{\mathbf{can}} \subseteq \mathbb{D}^\sharp$ is an **upper closure operator** (uco) iff it is monotone, extensive and idempotent.

Canonicalization

- **Disjunctive completion:** $\mathbb{D}^\sharp_\vee$ = finite disjunctions over $\mathbb{D}^\sharp$
- **Canonicalization operator** $\rho_\vee$ defined by $\rho_\vee : \mathbb{D}^\sharp_\vee \longrightarrow \mathbb{D}^\sharp_{\mathbf{can}\vee}$ and $\rho_\vee(X^\sharp) = \{\rho(x^\sharp) \mid x^\sharp \in X^\sharp\}$ where ρ is an uco and $\mathbb{D}^\sharp_{\mathbf{can}}$ has finite height

- Canonicalization is used in **many shape analysis tools**: **TVLA** (truth blurring), most **separation logic** based analysis tools
- **Easier to compute** but **less powerful** than widening: does not exploit history of computation

Weakening: definition

To design **inclusion test**, **join** and **widening** algorithms, we first study a more general notion of **weakening**:

Weakening

We say that $S_0^\sharp$ **can be weakened into** $S_1^\sharp$ if and only if

$$\forall(\hbar, \nu) \in \gamma_{\mathrm{S}}(S_0^\sharp),\ \exists \nu' \in \mathbf{Val},\ (\hbar, \nu') \in \gamma_{\mathrm{S}}(S_1^\sharp)$$

We then note $S_0^\sharp \preccurlyeq S_1^\sharp$

Applications:

- **inclusion test** (comparison) inputs $S_0^\sharp, S_1^\sharp$; if returns true $S_0^\sharp \preccurlyeq S_1^\sharp$
- **canonicalization** (unary weakening) inputs $S_0^\sharp$ and returns $\rho(S_0^\sharp)$ such that $S_0^\sharp \preccurlyeq \rho(S_0^\sharp)$
- **widening** / **join** (binary weakening ensuring termination or not) inputs $S_0^\sharp, S_1^\sharp$ and returns $S_{\mathrm{up}}^\sharp$ such that $S_i^\sharp \preccurlyeq S_{\mathrm{up}}^\sharp$

Local weakening: separating conjunction rule

We can apply the local reasoning principle to weakening

If $S_0^\sharp \preccurlyeq S_{0,\mathbf{weak}}^\sharp$ and $S_1^\sharp \preccurlyeq S_{1,\mathbf{weak}}^\sharp$ then:

Separating conjunction rule ($\preccurlyeq_*$)

Let us assume that

- $S_0^\sharp$ and $S_1^\sharp$ have distinct set of **source nodes**
- we can weaken $S_0^\sharp$ into $S_{0,\mathbf{weak}}^\sharp$
- we can weaken $S_1^\sharp$ into $S_{1,\mathbf{weak}}^\sharp$

then: we can weaken $S_0^\sharp * S_1^\sharp$ into $S_{0,\mathbf{weak}}^\sharp * S_{1,\mathbf{weak}}^\sharp$

Local weakening: unfolding rule

Weakening unfolded region ($\preccurlyeq_{\mathcal{U}}$)

Let us assume that $S_0^\sharp \xrightarrow{\mathcal{U}} S_1^\sharp$. Then, by definition of the concretization of unfolding

we can weaken $S_1^\sharp$ into $S_0^\sharp$

- the proof follows from the definition of unfolding
- it can be applied locally, on graph regions that differ due to unfolding of inductive definitions

Local weakening: identity rule

Identity weakening ($\preccurlyeq_{\mathbf{Id}}$)

we can weaken $S^\sharp$ into $S^\sharp$

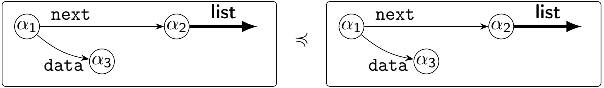
- the proof is trivial:

$$\gamma_{\mathrm{S}}(S^\sharp) \subseteq \gamma_{\mathrm{S}}(S^\sharp)$$

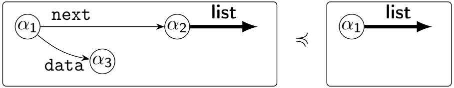
- on itself, this principle is not very useful, but it can be applied locally, and combined with $(\preccurlyeq_{\mathcal{U}})$ on graph regions that are not equal

Local weakening: example

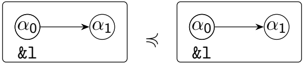
By **rule** ($\preccurlyeq_{\mathbf{Id}}$):



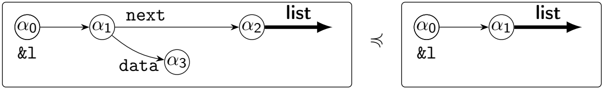
Thus, by **rule** ($\preccurlyeq_{\mathcal{U}}$):



Additionally, by **rule** ($\preccurlyeq_{\mathbf{Id}}$):



Thus, by **rule** ($\preccurlyeq_{*}$):



Inclusion checking rules in the shape domain

Graphs to compare have distinct sets of nodes, thus inclusion check should carry out a **valuation transformer** $\Psi : \mathbb{V}^\sharp(S_1^\sharp) \longrightarrow \mathbb{V}^\sharp(S_0^\sharp)$

Using (and extending) the weakening principles, we obtain the following rules (considering only inductive definition **list**, though these rules would extend to other definitions straightforwardly):

- **Identity rules:**

$$\begin{array}{rcrcl}
\forall i,\ \Psi(\beta_i) = \alpha_i & \Longrightarrow & \alpha_0 \cdot \mathtt{f} \mapsto \alpha_1 & \sqsubseteq^\sharp{}_\Psi & \beta_0 \cdot \mathtt{f} \mapsto \beta_1 \\
\Psi(\beta) = \alpha & \Longrightarrow & \alpha \cdot \mathbf{list} & \sqsubseteq^\sharp{}_\Psi & \beta \cdot \mathbf{list} \\
\forall i,\ \Psi(\beta_i) = \alpha_i & \Longrightarrow & \alpha_0 \cdot \mathbf{list_endp}(\alpha_1) & \sqsubseteq^\sharp{}_\Psi & \beta_0 \cdot \mathbf{list_endp}(\beta_1)
\end{array}$$

- **Rules on inductives:**

$$\begin{array}{rcrcl}
\forall i,\ \Psi(\beta_i) = \alpha & \Longrightarrow & \mathbf{emp} & \sqsubseteq^\sharp{}_\Psi & \beta_0 \cdot \mathbf{list_endp}(\beta_1) \\
S_0^\sharp \sqsubseteq^\sharp{}_\Psi S_1^\sharp \wedge \beta \cdot \iota \xrightarrow{\mathcal{U}} S_1^\sharp & \Longrightarrow & S_0^\sharp & \sqsubseteq^\sharp{}_\Psi & \beta \cdot \iota
\end{array}$$

if β_1 fresh , $\Psi' = \Psi[\beta_1 \mapsto \alpha_1]$ and $\Psi(\beta_0) = \alpha_0$ then,

$$S_0^\sharp \sqsubseteq^\sharp{}_{\Psi'} \beta_1 \cdot \mathbf{list} \quad \Longrightarrow \quad \alpha_0 \cdot \mathbf{list\_endp}(\alpha_1) * S_0^\sharp \quad \sqsubseteq^\sharp{}_\Psi \quad \beta_0 \cdot \iota$$

Inclusion checking algorithm

Comparison of $(e_0^\sharp, S_0^\sharp, N_0^\sharp)$ and $(e_1^\sharp, S_1^\sharp, N_1^\sharp)$

1. start with Ψ defined by $\Psi(\beta) = \alpha$ if and only if there exists a variable $\mathtt{x}$ such that $e_0^\sharp(\mathtt{x}) = \alpha \wedge e_1^\sharp(\mathtt{x}) = \beta$
2. iteratively **apply local rules**, and extend Ψ when needed
3. if the algorithm establishes $S_0^\sharp \preccurlyeq S_1^\sharp$, **compare** $N_0^\sharp \circ \Psi$ and $N_1^\sharp$ **in** $\mathbb{D}^\sharp_{\mathrm{num}}$

- the first step ensures both environments are consistent
- in the last step, composing with Ψ ensures we are comparing consistent numerical values (note that $N_0^\sharp$ and $N_1^\sharp$ may have distinct sets of nodes)

This algorithm is sound:

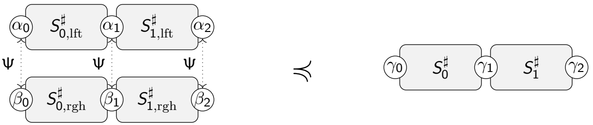
Soundness

$$(e_0^\sharp, S_0^\sharp, N_0^\sharp) \sqsubseteq^\sharp (e_1^\sharp, S_1^\sharp, N_1^\sharp) \implies \gamma(e_0^\sharp, S_0^\sharp, N_0^\sharp) \subseteq \gamma(e_1^\sharp, S_1^\sharp, N_1^\sharp)$$

Over-approximation of union

The principle of join and widening algorithm **is similar to that of $\sqsubseteq^\sharp$:**

- It can be computed **region by region**, as for weakening in general :
  If $\forall i \in \{0,1\},\ \forall s \in \{\mathrm{lft}, \mathrm{rgh}\},\ S^\sharp_{i,s} \preccurlyeq S^\sharp_s$,



  The partitioning of inputs / different nodes sets requires a **node correspondence function**

$$\Psi : \mathbb{V}^\sharp(S^\sharp_{\mathrm{lft}}) \times \mathbb{V}^\sharp(S^\sharp_{\mathrm{rgh}}) \longrightarrow \mathbb{V}^\sharp(S^\sharp)$$

- The computation of the shape join progresses by the application of **local join rules**, that produce a **new (output) shape graph, that weakens both inputs**

Over-approximation of union: syntactic identity rules

In the next few slides, we focus on $\triangledown$
though the abstract union would be defined similarly in the shape domain

Several rules derive **from** $(\preccurlyeq_{\mathbf{Id}})$:

- If $S^{\sharp}_{\mathrm{lft}} = \alpha_0 \cdot \mathtt{f} \mapsto \alpha_1$
  and $S^{\sharp}_{\mathrm{lft}} = \beta_0 \cdot \mathtt{f} \mapsto \beta_1$
  and $\Psi(\alpha_0, \beta_0) = \delta_0$, $\Psi(\alpha_1, \beta_1) = \delta_1$, then:

$$S^{\sharp}_{\mathrm{lft}} \triangledown S^{\sharp}_{\mathrm{rgh}} = \delta_0 \cdot \mathtt{f} \mapsto \delta_1$$

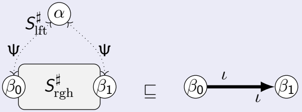
- If $S^{\sharp}_{\mathrm{lft}} = \alpha_0 \cdot \mathbf{list}$
  and $S^{\sharp}_{\mathrm{lft}} = \beta_0 \cdot \mathbf{list}_1$
  and $\Psi(\alpha_0, \beta_0) = \delta_0$, then:

$$S^{\sharp}_{\mathrm{lft}} \triangledown S^{\sharp}_{\mathrm{rgh}} = \delta_0 \cdot \mathbf{list}$$

Over-approximation of union: segment introduction rule

Rule

if

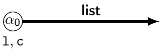


then

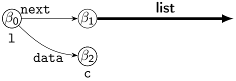
$$\left\{ \begin{array}{l} S^\sharp_{\mathrm{lft}} \triangledown S^\sharp_{\mathrm{rgh}} = \gamma_0 \xrightarrow[\iota]{\iota} \gamma_1 \\ (\alpha, \beta_0) \overset{\Psi}{\longleftrightarrow} \gamma_0 \\ (\alpha, \beta_1) \overset{\Psi}{\longleftrightarrow} \gamma_1 \end{array} \right.$$

Application to list traversal, at the end of iteration 1:

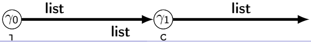
- **before iteration 0:**



- **end of iteration 0:**



- **join, before iteration 1:**

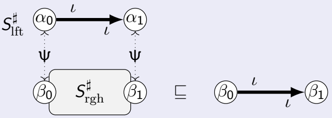


$$\left\{ \begin{array}{lcl} \Psi(\alpha_0, \beta_0) & = & \gamma_0 \\ \Psi(\alpha_0, \beta_1) & = & \gamma_1 \end{array} \right.$$

Over-approximation of union: segment extension rule

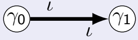
Rule

if



then

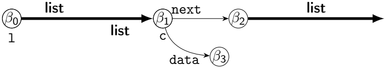
$$\begin{cases} S^\sharp_{\rm lft} \triangledown S^\sharp_{\rm rgh} = \\ (\alpha_0, \beta_0) \stackrel{\Psi}{\longleftrightarrow} \gamma_0 \\ (\alpha_1, \beta_1) \stackrel{\Psi}{\longleftrightarrow} \gamma_1 \end{cases}$$



Application to list traversal, at the end of iteration 1:

- previous invariant before iteration 1:

- end of iteration 1:



- join, before iteration 1:

$$\begin{cases} \Psi(\alpha_0, \beta_0) & = & \gamma_0 \\ \Psi(\alpha_1, \beta_2) & = & \gamma_1 \end{cases}$$

Over-approximation of union: rewrite system properties

- Comparison, canonicalization and widening algorithms can be considered **rewriting systems over tuples of graphs**
- **Success configuration**: weakening applies on all components, *i.e.*, the inputs are fully **"consumed"** in the weakening process
- **Failure configuration**: some components **cannot be weakened** *i.e.*, the algorithm should return the conservative answer (*i.e.*, $\top$)

Termination

- The systems are **terminating**
- This ensures comparison, canonicalization, widening are **computable**

Non confluence !

- The results depends on the order of application of the rules
- Implementation requires the choice of an **adequate strategy**

Over-approximation of union in the combined domain

Widening of $(e_0^\sharp, S_0^\sharp, N_0^\sharp)$ and $(e_1^\sharp, S_1^\sharp, N_1^\sharp)$

1. define Ψ, e by $\Psi(\alpha, \beta) = e(\mathrm{x}) = \delta$ (where δ is a fresh node) if and only if $e_0^\sharp(\mathrm{x}) = \alpha \wedge e_1^\sharp(\mathrm{x}) = \beta$
2. iteratively **apply join local rules**, and extend Ψ when new relations are inferred (for instance for points-to edges)
3. if the algorithm computes $S_0^\sharp \triangledown S_1^\sharp = S^\sharp$, compute the widening in the numeric domain: $N^\sharp = N_0^\sharp \circ \Psi_{\mathrm{lft}} \triangledown N_1^\sharp \circ \Psi_{\mathrm{rgh}}$

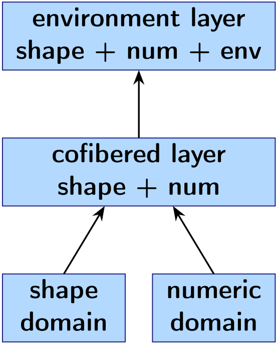
This algorithm is sound:

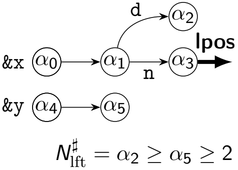
Soundness

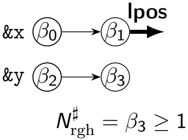
$$\gamma(e_0^\sharp, S_0^\sharp, N_0^\sharp) \cup \gamma(e_1^\sharp, S_1^\sharp, N_1^\sharp) \subseteq \gamma(e^\sharp, S^\sharp, N^\sharp)$$

Widening also enforces **termination** (it only introduces segments, and the growth induced by the introduction of segments is bounded)

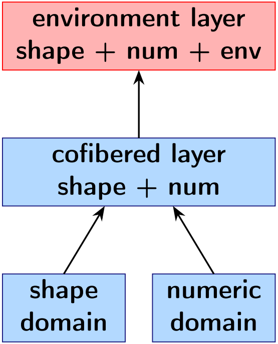
Widening / join in the combined domain

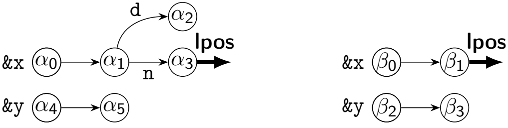






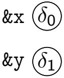
Widening / join in the combined domain





$N^\sharp_{\mathrm{lft}} = \alpha_2 \geq \alpha_5 \geq 2$

$N^\sharp_{\mathrm{rgh}} = \beta_3 \geq 1$

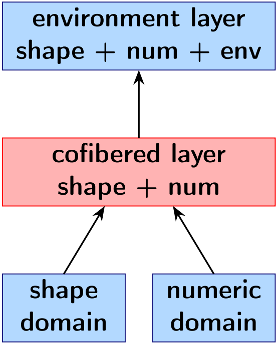


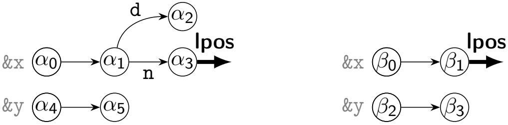
$$\begin{array}{lcl} \delta_0 & \equiv & (\alpha_0, \beta_0) \\ \delta_1 & \equiv & (\alpha_4, \beta_2) \end{array}$$

Stage 1: abstract environment

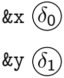
- compute new abstract environment and initial node relation
  e.g., α_0, β_0 both denote &x

Widening / join in the combined domain





$N^{\sharp}_{\mathrm{lft}} = \alpha_2 \geq \alpha_5 \geq 2$ $\qquad$ $N^{\sharp}_{\mathrm{rgh}} = \beta_3 \geq 1$



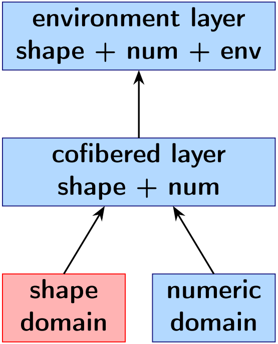
$$\begin{array}{lcl} \delta_0 & \equiv & (\alpha_0, \beta_0) \\ \delta_1 & \equiv & (\alpha_4, \beta_2) \end{array}$$

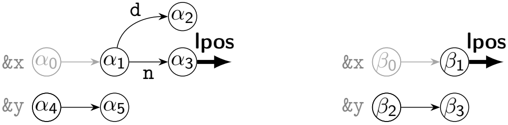
Stage 2: join in the "cofibered" layer

operations to perform:

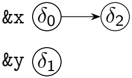
1. compute the join in the graph
2. convert value abstractions, and join the resulting lattice

Widening / join in the combined domain





$N^\sharp_{\mathrm{lft}} = \alpha_2 \geq \alpha_5 \geq 2$ $\quad$ $N^\sharp_{\mathrm{rgh}} = \beta_3 \geq 1$

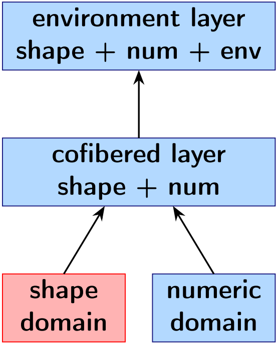


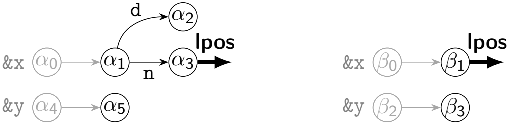
$$\begin{array}{lcl} \delta_0 & \equiv & (\alpha_0, \beta_0) \\ \delta_1 & \equiv & (\alpha_4, \beta_2) \\ \delta_2 & \equiv & (\alpha_1, \beta_1) \end{array}$$

Stage 2: graph join

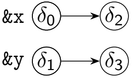
- apply local join rules
  ex: **points-to matching**, **weakening to inductive**...
- incremental algorithm

Widening / join in the combined domain





$$N^{\sharp}_{\mathrm{lft}} = \alpha_2 \geq \alpha_5 \geq 2 \qquad N^{\sharp}_{\mathrm{rgh}} = \beta_3 \geq 1$$

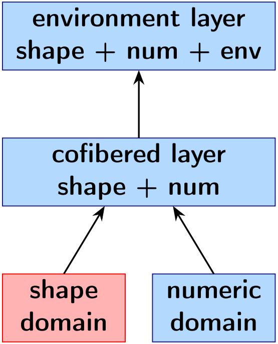


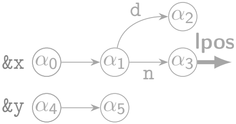
$$\begin{array}{lcl} \delta_0 & \equiv & (\alpha_0, \beta_0) \\ \delta_1 & \equiv & (\alpha_4, \beta_2) \\ \delta_2 & \equiv & (\alpha_1, \beta_1) \\ \delta_3 & \equiv & (\alpha_5, \beta_3) \end{array}$$

Stage 2: graph join

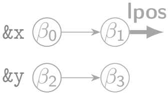
- apply local join rules
  ex: **points-to matching**, **weakening to inductive**...
- incremental algorithm

Widening / join in the combined domain

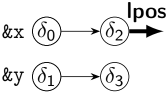




$N^\sharp_{\mathrm{lft}} = \alpha_2 \geq \alpha_5 \geq 2$



$N^\sharp_{\mathrm{rgh}} = \beta_3 \geq 1$

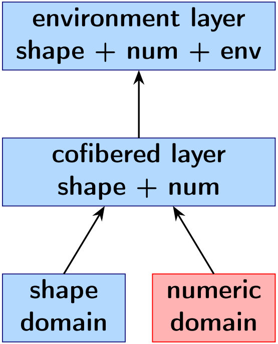


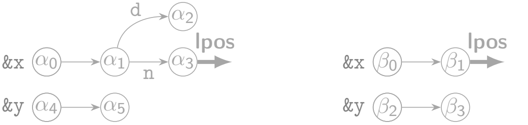
$$\begin{array}{lcl} \delta_0 & \equiv & (\alpha_0, \beta_0) \\ \delta_1 & \equiv & (\alpha_4, \beta_2) \\ \delta_2 & \equiv & (\alpha_1, \beta_1) \\ \delta_3 & \equiv & (\alpha_5, \beta_3) \end{array}$$

Stage 2: graph join

- apply local join rules
  ex: **points-to matching**, **weakening to inductive**...
- incremental algorithm

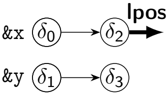
Widening / join in the combined domain





$N^\sharp_{\mathrm{lft}} = \alpha_2 \geq \alpha_5 \geq 2$

$N^\sharp_{\mathrm{rgh}} = \beta_3 \geq 1$



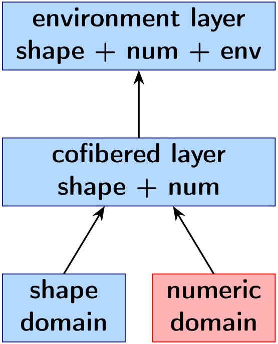
$$\begin{array}{lcl}
\delta_0 & \equiv & (\alpha_0, \beta_0) \\
\delta_1 & \equiv & (\alpha_4, \beta_2) \\
\delta_2 & \equiv & (\alpha_1, \beta_1) \\
\delta_3 & \equiv & (\alpha_5, \beta_3)
\end{array}$$

$N^\sharp_{\sqcup} = [\delta_3 \geq 2] \sqcup [\delta_3 \geq 1]$

Stage 3: conversion function application in numerics

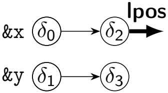
- remove nodes that were abstracted away
- rename other nodes

Widening / join in the combined domain



$N^\sharp_{\mathrm{lft}} = \alpha_2 \geq \alpha_5 \geq 2$

$N^\sharp_{\mathrm{rgh}} = \beta_3 \geq 1$



$N^\sharp_{\sqcup} = [\delta_3 \geq 1]$

$$\begin{array}{lcl}
\delta_0 & \equiv & (\alpha_0, \beta_0) \\
\delta_1 & \equiv & (\alpha_4, \beta_2) \\
\delta_2 & \equiv & (\alpha_1, \beta_1) \\
\delta_3 & \equiv & (\alpha_5, \beta_3)
\end{array}$$

Stage 4: join in the numeric domain

- apply $\sqcup$ for regular join, $\triangledown$ for a widening

Outline

Assumptions

What assumptions do we make ?
How do we prove soundness of the analysis of a loop ?

- **Assumptions in the concrete level**, and for block b:

$$
\begin{array}{rl}
(\mathcal{P}(\mathbb{M}), \subseteq) & \text{is a complete lattice, hence a CPO} \\
F : \mathcal{P}(\mathbb{M}) \to \mathcal{P}(\mathbb{M}) & \text{is the concrete semantic ("post") function of } \mathtt{b}
\end{array}
$$

  thus, the concrete semantics writes down as $[\![\mathtt{b}]\!] = \mathbf{lfp}_{\emptyset} F$

- **Assumptions in the abstract level:**

$$
\begin{array}{rl}
\mathbb{M}^\sharp & \text{set of abstract elements, no order a priori} \\
\gamma_{\mathrm{mem}} : \mathbb{M}^\sharp \to \mathcal{P}(\mathbb{M}) & \text{concretization} \\
F^\sharp : \mathbb{M}^\sharp \to \mathbb{M}^\sharp & \text{sound abstract semantic function} \\
 & \textit{i.e.}\text{, such that } F \circ \gamma_{\mathrm{mem}} \subseteq \gamma_{\mathrm{mem}} \circ F^\sharp \\
\nabla : \mathbb{M}^\sharp \times \mathbb{M}^\sharp \to \mathbb{M}^\sharp & \text{widening operator, terminates, and such that} \\
 & \gamma_{\mathrm{mem}}(m_0^\sharp) \cup \gamma_{\mathrm{mem}}(m_1^\sharp) \subseteq \gamma_{\mathrm{mem}}(m_0^\sharp \nabla m_1^\sharp)
\end{array}
$$

Computing a loop abstract post-condition

Loop abstract semantics

The abstract semantics of loop **while**(**rand**()){b} is calculated as the limit of the sequence of abstract iterates below:

$$\left\{ \begin{array}{rcl} m_0^\sharp & = & \perp \\ m_{n+1}^\sharp & = & m_n^\sharp \triangledown F^\sharp(m_n^\sharp) \end{array} \right.$$

Soundness proof:

- by induction over n, $\bigcup_{k \leq n} F^k(\emptyset) \subseteq \gamma_{\mathrm{mem}}(m_n^\sharp)$
- by the property of widening, the abstract sequence converges at a rank N: $\forall k \geq N,\ m_k^\sharp = m_N^\sharp$, thus

$$\mathbf{lfp}_\emptyset F = \bigcup_k F^k(\emptyset) \subseteq \gamma_{\mathrm{mem}}(m_N^\sharp)$$

Discussion on the abstract ordering

How about the abstract ordering ? We assumed *NONE* so far...

- **Logical ordering**, induced by concretization, used for **proofs**

$$m_0^\sharp \sqsubseteq m_1^\sharp \quad ::= \quad \text{"}\gamma_{\mathrm{mem}}(m_0^\sharp) \subseteq \gamma_{\mathrm{mem}}(m_1^\sharp)\text{"}$$

- **Approximation of the logical ordering**, **implemented** as a function **is_le** : $\mathbb{M}^\sharp \times \mathbb{M}^\sharp \to \{\textbf{true}, \top\}$, used to **test the convergence of abstract iterates**

$$\textbf{is\_le}(m_0^\sharp, m_1^\sharp) = \textbf{true} \quad \Longrightarrow \quad \gamma_{\mathrm{mem}}(m_0^\sharp) \subseteq \gamma_{\mathrm{mem}}(m_1^\sharp)$$

Abstract semantics is not assumed (and is actually most likely NOT) monotone with respect to either of these orders...

- Also, **computational ordering** would be used for **proving widening termination**

Outline

Updates and summarization

Weak updates cause significant precision loss...
Separation logic makes updates strong

Separation logic

Separating conjunction combines properties on disjoint stores

- Fundamental idea: $*$ **forces to identify what is modified**
- Before an **update** (or a **read**) takes place, memory cells need to be **materialized**
- **Local reasoning**: properties on unmodified cells pertain

Summaries

Inductive predicates describe unbounded memory regions

- Last lecture: **array segments** and **transitive closure** (TVLA)

Partial concretization, Global abstraction

Separation and summaries should be maintained by the analysis

Last lecture:

$$\begin{tikzcd} \mathcal{S}^\sharp_{\mathrm{pre}} \arrow[d, "\text{partially concretize}"'] & \mathcal{S}^\sharp_{\mathrm{post}} \\ \mathcal{S}_{\mathrm{pre}} \arrow[r, "f"] & \mathcal{S}_{\mathrm{post}} \arrow[u, "\text{abstract}"'] \end{tikzcd}$$

Today, two separate processes:

Local (partial) concretization

For **materialization**:

$$\begin{tikzcd} \mathcal{S}^\sharp_{\mathrm{pre}} \arrow[d, "\text{unfold (materialize)}"'] & \\ \mathcal{S}^\sharp_{\mathrm{pre,ref}} \arrow[r, "f"] & \mathcal{S}^\sharp_{\mathrm{post}} \end{tikzcd}$$

Global abstraction

Widening on loop heads:

$$\begin{tikzcd} & \mathcal{S}^\sharp_0 \triangledown \mathcal{S}^\sharp_1 & \\ \mathcal{S}^\sharp_0 \arrow[r] & \triangledown \arrow[u] & \mathcal{S}^\sharp_1 \arrow[l] \end{tikzcd}$$

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Internship topics:

- **Modular interprocedural analysis** using separation logic
- **Deciding implications among inductive predicates** and reduction