Non-Relational Numerical Abstract Domains

MPRI 2–6: Abstract Interpretation,
application to verification and static analysis

Antoine Miné

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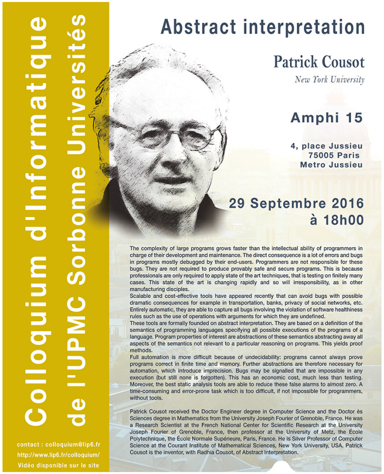
Abstract interpretation: LIP6 Colloquium



Colloquium d'Informatique de l'UPMC Sorbonne Universités

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Vidéo disponible sur le site

Abstract interpretation

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29 Septembre 2016
à 18h00

The complexity of large programs grows faster than the intellectual ability of programmers in charge of their development and maintenance. The direct consequence is a lot of errors and bugs in programs mostly debugged by their end-users. Programmers are not responsible for these bugs. They are not required to produce provably safe and secure programs. This is because professionals are only required to apply state of the art techniques, that is testing on finitely many cases. This state of the art is changing rapidly and so will irresponsibility, as in other manufacturing disciples.

Scalable and cost-effective tools have appeared recently that can avoid bugs with possible dramatic consequences for example in transportation, banks, privacy of social networks, etc. Entirely automatic, they are able to capture all bugs involving the violation of software healthiness rules such as the use of operations with arguments for which they are undefined.

These tools are formally founded on abstract interpretation. They are based on a definition of the semantics of programming languages specifying all possible executions of the programs of a language. Program properties of interest are abstractions of these semantics abstracting away all aspects of the semantics not relevant to a particular reasoning on programs. This yields proof methods.

Full automation is more difficult because of undecidability: programs cannot always prove programs correct in finite time and memory. Further abstractions are therefore necessary for automation, which introduce imprecision. Bugs may be signalled that are impossible in any execution (but still none is forgotten). This has an economic cost, much less than testing. Moreover, the best static analysis tools are able to reduce these false alarms to almost zero. A time-consuming and error-prone task which is too difficult, if not impossible for programmers, without tools.

Patrick Cousot received the Doctor Engineer degree in Computer Science and the Doctor ès Sciences degree in Mathematics from the University Joseph Fourier of Grenoble, France. He was a Research Scientist at the French National Center for Scientific Research at the University Joseph Fourier of Grenoble, France, then professor at the University of Metz, the École Polytechnique, the École Normale Supérieure, Paris, France. He is Silver Professor of Computer Science at the Courant Institute of Mathematical Sciences, New York University, USA. Patrick Cousot is the inventor, with Radhia Cousot, of Abstract Interpretation.

Talk by Patrick Cousot at Paris 6, 29 September 2016, 18h00
https://www.lip6.fr/colloquium/

Introduction

Invariant discovery

Goal: find intermittent numerical invariants

(at each program point, properties of numerical variables)

Example

```
X:=[0,10]; Y:=100;

while X>=0 do
        // loop invariant?
    X:=X-1;

    Y:=Y+10

done
// value of X and Y?
```

Invariant discovery

Goal: find intermittent numerical invariants

(at each program point, properties of numerical variables)

Example

```
X:=[0,10]; Y:=100;
   // $X \in [0,10],\ Y = 100$
while X>=0 do
      // $X \in [0,10],\ Y \in [100,200]$
   X:=X-1;
      // $X \in [-1,9],\ Y \in [100,200]$
   Y:=Y+10
      // $X \in [-1,9],\ Y \in [110,210]$
done
// $X = -1,\ Y \in [110,210]$
```

Variable bounds

Invariant discovery

Hope: find **the strongest** intermittent numerical invariants

(at each program point, **the strongest** properties of numerical variables)

Example

```
X:=[0,10]; Y:=100;
   // $X \in [0,10],\ Y = 100$
while X>=0 do
      // $X \in [0,10],\ 10X + Y \in [100,200] \cap 10\mathbb{Z}$
   X:=X-1;
      // $X \in [-1,9],\ 10X + Y \in [90,190] \cap 10\mathbb{Z}$
   Y:=Y+10
      // $X \in [-1,9],\ 10X + Y \in [100,200] \cap 10\mathbb{Z}$
done
// $X = -1,\ Y \in [110,210] \cap 10\mathbb{Z}$
```

Variable bounds, linear relations and congruences

Application: prove the absence of run-time error (overflow, array access, . . .)

Forward–backward analysis

sign function

```
X:=[-100,100];
if X=0 then Z:=0 else
  Y:=X;
  if Y < 0 then Y:=-Y;
  Z:=X/Y
fi
```

Forward–backward analysis

sign function

```
X:=[-100,100]; $(X \in [-100, 100])$
if X=0 then Z:=0 else $(X \in [-100, 100])$
  Y:=X; $(X, Y \in [-100, 100])$
  if Y < 0 then Y:=-Y; $(X \in [-100, 100], Y \in [0, 100])$
  Z:=X/Y $(X \in [-100, 100], Y \in [0, 100])$
fi
```

Forward interval analysis
(possible division by 0)

Forward–backward analysis

sign function

```
X:=[-100,100]; $(\perp)$
if X=0 then Z:=0 else $(X = 0)$
  Y:=X; $(Y = 0)$
  if Y < 0 then Y:=-Y; $(Y = 0)$
  Z:=X/Y $(Y = 0)$
fi
```

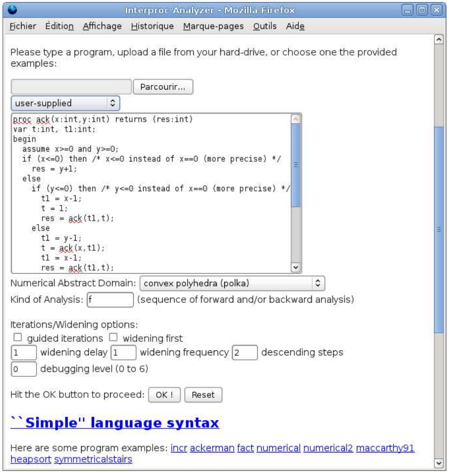
Backward interval analysis

- infer (tight) necessary conditions on inputs
  to reach a given point in a given state
  ($Y = 0$ at the end of the program)
- refine and focus the result of a forward analysis
  (prove the absence of division by zero) [Bour93b]

Academic implementation: Apron and Interproc

Apron: library of numerical abstractions [Jean09]

Interproc: on-line analyzer for a toy language, based on Apron


Interproc Analyzer - Mozilla Firefox
Fichier Édition Affichage Historique Marque-pages Outils Aide
Parcourir...

Please type a program, upload a file from your hard-drive, or choose one the provided examples:

user-supplied

```
proc ack(x:int,y:int) returns (res:int)
var t:int, t1:int;
begin
  assume x>=0 and y>=0;
  if (x<=0) then /* x<=0 instead of x==0 (more precise) */
    res = y+1;
  else
    if (y<=0) then /* y<=0 instead of x==0 (more precise) */
      t1 = x-1;
      t = 1;
      res = ack(t1,t);
    else
      t1 = y-1;
      t = ack(x,t1);
      t1 = x-1;
      res = ack(t1,t);
```

Numerical Abstract Domain: convex polyhedra (polka)

Kind of Analysis: f (sequence of forward and/or backward analysis)

Iterations/Widening options:

☐ guided iterations ☐ widening first

1 widening delay 1 widening frequency 2 descending steps

0 debugging level (0 to 6)

Hit the OK button to proceed: OK ! Reset

``Simple'' language syntax

Here are some program examples: incr ackerman fact numerical numerical2 maccarthy91 heapsort symmetricalstairs

http://pop-art.inrialpes.fr/interproc/interprocweb.cgi

Outline

- Generalities, notations
- Presentation of a few numerical abstract domains (non-relational)
  - sign domains
  - constant domain
  - interval domain
  - simple congruence domain
- Reduced products of domains
- Bibliography

Generalities and notations

Syntax

Expression syntax

Toy language:

- fixed, finite set of variables $\mathbb{V}$,
- one datatype: scalars in $\mathbb{I}$, with $\mathbb{I} \in \{\, \mathbb{Z}, \mathbb{Q}, \mathbb{R} \,\}$
  (and later, floating-point numbers $\mathbb{F}$)
- no procedure

arithmetic expressions:

$$
\begin{array}{lcll}
\texttt{exp} & ::= & \texttt{V} & \text{variable } \texttt{V} \in \mathbb{V} \\
 & | & -\texttt{exp} & \text{negation} \\
 & | & \texttt{exp} \diamond \texttt{exp} & \text{binary operation: } \diamond \in \{\, +, -, \times, / \,\} \\
 & | & [c, c'] & \text{constant range, } c, c' \in \mathbb{I} \cup \{\, \pm\infty \,\} \\
 & & & c \text{ is a shorthand for } [c, c]
\end{array}
$$

Programs (as control-flow graphs)

commands:

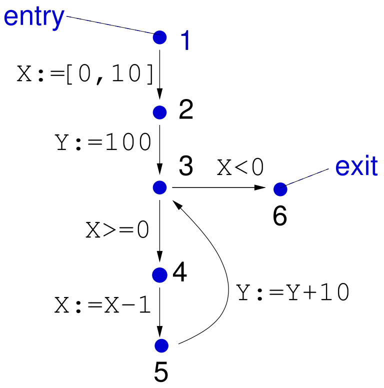
$$\begin{array}{lcll} \texttt{com} & ::= & \texttt{V := exp} & \text{assignment into } \texttt{V} \in \mathbb{V} \\ & | & \texttt{exp} \bowtie 0 & \text{test, } \bowtie \in \{=,<,>,<=,>=,<>\} \end{array}$$

programs: as control-flow graphs

$$P \overset{\text{def}}{=} (L, e, x, A) \quad \left| \quad \begin{array}{ll} L & \text{program points (labels)} \\ e & \text{entry point: } e \in L \\ x & \text{exit point: } x \in L \\ A & \text{arcs: } A \subseteq L \times \texttt{com} \times L \end{array} \right.$$

Example

```
¹X:=[0,10];²
Y:=100;
while ³X>=0 do⁴
  X:=X-1;⁵
  Y:=Y+10
done⁶
```



structured program

control flow graph

Structured programs can be easily compiled into a CFG.
We use structured program as examples, but present our analysis formally on CFG.

Concrete semantics

Forward concrete semantics

Semantics of expressions: $\mathsf{E}[\![\, e \,]\!] : (\mathbb{V} \to \mathbb{I}) \to \mathcal{P}(\mathbb{I})$

The evaluation of e in ρ gives a set of values:

$$\begin{array}{lcl}
\mathsf{E}[\![\, [c, c'] \,]\!]\, \rho & \overset{\text{def}}{=} & \{\, x \in \mathbb{I} \mid c \leq x \leq c' \,\} \\
\mathsf{E}[\![\, \mathtt{V} \,]\!]\, \rho & \overset{\text{def}}{=} & \{\, \rho(\mathtt{V}) \,\} \\
\mathsf{E}[\![\, -e \,]\!]\, \rho & \overset{\text{def}}{=} & \{\, -v \mid v \in \mathsf{E}[\![\, e \,]\!]\, \rho \,\} \\
\mathsf{E}[\![\, e_1 + e_2 \,]\!]\, \rho & \overset{\text{def}}{=} & \{\, v_1 + v_2 \mid v_1 \in \mathsf{E}[\![\, e_1 \,]\!]\, \rho, v_2 \in \mathsf{E}[\![\, e_2 \,]\!]\, \rho \,\} \\
\mathsf{E}[\![\, e_1 - e_2 \,]\!]\, \rho & \overset{\text{def}}{=} & \{\, v_1 - v_2 \mid v_1 \in \mathsf{E}[\![\, e_1 \,]\!]\, \rho, v_2 \in \mathsf{E}[\![\, e_2 \,]\!]\, \rho \,\} \\
\mathsf{E}[\![\, e_1 \times e_2 \,]\!]\, \rho & \overset{\text{def}}{=} & \{\, v_1 \times v_2 \mid v_1 \in \mathsf{E}[\![\, e_1 \,]\!]\, \rho, v_2 \in \mathsf{E}[\![\, e_2 \,]\!]\, \rho \,\} \\
\mathsf{E}[\![\, e_1 / e_2 \,]\!]\, \rho & \overset{\text{def}}{=} & \{\, v_1 / v_2 \mid v_1 \in \mathsf{E}[\![\, e_1 \,]\!]\, \rho, v_2 \in \mathsf{E}[\![\, e_2 \,]\!]\, \rho,\ v_2 \neq 0 \,\}
\end{array}$$

Forward concrete semantics (cont.)

Semantics of commands: $\mathsf{C}[\![\, c \,]\!] : \mathcal{P}(\mathbb{V} \to \mathbb{I}) \to \mathcal{P}(\mathbb{V} \to \mathbb{I})$

A transfer function for c defines a relation on environments:

$$\begin{array}{lcl} \mathsf{C}[\![\, \mathtt{V} := e \,]\!]\, \mathcal{X} & \overset{\text{def}}{=} & \{\, \rho[\, \mathtt{V} \mapsto v \,] \mid \rho \in \mathcal{X},\ v \in \mathsf{E}[\![\, e \,]\!]\, \rho \,\} \\ \mathsf{C}[\![\, e \bowtie 0 \,]\!]\, \mathcal{X} & \overset{\text{def}}{=} & \{\, \rho \mid \rho \in \mathcal{X},\ \exists v \in \mathsf{E}[\![\, e \,]\!]\, \rho,\ v \bowtie 0 \,\} \end{array}$$

It relates the environments after the execution of a command to the environments before.

Complete join morphism: $\mathsf{C}[\![\, c \,]\!]\, \mathcal{X} = \bigcup_{\rho \in \mathcal{X}} \mathsf{C}[\![\, c \,]\!]\, \{\, \rho \,\}$.

Forward concrete semantics (cont.)

Semantics of programs: $\mathsf{P}[\![\,(L,e,x,A)\,]\!] : L \to \mathcal{P}(\mathbb{V} \to \mathbb{I})$

$\mathsf{P}[\![\,(L,e,x,A)\,]\!]\,\ell$ is the most precise invariant at $\ell \in L$.

It is the smallest solution of a recursive equation system $(\mathcal{X}_\ell)_{\ell \in L}$:

Semantic equation system

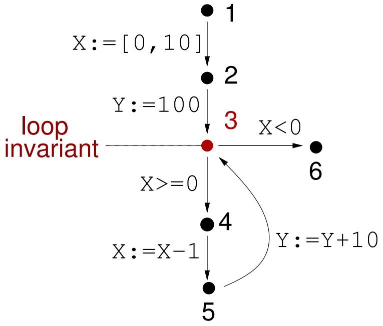
$$\begin{array}{lll} \mathcal{X}_e & & \text{(given initial state)} \\ \mathcal{X}_{\ell \neq e} = \displaystyle\bigcup_{(\ell',c,\ell)\in A} \mathsf{C}[\![\,c\,]\!]\,\mathcal{X}_{\ell'} & & \text{(transfer function)} \end{array}$$

Tarski's Theorem: this smallest solution exists and is unique.

- $\mathcal{D} \overset{\text{def}}{=} (\mathcal{P}(\mathbb{V} \to \mathbb{I}), \subseteq, \cup, \cap, \emptyset, (\mathbb{V} \to \mathbb{I}))$ is a complete lattice,
- each $M_\ell : \mathcal{X}_\ell \mapsto \displaystyle\bigcup_{(\ell',c,\ell)\in A} \mathsf{C}[\![\,c\,]\!]\,\mathcal{X}_{\ell'}$ is monotonic in $\mathcal{D}$.

$\Rightarrow$ the solution is the least fixpoint of $(M_\ell)_{\ell \in L}$.

Forward concrete semantics (example)



control flow graph

$$\begin{cases} \mathcal{X}_1 = (\{\,\mathtt{X},\mathtt{Y}\,\} \to \mathbb{Z}) \\ \mathcal{X}_2 = \mathsf{C}[\![\,\mathtt{X} := [0,10]\,]\!]\,\mathcal{X}_1 \\ \mathcal{X}_3 = \mathsf{C}[\![\,\mathtt{Y} := 100\,]\!]\,\mathcal{X}_2 \cup \\ \qquad \mathsf{C}[\![\,\mathtt{Y} := \mathtt{Y} + 10\,]\!]\,\mathcal{X}_5 \\ \mathcal{X}_4 = \mathsf{C}[\![\,\mathtt{X} \geq 0\,]\!]\,\mathcal{X}_3 \\ \mathcal{X}_5 = \mathsf{C}[\![\,\mathtt{X} := \mathtt{X} - 1\,]\!]\,\mathcal{X}_4 \\ \mathcal{X}_6 = \mathsf{C}[\![\,\mathtt{X} < 0\,]\!]\,\mathcal{X}_3 \end{cases}$$

equation system

Loop invariant:

$$\mathcal{X}_3 = \{\,\rho \mid \rho(\mathtt{X}) \in [0,10],\ 10\rho(\mathtt{X}) + \rho(\mathtt{Y}) \in [100,200] \cap 10\mathbb{Z}\,\}$$

Resolution

Resolution by increasing iterations:

$$\begin{cases} \mathcal{X}_e^0 & \stackrel{\text{def}}{=} & \mathcal{X}_e \\ \mathcal{X}_{\ell\neq e}^0 & \stackrel{\text{def}}{=} & \emptyset \end{cases} \qquad \begin{cases} \mathcal{X}_e^{n+1} & \stackrel{\text{def}}{=} & \mathcal{X}_e \\ \mathcal{X}_{\ell\neq e}^{n+1} & \stackrel{\text{def}}{=} & \bigcup_{(\ell',c,\ell)\in A} \mathsf{C}[\![\, c \,]\!]\, \mathcal{X}_{\ell'}^n \end{cases}$$

Converges in ω iterations to a least solution,
because each $\mathsf{C}[\![\, c \,]\!]$ is continuous in the CPO $\mathcal{D}$.

(Kleene fixpoint theorem)

Resolution (example)

iteration 0

$$\begin{cases} \mathcal{X}_1 = \mathbb{Z}^2 & \mathbb{Z}^2 \\ \mathcal{X}_2 = \mathsf{C}[\![\, \mathtt{X} := [0,10] \,]\!]\, \mathcal{X}_1 & \emptyset \\ \mathcal{X}_3 = \mathsf{C}[\![\, \mathtt{Y} := 100 \,]\!]\, \mathcal{X}_2 \cup \mathsf{C}[\![\, \mathtt{Y} := \mathtt{Y} + 10 \,]\!]\, \mathcal{X}_5 & \emptyset \\ \mathcal{X}_4 = \mathsf{C}[\![\, \mathtt{X} \geq 0 \,]\!]\, \mathcal{X}_3 & \emptyset \\ \mathcal{X}_5 = \mathsf{C}[\![\, \mathtt{X} := \mathtt{X} - 1 \,]\!]\, \mathcal{X}_4 & \emptyset \\ \mathcal{X}_6 = \mathsf{C}[\![\, \mathtt{X} < 0 \,]\!]\, \mathcal{X}_3 & \emptyset \end{cases}$$

Resolution (example)

$$\begin{cases}
 & \text{iteration 1} \\
\mathcal{X}_1 = \mathbb{Z}^2 & \mathbb{Z}^2 \\
\mathcal{X}_2 = \mathsf{C}[\![\, \mathtt{X} := [0,10] \,]\!]\, \mathcal{X}_1 & [0,10] \times \mathbb{Z} \\
\mathcal{X}_3 = \;\; \mathsf{C}[\![\, \mathtt{Y} := 100 \,]\!]\, \mathcal{X}_2 \cup & \emptyset \\
\qquad\;\; \mathsf{C}[\![\, \mathtt{Y} := \mathtt{Y} + 10 \,]\!]\, \mathcal{X}_5 & \\
\mathcal{X}_4 = \;\; \mathsf{C}[\![\, \mathtt{X} \geq 0 \,]\!]\, \mathcal{X}_3 & \emptyset \\
\mathcal{X}_5 = \;\; \mathsf{C}[\![\, \mathtt{X} := \mathtt{X} - 1 \,]\!]\, \mathcal{X}_4 & \emptyset \\
\mathcal{X}_6 = \mathsf{C}[\![\, \mathtt{X} < 0 \,]\!]\, \mathcal{X}_3 & \emptyset
\end{cases}$$

Resolution (example)

$$\begin{cases}
\mathcal{X}_1 = \mathbb{Z}^2 & \text{iteration 2} \quad \mathbb{Z}^2 \\
\mathcal{X}_2 = \mathsf{C}[\![\, \mathtt{X} := [0,10] \,]\!]\, \mathcal{X}_1 & [0,10] \times \mathbb{Z} \\
\mathcal{X}_3 = \begin{array}{l} \mathsf{C}[\![\, \mathtt{Y} := 100 \,]\!]\, \mathcal{X}_2 \cup \\ \mathsf{C}[\![\, \mathtt{Y} := \mathtt{Y} + 10 \,]\!]\, \mathcal{X}_5 \end{array} & \{\, (0,100), \ldots, (10,100) \,\} \\
\mathcal{X}_4 = \mathsf{C}[\![\, \mathtt{X} \geq 0 \,]\!]\, \mathcal{X}_3 & \emptyset \\
\mathcal{X}_5 = \mathsf{C}[\![\, \mathtt{X} := \mathtt{X} - 1 \,]\!]\, \mathcal{X}_4 & \emptyset \\
\mathcal{X}_6 = \mathsf{C}[\![\, \mathtt{X} < 0 \,]\!]\, \mathcal{X}_3 & \emptyset
\end{cases}$$

Resolution (example)

iteration 3

$$\begin{cases} \mathcal{X}_1 = \mathbb{Z}^2 & \mathbb{Z}^2 \\ \mathcal{X}_2 = \mathsf{C}[\![\, \mathtt{X} := [0,10] \,]\!]\, \mathcal{X}_1 & [0,10] \times \mathbb{Z} \\ \mathcal{X}_3 = \begin{array}{l} \mathsf{C}[\![\, \mathtt{Y} := 100 \,]\!]\, \mathcal{X}_2 \cup \\ \mathsf{C}[\![\, \mathtt{Y} := \mathtt{Y} + 10 \,]\!]\, \mathcal{X}_5 \end{array} & \{\, (0,100), \ldots, (10,100) \,\} \\ \mathcal{X}_4 = \mathsf{C}[\![\, \mathtt{X} \geq 0 \,]\!]\, \mathcal{X}_3 & \{\, (0,100), \ldots, (10,100) \,\} \\ \mathcal{X}_5 = \mathsf{C}[\![\, \mathtt{X} := \mathtt{X} - 1 \,]\!]\, \mathcal{X}_4 & \emptyset \\ \mathcal{X}_6 = \mathsf{C}[\![\, \mathtt{X} < 0 \,]\!]\, \mathcal{X}_3 & \emptyset \end{cases}$$

Resolution (example)

iteration 4

$$\begin{cases} \mathcal{X}_1 = \mathbb{Z}^2 & \mathbb{Z}^2 \\ \mathcal{X}_2 = \mathsf{C}[\![\, \texttt{X} := [0,10] \,]\!]\, \mathcal{X}_1 & [0,10] \times \mathbb{Z} \\ \mathcal{X}_3 = \mathsf{C}[\![\, \texttt{Y} := 100 \,]\!]\, \mathcal{X}_2 \cup \mathsf{C}[\![\, \texttt{Y} := \texttt{Y} + 10 \,]\!]\, \mathcal{X}_5 & \{ (0,100), \ldots, (10,100) \} \\ \mathcal{X}_4 = \mathsf{C}[\![\, \texttt{X} \geq 0 \,]\!]\, \mathcal{X}_3 & \{ (0,100), \ldots, (10,100) \} \\ \mathcal{X}_5 = \mathsf{C}[\![\, \texttt{X} := \texttt{X} - 1 \,]\!]\, \mathcal{X}_4 & \{ (-1,100), \ldots, (9,100) \} \\ \mathcal{X}_6 = \mathsf{C}[\![\, \texttt{X} < 0 \,]\!]\, \mathcal{X}_3 & \emptyset \end{cases}$$

Resolution (example)

$$\begin{cases}
& \text{iteration 5} \\
\mathcal{X}_1 = \mathbb{Z}^2 & \mathbb{Z}^2 \\
\mathcal{X}_2 = \mathsf{C}[\![\, \mathtt{X} := [0,10] \,]\!]\, \mathcal{X}_1 & [0,10] \times \mathbb{Z} \\
\mathcal{X}_3 = \begin{array}{l} \mathsf{C}[\![\, \mathtt{Y} := 100 \,]\!]\, \mathcal{X}_2 \cup \\ \mathsf{C}[\![\, \mathtt{Y} := \mathtt{Y} + 10 \,]\!]\, \mathcal{X}_5 \end{array} & \begin{array}{l} \{\, (0,100), \ldots, (10,100), \\ \ (-1,110), \ldots, (9,110) \,\} \end{array} \\
\mathcal{X}_4 = \mathsf{C}[\![\, \mathtt{X} \geq 0 \,]\!]\, \mathcal{X}_3 & \{\, (0,100), \ldots, (10,100) \,\} \\
\mathcal{X}_5 = \mathsf{C}[\![\, \mathtt{X} := \mathtt{X} - 1 \,]\!]\, \mathcal{X}_4 & \{\, (-1,100), \ldots, (9,100) \,\} \\
\mathcal{X}_6 = \mathsf{C}[\![\, \mathtt{X} < 0 \,]\!]\, \mathcal{X}_3 & \emptyset
\end{cases}$$

Resolution (example)

iteration 6

$$\begin{cases}
\mathcal{X}_1 = \mathbb{Z}^2 & \mathbb{Z}^2 \\
\mathcal{X}_2 = \mathsf{C}[\![\, \mathtt{X} := [0,10] \,]\!]\, \mathcal{X}_1 & [0,10] \times \mathbb{Z} \\
\mathcal{X}_3 = \begin{array}{l} \mathsf{C}[\![\, \mathtt{Y} := 100 \,]\!]\, \mathcal{X}_2 \cup \\ \mathsf{C}[\![\, \mathtt{Y} := \mathtt{Y} + 10 \,]\!]\, \mathcal{X}_5 \end{array} & \begin{array}{l} \{\, (0,100), \ldots, (10,100), \\ \;(-1,110), \ldots, (9,110) \,\} \end{array} \\
\mathcal{X}_4 = \mathsf{C}[\![\, \mathtt{X} \geq 0 \,]\!]\, \mathcal{X}_3 & \begin{array}{l} \{\, (0,100), \ldots, (10,100), \\ \;(0,110), \ldots, (9,110) \,\} \end{array} \\
\mathcal{X}_5 = \mathsf{C}[\![\, \mathtt{X} := \mathtt{X} - 1 \,]\!]\, \mathcal{X}_4 & \{\, (-1,100), \ldots, (9,100) \,\} \\
\mathcal{X}_6 = \mathsf{C}[\![\, \mathtt{X} < 0 \,]\!]\, \mathcal{X}_3 & \{\, (-1,110) \,\}
\end{cases}$$

Resolution (example)

$$
\begin{cases}
 & \text{iteration 7} \\
\mathcal{X}_1 = \mathbb{Z}^2 & \mathbb{Z}^2 \\
\mathcal{X}_2 = \mathsf{C}[\![\, \mathtt{X} := [0,10] \,]\!]\, \mathcal{X}_1 & [0,10] \times \mathbb{Z} \\
\mathcal{X}_3 = \begin{array}{l} \mathsf{C}[\![\, \mathtt{Y} := 100 \,]\!]\, \mathcal{X}_2 \cup \\ \mathsf{C}[\![\, \mathtt{Y} := \mathtt{Y} + 10 \,]\!]\, \mathcal{X}_5 \end{array} & \begin{array}{l} \{\, (0,100), \ldots, (10,100), \\ \ (-1,110), \ldots, (9,110) \,\} \end{array} \\
\mathcal{X}_4 = \mathsf{C}[\![\, \mathtt{X} \geq 0 \,]\!]\, \mathcal{X}_3 & \begin{array}{l} \{\, (0,100), \ldots, (10,100), \\ \ (0,110), \ldots, (9,110) \,\} \end{array} \\
\mathcal{X}_5 = \mathsf{C}[\![\, \mathtt{X} := \mathtt{X} - 1 \,]\!]\, \mathcal{X}_4 & \begin{array}{l} \{\, (-1,100), \ldots, (9,100), \\ \ (-1,110), \ldots, (8,110) \,\} \end{array} \\
\mathcal{X}_6 = \mathsf{C}[\![\, \mathtt{X} < 0 \,]\!]\, \mathcal{X}_3 & \{\, (-1,110) \,\}
\end{cases}
$$

Resolution (example)

iteration 8

$$\begin{cases}
\mathcal{X}_1 = \mathbb{Z}^2 & \mathbb{Z}^2 \\[2em]
\mathcal{X}_2 = \mathsf{C}[\![\, \mathtt{X} := [0,10] \,]\!]\, \mathcal{X}_1 & [0,10] \times \mathbb{Z} \\[2em]
\mathcal{X}_3 = \begin{array}{l} \mathsf{C}[\![\, \mathtt{Y} := 100 \,]\!]\, \mathcal{X}_2 \cup \\ \mathsf{C}[\![\, \mathtt{Y} := \mathtt{Y} + 10 \,]\!]\, \mathcal{X}_5 \end{array} & \begin{array}{l} \{\, (0,100), \ldots, (10,100), \\ \;(-1,110), \ldots, (9,110), \\ \;(-1,120), \ldots, (8,120) \,\} \end{array} \\
\mathcal{X}_4 = \mathsf{C}[\![\, \mathtt{X} \geq 0 \,]\!]\, \mathcal{X}_3 & \begin{array}{l} \{\, (0,100), \ldots, (10,100), \\ \;(0,110), \ldots, (9,110) \,\} \end{array} \\[2em]
\mathcal{X}_5 = \mathsf{C}[\![\, \mathtt{X} := \mathtt{X} - 1 \,]\!]\, \mathcal{X}_4 & \begin{array}{l} \{\, (-1,100), \ldots, (9,100), \\ \;(-1,110), \ldots, (8,110) \,\} \end{array} \\[2em]
\mathcal{X}_6 = \mathsf{C}[\![\, \mathtt{X} < 0 \,]\!]\, \mathcal{X}_3 & \{\, (-1,110) \,\}
\end{cases}$$

Resolution (example)

$$\begin{cases} \mathcal{X}_1 = \mathbb{Z}^2 & \text{iteration 9} \\ & \mathbb{Z}^2 \\ \mathcal{X}_2 = \mathsf{C}[\![\, \texttt{X} := [0,10] \,]\!]\, \mathcal{X}_1 & [0,10] \times \mathbb{Z} \\ \mathcal{X}_3 = \mathsf{C}[\![\, \texttt{Y} := 100 \,]\!]\, \mathcal{X}_2 \cup & \{ (0,100), \ldots, (10,100), \\ \qquad \mathsf{C}[\![\, \texttt{Y} := \texttt{Y} + 10 \,]\!]\, \mathcal{X}_5 & (-1,110), \ldots, (9,110), \\ & (-1,120), \ldots, (8,120) \} \\ \mathcal{X}_4 = \mathsf{C}[\![\, \texttt{X} \geq 0 \,]\!]\, \mathcal{X}_3 & \{ (0,100), \ldots, (10,100), \\ & (0,110), \ldots, (9,110), \\ & (0,120), \ldots, (8,120) \} \\ \mathcal{X}_5 = \mathsf{C}[\![\, \texttt{X} := \texttt{X} - 1 \,]\!]\, \mathcal{X}_4 & \{ (-1,100), \ldots, (9,100), \\ & (-1,110), \ldots, (8,110) \} \\ \mathcal{X}_6 = \mathsf{C}[\![\, \texttt{X} < 0 \,]\!]\, \mathcal{X}_3 & \{ (-1,110), (-1,120) \} \end{cases}$$

Resolution (example)

iteration 10

$$\begin{cases}
\mathcal{X}_1 = \mathbb{Z}^2 & \mathbb{Z}^2 \\
\mathcal{X}_2 = \mathsf{C}[\![\, \mathtt{X} := [0,10] \,]\!]\, \mathcal{X}_1 & [0,10] \times \mathbb{Z} \\
\mathcal{X}_3 = \begin{array}{l} \mathsf{C}[\![\, \mathtt{Y} := 100 \,]\!]\, \mathcal{X}_2 \cup \\ \mathsf{C}[\![\, \mathtt{Y} := \mathtt{Y} + 10 \,]\!]\, \mathcal{X}_5 \end{array} & \begin{array}{l} \{\, (0,100), \ldots, (10,100), \\ \;(-1,110), \ldots, (9,110), \\ \;(-1,120), \ldots, (8,120) \,\} \end{array} \\
\mathcal{X}_4 = \mathsf{C}[\![\, \mathtt{X} \geq 0 \,]\!]\, \mathcal{X}_3 & \begin{array}{l} \{\, (0,100), \ldots, (10,100), \\ \;(0,110), \ldots, (9,110), \\ \;(0,120), \ldots, (8,120) \,\} \end{array} \\
\mathcal{X}_5 = \mathsf{C}[\![\, \mathtt{X} := \mathtt{X} - 1 \,]\!]\, \mathcal{X}_4 & \begin{array}{l} \{\, (-1,100), \ldots, (9,100), \\ \;(-1,110), \ldots, (8,110), \\ \;(-1,120), \ldots, (7,120) \,\} \end{array} \\
\mathcal{X}_6 = \mathsf{C}[\![\, \mathtt{X} < 0 \,]\!]\, \mathcal{X}_3 & \{\, (-1,110), (-1,120) \,\}
\end{cases}$$

Resolution (example)

$$\begin{cases}
 & \text{iteration } \dots \\
\mathcal{X}_1 = \mathbb{Z}^2 & \mathbb{Z}^2 \\
\mathcal{X}_2 = \mathsf{C}[\![\, \mathtt{X} := [0,10] \,]\!]\, \mathcal{X}_1 & [0,10] \times \mathbb{Z} \\
\mathcal{X}_3 = \ \mathsf{C}[\![\, \mathtt{Y} := 100 \,]\!]\, \mathcal{X}_2 \cup & \{\, (0,100), \dots, (10,100), \\
\qquad\ \ \mathsf{C}[\![\, \mathtt{Y} := \mathtt{Y} + 10 \,]\!]\, \mathcal{X}_5 & \ \ (-1,110), \dots, (9,110), \\
 & \ \ (-1,120), \dots, (8,120), \dots \,\} \\
\mathcal{X}_4 = \ \mathsf{C}[\![\, \mathtt{X} \geq 0 \,]\!]\, \mathcal{X}_3 & \{\, (0,100), \dots, (10,100), \\
 & \ \ (0,110), \dots, (9,110), \\
 & \ \ (0,120), \dots, (8,120), \dots \,\} \\
\mathcal{X}_5 = \ \mathsf{C}[\![\, \mathtt{X} := \mathtt{X} - 1 \,]\!]\, \mathcal{X}_4 & \{\, (-1,100), \dots, (9,100), \\
 & \ \ (-1,110), \dots, (8,110), \\
 & \ \ (-1,120), \dots, (7,120), \dots \,\} \\
\mathcal{X}_6 = \mathsf{C}[\![\, \mathtt{X} < 0 \,]\!]\, \mathcal{X}_3 & \{\, (-1,110), (-1,120), \dots \,\}
\end{cases}$$

Backward concrete semantics

Semantics of commands: $\overleftarrow{C}[\![c]\!] : \mathcal{P}(\mathbb{V} \to \mathbb{I}) \to \mathcal{P}(\mathbb{V} \to \mathbb{I})$

$$\begin{array}{lcl}
\overleftarrow{C}[\![\texttt{V} := e]\!]\,\mathcal{X} & \stackrel{\text{def}}{=} & \{\, \rho \mid \exists v \in \mathsf{E}[\![e]\!]\,\rho,\ \rho[\,\texttt{V} \mapsto v\,] \in \mathcal{X} \,\} \\
\overleftarrow{C}[\![e \bowtie 0]\!]\,\mathcal{X} & \stackrel{\text{def}}{=} & \mathsf{C}[\![e \bowtie 0]\!]\,\mathcal{X}
\end{array}$$

(necessary conditions on ρ to have a successor in $\mathcal{X}$ by c)

Refinement decreasing iterations: given:

- a solution $(\mathcal{X}_\ell)_{\ell \in L}$ of the forward system
- an output criterion $\mathcal{Y}_x$

compute a least fixpoint by decreasing iterations [Bour93b]

$$\begin{cases}
\mathcal{Y}^0_x & \stackrel{\text{def}}{=} & \mathcal{X}_x \cap \mathcal{Y}_x \\
\mathcal{Y}^0_{\ell \neq x} & \stackrel{\text{def}}{=} & \mathcal{X}_\ell
\end{cases}$$

$$\begin{cases}
\mathcal{Y}^{n+1}_x & \stackrel{\text{def}}{=} & \mathcal{X}_x \cap \mathcal{Y}_x \\
\mathcal{Y}^{n+1}_{\ell \neq x} & \stackrel{\text{def}}{=} & \mathcal{X}_\ell \cap \left(\bigcup_{(\ell, c, \ell') \in A} \overleftarrow{C}[\![c]\!]\,\mathcal{Y}^n_{\ell'} \right)
\end{cases}$$

Limit to automation

We wish to perform automatic numerical invariant discovery.

Theoretical problems

- elements of $\mathcal{P}(\mathbb{V} \to \mathbb{I})$ are not computer representable
- transfer functions $\mathsf{C}[\![\, c \,]\!]$, $\overleftarrow{C}[\![\, c \,]\!]$ are not computable
- lattice iterations in $\mathcal{P}(\mathbb{V} \to \mathbb{I})$ are transfinite

Finding the best invariant is an undecidable problem

Note:

Even when $\mathbb{I}$ is finite, a concrete analysis is not tractable:

- representing elements in $\mathcal{P}(\mathbb{V} \to \mathbb{I})$ in extension is expensive
- computing $\mathsf{C}[\![\, c \,]\!]$, $\overleftarrow{C}[\![\, c \,]\!]$ explicitly is expensive
- the lattice $\mathcal{P}(\mathbb{V} \to \mathbb{I})$ has a large height ($\Rightarrow$ many iterations)

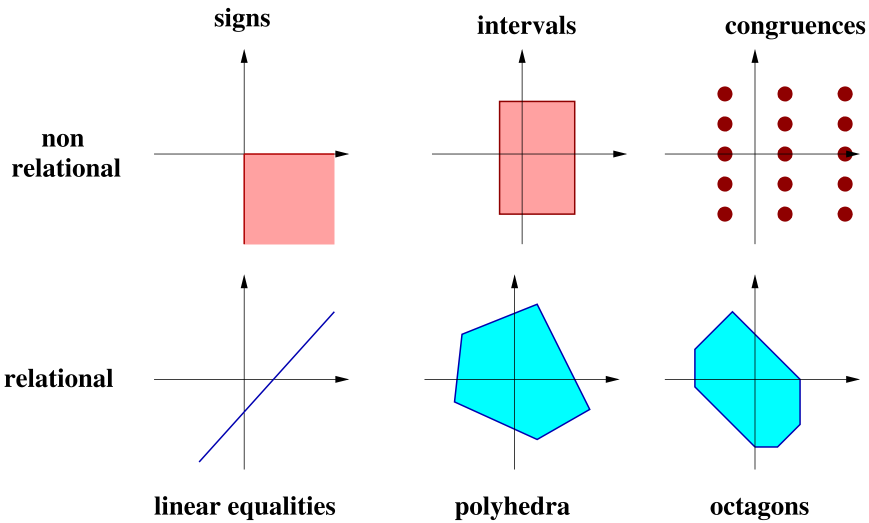
Abstraction

Numerical abstract domains

A numerical abstract domain is given by:

- a subset of $\mathcal{P}(\mathbb{V} \to \mathbb{I})$
  (a set of environment sets)
  together with a machine encoding,
- effective and sound abstract operators,
- an iteration strategy
  ensuring convergence in finite time.

Numerical abstract domain examples



Numerical abstract domains (cont.)

Representation: given by

- a set $\mathcal{D}^\sharp$ of machine-representable abstract values,
- a partial order $(\mathcal{D}^\sharp, \sqsubseteq, \bot^\sharp, \top^\sharp)$
  relating the amount of information given by abstract values,
- a concretization function γ: $\mathcal{D}^\sharp \to \mathcal{P}(\mathbb{V} \to \mathbb{I})$
  giving a concrete meaning to each abstract element.

Required algebraic properties:

- γ should be monotonic for $\sqsubseteq$: $\mathcal{X}^\sharp \sqsubseteq \mathcal{Y}^\sharp \implies \gamma(\mathcal{X}^\sharp) \subseteq \gamma(\mathcal{Y}^\sharp)$,
- $\gamma(\bot^\sharp) = \emptyset$,
- $\gamma(\top^\sharp) = \mathbb{V} \to \mathbb{I}$.

Note: γ need not be one-to-one.

Numerical abstract domains (cont.)

Abstract operators: we require:

- sound, effective, abstract transfer functions $C^\sharp[\![\, c \,]\!]$, $\overleftarrow{C}^\sharp[\![\, c \,]\!]$ for all commands c,
- sound, effective, abstract set operators $\cup^\sharp$, $\cap^\sharp$,
- an algorithm to decide the ordering $\sqsubseteq$.

Soundness criterion:

$F^\sharp$ is a sound abstraction of a $n-$ary operator F if:

$$\forall \mathcal{X}_1^\sharp, \ldots, \mathcal{X}_n^\sharp \in D^\sharp,\ F(\gamma(\mathcal{X}_1^\sharp), \ldots, \gamma(\mathcal{X}_n^\sharp)) \ \subseteq\ \gamma(F^\sharp(\mathcal{X}_1^\sharp, \ldots, \mathcal{X}_n^\sharp))$$

Both semantic and algorithmic aspects.

Abstract semantics

Abstract semantic equation system

$\mathcal{X}^\sharp : L \to \mathcal{D}^\sharp$

$$\mathcal{X}^\sharp_\ell \sqsupseteq \begin{cases} \mathcal{X}^\sharp_e & \text{if } \ell = e \qquad \text{(where } \mathcal{X}_e \subseteq \gamma(\mathcal{X}^\sharp_e)\text{)} \\ \bigcup^\sharp_{(\ell',c,\ell)\in A} \mathsf{C}^\sharp[\![\, c \,]\!]\, \mathcal{X}^\sharp_{\ell'} & \text{if } \ell \neq e \qquad \text{(abstract transfer function)} \end{cases}$$

Soundness Theorem

Any solution $(\mathcal{X}^\sharp_\ell)_{\ell \in L}$ is a **sound over-approximation** of the concrete collecting semantics:

$$\forall \ell \in L,\ \gamma(\mathcal{X}^\sharp_\ell) \supseteq \mathcal{X}_\ell$$

where $\mathcal{X}_\ell$ is the smallest solution of

$$\begin{cases} \mathcal{X}_e & \text{given} \\ \mathcal{X}_\ell = \bigcup_{(\ell',c,\ell)\in A} \mathsf{C}[\![\, c \,]\!]\, \mathcal{X}_{\ell'} & \text{if } \ell \neq e \end{cases}$$

Iteration strategy

Resolution by iterations in $\mathcal{D}^\sharp$:

To effectively solve the abstract system, we require:

- an iteration ordering on abstract equations
  (which equation(s) are applied at a given iteration)
- a widening operator $\triangledown$ to speed-up the convergence, if there are infinite strictly increasing chains in $D^\sharp$.

$\triangledown : (\mathcal{D}^\sharp \times \mathcal{D}^\sharp) \to \mathcal{D}^\sharp$ is a widening if:

- it is sound: $\gamma(\mathcal{X}^\sharp) \cup \gamma(\mathcal{Y}^\sharp) \subseteq \gamma(\mathcal{X}^\sharp \triangledown \mathcal{Y}^\sharp)$
- it enforces termination:
  $\forall$ sequence $(\mathcal{Y}_i^\sharp)_{i\in\mathbb{N}}$
  the sequence $\mathcal{X}_0^\sharp = \mathcal{Y}_0^\sharp,\ \mathcal{X}_{i+1}^\sharp = \mathcal{X}_i^\sharp \triangledown \mathcal{Y}_{i+1}^\sharp$
  stabilizes in finite time: $\exists n < \omega,\ \mathcal{X}_{n+1}^\sharp = \mathcal{X}_n^\sharp$
  (note: $\exists n,\ \forall m \geq n,\ \mathcal{X}_{m+1}^\sharp = \mathcal{X}_m^\sharp$ is not required)

Abstract analysis

$\mathcal{W} \subseteq L$ is a set of widening points if every CFG cycle has a point in $\mathcal{W}$.

Forward analysis:

$$\mathcal{X}_e^{\sharp 0} \overset{\text{def}}{=} \mathcal{X}_e^{\sharp} \quad \text{given, such that } \mathcal{X}_e \subseteq \gamma(\mathcal{X}_e^{\sharp})$$

$$\mathcal{X}_{\ell \neq e}^{\sharp 0} \overset{\text{def}}{=} \perp^{\sharp}$$

$$\mathcal{X}_\ell^{\sharp n+1} \overset{\text{def}}{=} \begin{cases} \mathcal{X}_e^{\sharp} & \text{if } \ell = e \\ \bigcup^{\sharp}_{(\ell',c,\ell)\in A} \mathsf{C}^{\sharp}[\![\, c \,]\!]\, \mathcal{X}_{\ell'}^{\sharp n} & \text{if } \ell \notin \mathcal{W},\ \ell \neq e \\ \mathcal{X}_\ell^{\sharp n} \triangledown \bigcup^{\sharp}_{(\ell',c,\ell)\in A} \mathsf{C}^{\sharp}[\![\, c \,]\!]\, \mathcal{X}_{\ell'}^{\sharp n} & \text{if } \ell \in \mathcal{W},\ \ell \neq e \end{cases}$$

- termination: for some δ, $\forall \ell,\ \mathcal{X}_\ell^{\sharp \delta+1} = \mathcal{X}_\ell^{\sharp \delta}$
- soundness: $\forall \ell \in L,\ \mathcal{X}_\ell \subseteq \gamma(\mathcal{X}_\ell^{\sharp \delta})$
- can be refined by decreasing iterations with narrowing $\triangle$ (presented later)
- here, apply every equation at each step, but other iteration scheme are possible (worklist, chaotic iterations, see [Bour93a])

Abstract analysis (proof)

Proof of soundness:

Suppose that $\forall \ell$, $\mathcal{X}_\ell^{\sharp\delta+1} = \mathcal{X}_\ell^{\sharp\delta}$.

If $\ell = e$, by definition: $\mathcal{X}_e^{\sharp\delta} = \mathcal{X}_e^{\sharp}$ and $\mathcal{X}_e \subseteq \gamma(\mathcal{X}_e^{\sharp\delta})$.

If $\ell \neq e$, $\ell \notin \mathcal{W}$, then $\mathcal{X}_\ell^{\sharp\delta} = \mathcal{X}_\ell^{\sharp\delta+1} = \cup^\sharp_{(\ell',c,\ell)\in A} \mathsf{C}^\sharp[\![c]\!] \mathcal{X}_{\ell'}^{\sharp\,\delta}$.

By soundness of $\cup^\sharp$ and $\mathsf{C}^\sharp[\![c]\!]$, $\gamma(\mathcal{X}_\ell^{\sharp\delta}) \supseteq \cup_{(\ell',c,\ell)\in A} \mathsf{C}[\![c]\!] \gamma(\mathcal{X}_{\ell'}^{\sharp\,\delta})$.

If $\ell \neq e$, $\ell \in \mathcal{W}$, then $\mathcal{X}_\ell^{\sharp\delta} = \mathcal{X}_\ell^{\sharp\delta+1} = \mathcal{X}_\ell^{\sharp\delta} \mathbin{\triangledown} \cup^\sharp_{(\ell',c,\ell)\in A} \mathsf{C}^\sharp[\![c]\!] \mathcal{X}_{\ell'}^{\sharp\,\delta}$.

By soundness of $\triangledown$, $\gamma(\mathcal{X}_\ell^{\sharp\delta}) \supseteq \gamma(\cup^\sharp_{(\ell',c,\ell)\in A} \mathsf{C}^\sharp[\![c]\!] \mathcal{X}_{\ell'}^{\sharp\,\delta})$,

and so we also have $\gamma(\mathcal{X}_\ell^{\sharp\delta}) \supseteq \cup_{(\ell',c,\ell)\in A} \mathsf{C}[\![c]\!] \gamma(\mathcal{X}_{\ell'}^{\sharp\,\delta})$.

We have proved that $\lambda\ell.\gamma(\mathcal{X}_\ell^{\sharp\delta})$ is a postfixpoint of the concrete equation system.
Hence, it is greater than its least solution.

Abstract analysis (proof)

Proof of termination:

Suppose that the iteration does not terminate in finite time.

Given a label $\ell \in L$, we denote by $i_\ell^1, \ldots, i_\ell^k, \ldots$ the increasing sequence of unstable indices, i.e., such that $\forall k$, $\mathcal{X}_\ell^{\sharp\, i_\ell^k+1} \neq \mathcal{X}_\ell^{\sharp\, i_\ell^k}$.
As the iteration is not stable, $\forall n$, $\exists \ell$, $\mathcal{X}_\ell^{\sharp\, n} \neq \mathcal{X}_\ell^{\sharp\, n+1}$.
Hence, the sequence $(i_\ell^k)_k$ is infinite for at least one $\ell \in L$.

We argue that $\exists \ell \in \mathcal{W}$ such that $(i_\ell^k)_k$ is infinite as, otherwise,
$N = \max \{\, i_\ell^k \mid \ell \in \mathcal{W} \,\} + |L|$ is finite and satisfies: $\forall n \geq N, \forall \ell \in L$, $\mathcal{X}_\ell^{\sharp\, n} = \mathcal{X}_\ell^{\sharp\, n+1}$,
contradicting our assumption.

For such a $\ell \in \mathcal{W}$, consider the subsequence $\mathcal{Y}_k^\sharp = \mathcal{X}_\ell^{\sharp\, i_\ell^k}$ comprised of the unstable iterates of $\mathcal{X}_\ell^\sharp$.
Then $\mathcal{Y}^{\sharp\, k+1} = \mathcal{Y}^{\sharp\, k} \triangledown \mathcal{Z}^{\sharp\, k}$ for some sequence $\mathcal{Z}^{\sharp\, k}$.
The subsequence is infinite and $\forall k$, $\mathcal{Y}^{\sharp\, k+1} \neq \mathcal{Y}^{\sharp\, k}$, which contradicts the definition of $\triangledown$.

Hence, the iteration must terminate in finite time.

Abstract analysis (cont.)

Backward refinement:

Given a forward analysis result $\mathcal{X}^\sharp$ and an abstract output $\mathcal{Y}^\sharp_x$.

$$\mathcal{Y}^{\sharp 0}_x \stackrel{\text{def}}{=} \mathcal{X}^\sharp_x \cap^\sharp \mathcal{Y}^\sharp_x$$

$$\mathcal{Y}^{\sharp 0}_{\ell \neq x} \stackrel{\text{def}}{=} \mathcal{X}^\sharp_\ell$$

$$\mathcal{Y}^{\sharp n+1}_\ell \stackrel{\text{def}}{=} \begin{cases} \mathcal{X}^\sharp_x \cap^\sharp \mathcal{Y}^\sharp_x & \text{if } \ell = x \\ \mathcal{X}^\sharp_\ell \cap^\sharp \bigcup^\sharp_{(\ell,c,\ell')\in A} \overleftarrow{C}^\sharp[\![c]\!] \mathcal{Y}^{\sharp n}_{\ell'} & \text{if } \ell \notin \mathcal{W},\ \ell \neq x \\ \mathcal{Y}^{\sharp n}_\ell \bigtriangleup (\mathcal{X}^\sharp_\ell \cap^\sharp \bigcup^\sharp_{(\ell,c,\ell')\in A} \overleftarrow{C}^\sharp[\![c]\!] \mathcal{Y}^{\sharp n}_{\ell'}) & \text{if } \ell \in \mathcal{W},\ \ell \neq x \end{cases}$$

$\bigtriangleup$ overapproximates $\cap$ while enforcing the convergence of decreasing iterations (the definition will be given later, on intervals)

Forward–backward analyses can be iterated [Bour93b].

Exact and best abstractions: Reminders

Galois connection: $(\mathcal{D},\subseteq) \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} (\mathcal{D}^\sharp,\sqsubseteq)$

- α, γ monotonic and $\forall \mathcal{X},\mathcal{Y}^\sharp$, $\alpha(\mathcal{X}) \sqsubseteq \mathcal{Y}^\sharp \iff \mathcal{X} \subseteq \gamma(\mathcal{Y}^\sharp)$
- $\Rightarrow$ elements $\mathcal{X}$ have a best abstraction: $\alpha(\mathcal{X})$
- $\Rightarrow$ operators F have a best abstraction: $F^\sharp = \alpha \circ F \circ \gamma$

Sometimes, no α exists:

- $\{\, \gamma(\mathcal{Y}^\sharp) \,|\, \mathcal{X} \subseteq \gamma(\mathcal{Y}^\sharp) \,\}$ has no greatest lower bound
- abstract elements with the same γ have no best representation

$\alpha \circ F \circ \gamma$ may still be defined for some F (partial α)

Concretization-based optimality:

- sound abstraction: $\gamma \circ F^\sharp \supseteq F \circ \gamma$
- exact abstraction: $\gamma \circ F^\sharp = F \circ \gamma$
- optimal abstraction: $\gamma(\mathcal{X}^\sharp)$ minimal in $\{\, \gamma(\mathcal{Y}^\sharp) \,|\, \mathcal{X} \subseteq \gamma(\mathcal{Y}^\sharp) \,\}$

Non-relational domains

Value abstract domain

Idea: start from an abstraction of values $\mathcal{P}(\mathbb{I})$

Numerical value abstract domain:

$\mathcal{B}^\sharp$	abstract values, machine-representable
γ_b: $\mathcal{B}^\sharp \to \mathcal{P}(\mathbb{I})$	concretization
$\sqsubseteq_b$	partial order
$\bot_b^\sharp$, $\top_b^\sharp$	represent $\emptyset$ and $\mathbb{I}$
$\cup_b^\sharp$, $\cap_b^\sharp$	abstractions of $\cup$ and $\cap$
$\triangledown_b$	extrapolation operator
α_b: $\mathcal{P}(\mathbb{I}) \to \mathcal{B}^\sharp$	abstraction (optional)

Derived abstract domain

$$\mathcal{D}^\sharp \stackrel{\text{def}}{=} (\mathbb{V} \to (\mathcal{B}^\sharp \setminus \{\,\bot_b^\sharp\,\})) \cup \{\,\bot^\sharp\,\}$$

- point-wise extension: $\mathcal{X}^\sharp \in \mathcal{D}^\sharp$ is a vector of elements in $\mathcal{B}^\sharp$
  (e.g. using arrays of size $|\mathbb{V}|$)
- smashed $\bot^\sharp$ (avoids redundant representations of $\emptyset$)

Definitions on $\mathcal{D}^\sharp$ derived from $\mathcal{B}^\sharp$:

$$\gamma(\mathcal{X}^\sharp) \stackrel{\text{def}}{=} \begin{cases} \emptyset & \text{if } \mathcal{X}^\sharp = \bot^\sharp \\ \{\,\rho \mid \forall \mathtt{V},\ \rho(\mathtt{V}) \in \gamma_b(\mathcal{X}^\sharp(\mathtt{V}))\,\} & \text{otherwise} \end{cases}$$

$$\alpha(\mathcal{X}) \stackrel{\text{def}}{=} \begin{cases} \bot^\sharp & \text{if } \mathcal{X} = \emptyset \\ \lambda \mathtt{V}.\alpha_b(\{\,\rho(\mathtt{V}) \mid \rho \in \mathcal{X}\,\}) & \text{otherwise} \end{cases}$$

$$\top^\sharp \stackrel{\text{def}}{=} \lambda \mathtt{V}.\top_b^\sharp$$

Derived abstract domain (cont.)

$$\mathcal{X}^\sharp \sqsubseteq \mathcal{Y}^\sharp \overset{\text{def}}{\iff} \mathcal{X}^\sharp = \bot^\sharp \vee (\mathcal{X}^\sharp, \mathcal{Y}^\sharp \neq \bot^\sharp \wedge \forall \mathtt{V}, \mathcal{X}^\sharp(\mathtt{V}) \sqsubseteq_b \mathcal{Y}^\sharp(\mathtt{V}))$$

$$\mathcal{X}^\sharp \cup^\sharp \mathcal{Y}^\sharp \overset{\text{def}}{=} \begin{cases} \mathcal{Y}^\sharp & \text{if } \mathcal{X}^\sharp = \bot^\sharp \\ \mathcal{X}^\sharp & \text{if } \mathcal{Y}^\sharp = \bot^\sharp \\ \lambda \mathtt{V}.\mathcal{X}^\sharp(\mathtt{V}) \cup_b^\sharp \mathcal{Y}^\sharp(\mathtt{V}) & \text{otherwise} \end{cases}$$

$$\mathcal{X}^\sharp \triangledown \mathcal{Y}^\sharp \overset{\text{def}}{=} \begin{cases} \mathcal{Y}^\sharp & \text{if } \mathcal{X}^\sharp = \bot^\sharp \\ \mathcal{X}^\sharp & \text{if } \mathcal{Y}^\sharp = \bot^\sharp \\ \lambda \mathtt{V}.\mathcal{X}^\sharp(\mathtt{V}) \triangledown_b \mathcal{Y}^\sharp(\mathtt{V}) & \text{otherwise} \end{cases}$$

$$\mathcal{X}^\sharp \cap^\sharp \mathcal{Y}^\sharp \overset{\text{def}}{=} \begin{cases} \bot^\sharp & \text{if } \mathcal{X}^\sharp = \bot^\sharp \text{ or } \mathcal{Y}^\sharp = \bot^\sharp \\ \bot^\sharp & \text{if } \exists \mathtt{V}, \mathcal{X}^\sharp(\mathtt{V}) \cap_b^\sharp \mathcal{Y}^\sharp(\mathtt{V}) = \bot_b^\sharp \\ \lambda \mathtt{V}.\mathcal{X}^\sharp(\mathtt{V}) \cap_b^\sharp \mathcal{Y}^\sharp(\mathtt{V}) & \text{otherwise} \end{cases}$$

We will see later how to derive $\mathsf{C}^\sharp[\![c]\!]$, $\overleftarrow{C}^\sharp[\![c]\!]$ using:

- abstract operators $+_b^\sharp$, ... for $\mathsf{C}^\sharp[\![\mathtt{V} := e]\!]$
- backward abstract operators $\overleftarrow{+}_b^\sharp$, ...
  for $\overleftarrow{C}^\sharp[\![\mathtt{V} := e]\!]$ and $\mathsf{C}^\sharp[\![e \bowtie 0]\!]^\sharp$

Cartesian abstraction

Non-relational domains "forget" all relationships between variables.

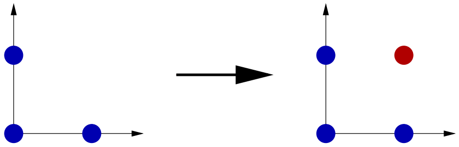
Cartesian abstraction:

Upper closure operator $\rho_c : \mathcal{P}(\mathbb{V} \to \mathbb{I}) \to \mathcal{P}(\mathbb{V} \to \mathbb{I})$

$$\rho_c(\mathcal{X}) \stackrel{\text{def}}{=} \{ \rho \in \mathbb{V} \to \mathbb{I} \mid \forall \mathtt{V} \in \mathbb{V},\ \exists \rho' \in \mathcal{X},\ \rho(\mathtt{V}) = \rho'(\mathtt{V}) \}$$

A domain is non relational if $\rho \circ \gamma = \gamma$,
i.e. it cannot distinguish between $\mathcal{X}$ and $\mathcal{X}'$ if $\rho_c(\mathcal{X}) = \rho_c(\mathcal{X}')$.

Example: $\rho_c(\{(X,Y) \mid X \in \{0,2\}, Y \in \{0,2\},\ X+Y \leq 2\}) = \{0,2\} \times \{0,2\}$.



Generic non-relational abstract assignments

Given: sound abstract versions in $\mathcal{B}^\sharp$ of all arithmetic operators:

$$\begin{array}{rccc}
[c,c']^\sharp_b : & \{\, x \mid c \leq x \leq c' \,\} & \subseteq & \gamma_b([c,c']^\sharp_b) \\
-^\sharp_b : & \{\, -x \mid x \in \gamma_b(\mathcal{X}^\sharp_b) \,\} & \subseteq & \gamma_b(-^\sharp_b\ \mathcal{X}^\sharp_b) \\
+^\sharp_b : & \{\, x+y \mid x \in \gamma_b(\mathcal{X}^\sharp_b), y \in \gamma_b(\mathcal{Y}^\sharp_b) \,\} & \subseteq & \gamma_b(\mathcal{X}^\sharp_b +^\sharp_b \mathcal{Y}^\sharp_b) \\
\vdots & & &
\end{array}$$

We can define:

- an abstract semantics of expressions: $\mathsf{E}^\sharp[\![\, e \,]\!] : \mathcal{D}^\sharp \to \mathcal{B}^\sharp$

$$\begin{array}{lcl}
\mathsf{E}^\sharp[\![\, e \,]\!]\, \bot^\sharp & \stackrel{\text{def}}{=} & \bot^\sharp_b \\
\text{if } \mathcal{X}^\sharp \neq \bot^\sharp : & & \\
\mathsf{E}^\sharp[\![\, [c,c'] \,]\!]\, \mathcal{X}^\sharp & \stackrel{\text{def}}{=} & [c,c']^\sharp_b \\
\mathsf{E}^\sharp[\![\, \mathtt{V} \,]\!]\, \mathcal{X}^\sharp & \stackrel{\text{def}}{=} & \mathcal{X}^\sharp(\mathtt{V}) \\
\mathsf{E}^\sharp[\![\, -e \,]\!]\, \mathcal{X}^\sharp & \stackrel{\text{def}}{=} & -^\sharp_b\ \mathsf{E}^\sharp[\![\, e \,]\!]\, \mathcal{X}^\sharp \\
\mathsf{E}^\sharp[\![\, e_1 + e_2 \,]\!]\, \mathcal{X}^\sharp & \stackrel{\text{def}}{=} & \mathsf{E}^\sharp[\![\, e_1 \,]\!]\, \mathcal{X}^\sharp +^\sharp_b \mathsf{E}^\sharp[\![\, e_2 \,]\!]\, \mathcal{X}^\sharp \\
\vdots & &
\end{array}$$

Generic non-relational abstract assignments (cont.)

We can then define:

- an abstract assignment:

$$C^\sharp[\![\, \mathtt{V} := e \,]\!]\, \mathcal{X}^\sharp \overset{\text{def}}{=} \begin{cases} \bot^\sharp & \text{if } \mathcal{V}_b^\sharp = \bot_b^\sharp \\ \mathcal{X}^\sharp[\mathtt{V} \mapsto \mathcal{V}_b^\sharp] & \text{otherwise} \end{cases}$$

where $\mathcal{V}_b^\sharp = E^\sharp[\![\, e \,]\!]\, \mathcal{X}^\sharp$.

Using a Galois connection (α_b, γ_b):

We can define best abstract arithmetic operators:

$$\begin{array}{rcl} [c, c']_b^\sharp & \overset{\text{def}}{=} & \alpha_b(\{\, x \mid c \leq x \leq c' \,\}) \\ -_b^\sharp\, \mathcal{X}_b^\sharp & \overset{\text{def}}{=} & \alpha_b(\{\, -x \mid x \in \gamma(\mathcal{X}_b^\sharp) \,\}) \\ \mathcal{X}_b^\sharp +_b^\sharp \mathcal{Y}_b^\sharp & \overset{\text{def}}{=} & \alpha_b(\{\, x+y \mid x \in \gamma(\mathcal{X}_b^\sharp),\ y \in \gamma(\mathcal{Y}_b^\sharp) \,\}) \\ \vdots & & \end{array}$$

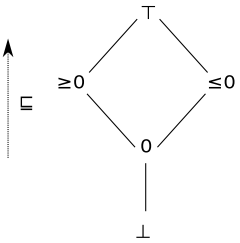
Note: in general, $E^\sharp[\![\, e \,]\!]$ is less precise than $\alpha_b \circ E[\![\, e \,]\!] \circ \gamma$

e.g. $e = \mathtt{V} - \mathtt{V}$ and $\gamma_b(\mathcal{X}^\sharp(\mathtt{V})) = [0, 1]$

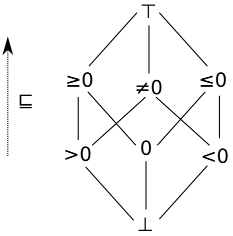
The sign domain

The sign lattices

Hasse diagram: for the lattice $(\mathcal{B}^\sharp, \sqsubseteq_b, \bot_b^\sharp, \top_b^\sharp)$



simple signs



extended signs

The extended sign domain is a refinement of the simple sign domain.

The diagram implicitly defines $\cup^\sharp$ and $\cap^\sharp$ as the least upper bound and greatest lower bound for $\sqsubseteq$.

Operations on simple signs

Abstraction α: there is a Galois connection between $\mathcal{B}^\sharp$ and $\mathcal{P}(\mathbb{I})$:

$$\alpha_b(S) \overset{\text{def}}{=} \begin{cases} \bot_b^\sharp & \text{if } S = \emptyset \\ 0 & \text{if } S = \{0\} \\ \geq 0 & \text{else if } \forall s \in S,\ s \geq 0 \\ \leq 0 & \text{else if } \forall s \in S,\ s \leq 0 \\ \top_b^\sharp & \text{otherwise} \end{cases}$$

Derived abstract arithmetic operators:

$$c_b^\sharp \overset{\text{def}}{=} \alpha_b(\{c\}) = \begin{cases} 0 & \text{if } c = 0 \\ \leq 0 & \text{if } c < 0 \\ \geq 0 & \text{if } c > 0 \end{cases}$$

$$X^\sharp +_b^\sharp Y^\sharp \overset{\text{def}}{=} \alpha_b(\{\, x + y \mid x \in \gamma_b(X^\sharp),\ y \in \gamma_b(Y^\sharp) \,\})$$

$$= \begin{cases} \bot_b^\sharp & \text{if } X \text{ or } Y^\sharp = \bot_b^\sharp \\ 0 & \text{if } X^\sharp = Y^\sharp = 0 \\ \leq 0 & \text{else if } X^\sharp \text{ and } Y^\sharp \in \{0, \leq 0\} \\ \geq 0 & \text{else if } X^\sharp \text{ and } Y^\sharp \in \{0, \geq 0\} \\ \top_b^\sharp & \text{otherwise} \end{cases}$$

Operations on simple signs (cont.)

Abstract test examples:

$$C^\sharp[\![\, \mathtt{X} \leq 0 \,]\!]\, \mathcal{X}^\sharp \overset{\text{def}}{=} \left(\begin{cases} \mathcal{X}^\sharp[\mathtt{X} \mapsto 0] & \text{if } \mathcal{X}^\sharp(\mathtt{X}) \in \{0, \geq 0\} \\ \mathcal{X}^\sharp[\mathtt{X} \mapsto \leq 0] & \text{if } \mathcal{X}^\sharp(\mathtt{X}) \in \{\top^\sharp_b, \leq 0\} \\ \bot^\sharp & \text{otherwise} \end{cases} \right)$$

$$C^\sharp[\![\, \mathtt{X} - c \leq 0 \,]\!]\, \mathcal{X}^\sharp \overset{\text{def}}{=} \left(\begin{cases} C^\sharp[\![\, \mathtt{X} \leq 0 \,]\!]\, \mathcal{X}^\sharp & \text{if } c \leq 0 \\ \mathcal{X}^\sharp & \text{otherwise} \end{cases} \right)$$

$$C^\sharp[\![\, \mathtt{X} - \mathtt{Y} \leq 0 \,]\!]\, \mathcal{X}^\sharp \overset{\text{def}}{=}$$
$$\begin{cases} C^\sharp[\![\, \mathtt{X} \leq 0 \,]\!]\, \mathcal{X}^\sharp & \text{if } \mathcal{X}^\sharp(\mathtt{Y}) \in \{0, \leq 0\} \\ \mathcal{X}^\sharp & \text{otherwise} \end{cases} \quad \cap^\sharp$$
$$\begin{cases} C^\sharp[\![\, \mathtt{Y} \geq 0 \,]\!]\, \mathcal{X}^\sharp & \text{if } \mathcal{X}^\sharp(\mathtt{X}) \in \{0, \geq 0\} \\ \mathcal{X}^\sharp & \text{otherwise} \end{cases}$$

Other cases: $C^\sharp[\![\, \mathit{expr} \bowtie 0 \,]\!]\, \mathcal{X}^\sharp \overset{\text{def}}{=} \mathcal{X}^\sharp$ is always a sound abstraction.

Simple sign analysis example

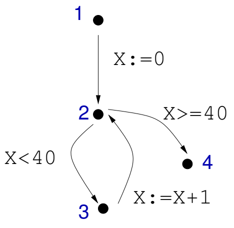
Example analysis using the simple sign domain:

```
X:=0;
while X<40 do
  X:=X+1
done
```

Program

$$\begin{cases} \mathcal{X}_2^{\sharp i+1} = \mathsf{C}^\sharp[\![\, \mathtt{X} := 0 \,]\!]\, \mathcal{X}_1^{\sharp i} \cup \\ \qquad\qquad \mathsf{C}^\sharp[\![\, \mathtt{X} := \mathtt{X}+1 \,]\!]\, \mathcal{X}_3^{\sharp i} \\ \mathcal{X}_3^{\sharp i+1} = \mathsf{C}^\sharp[\![\, \mathtt{X} < 40 \,]\!]\, \mathcal{X}_2^{\sharp i} \\ \mathcal{X}_4^{\sharp i+1} = \mathsf{C}^\sharp[\![\, \mathtt{X} \geq 40 \,]\!]\, \mathcal{X}_2^{\sharp i} \end{cases}$$

Iteration system



CFG

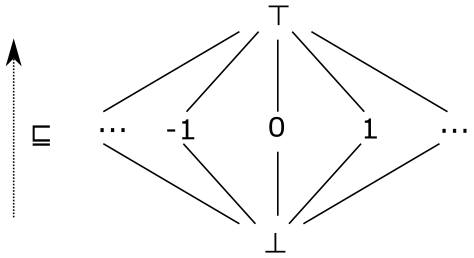
ℓ	$\mathcal{X}_\ell^{\sharp 0}$	$\mathcal{X}_\ell^{\sharp 1}$	$\mathcal{X}_\ell^{\sharp 2}$	$\mathcal{X}_\ell^{\sharp 3}$	$\mathcal{X}_\ell^{\sharp 4}$	$\mathcal{X}_\ell^{\sharp 5}$
1	$\top^\sharp$	$\top^\sharp$	$\top^\sharp$	$\top^\sharp$	$\top^\sharp$	$\top^\sharp$
2	$\perp^\sharp$	$\mathtt{X} = 0$	$\mathtt{X} = 0$	$\mathtt{X} \geq 0$	$\mathtt{X} \geq 0$	$\mathtt{X} \geq 0$
3	$\perp^\sharp$	$\perp^\sharp$	$\mathtt{X} = 0$	$\mathtt{X} = 0$	$\mathtt{X} \geq 0$	$\mathtt{X} \geq 0$
4	$\perp^\sharp$	$\perp^\sharp$	$\mathtt{X} = 0$	$\mathtt{X} = 0$	$\mathtt{X} \geq 0$	$\mathtt{X} \geq 0$

Iterations

The constant domain

The constant lattice

Hasse diagram:



$\mathcal{B}^\sharp = \mathbb{I} \cup \{\top_b^\sharp; \bot_b^\sharp\}$

The lattice is flat but infinite.

Operations on constants

Abstraction α: there is a Galois connection:

$$\alpha_b(S) \overset{\text{def}}{=} \begin{cases} \bot_b^\sharp & \text{if } S = \emptyset \\ c & \text{if } S = \{c\} \\ \top_b^\sharp & \text{otherwise} \end{cases}$$

Derived abstract arithmetic operators:

$$c_b^\sharp \overset{\text{def}}{=} c$$

$$(X^\sharp) +_b^\sharp (Y^\sharp) \overset{\text{def}}{=} \begin{cases} \bot_b^\sharp & \text{if } X^\sharp \text{ or } Y^\sharp = \bot_b^\sharp \\ \top_b^\sharp & \text{else if } X^\sharp \text{ or } Y^\sharp = \top_b^\sharp \\ X^\sharp + Y^\sharp & \text{otherwise} \end{cases}$$

$$(X^\sharp) \times_b^\sharp (Y^\sharp) \overset{\text{def}}{=} \begin{cases} \bot_b^\sharp & \text{if } X^\sharp \text{ or } Y^\sharp = \bot_b^\sharp \\ 0 & \text{else if } X^\sharp \text{ or } Y^\sharp = 0 \\ \top_b^\sharp & \text{else if } X^\sharp \text{ or } Y^\sharp = \top_b^\sharp \\ X^\sharp \times Y^\sharp & \text{otherwise} \end{cases}$$

Operations on constants (cont.)

Abstract test examples:

$$C^\sharp[\![\, X - c = 0 \,]\!]\, \mathcal{X}^\sharp \overset{\text{def}}{=} \begin{cases} \bot^\sharp & \text{if } \mathcal{X}^\sharp(X) \notin \{c, \top_b^\sharp\} \\ \mathcal{X}^\sharp[X \mapsto c] & \text{otherwise} \end{cases}$$

$$C^\sharp[\![\, X - Y - c = 0 \,]\!]\, \mathcal{X}^\sharp \overset{\text{def}}{=}$$

$$\left(\begin{cases} C^\sharp[\![\, X - (\mathcal{X}^\sharp(Y) + c) = 0 \,]\!]\, \mathcal{X}^\sharp & \text{if } \mathcal{X}^\sharp(Y) \notin \{\bot_b^\sharp, \top_b^\sharp\} \\ \mathcal{X}^\sharp & \text{otherwise} \end{cases} \right) \cap^\sharp$$
$$\left(\begin{cases} C^\sharp[\![\, Y - (\mathcal{X}^\sharp(X) - c) = 0 \,]\!]\, \mathcal{X}^\sharp & \text{if } \mathcal{X}^\sharp(X) \notin \{\bot_b^\sharp, \top_b^\sharp\} \\ \mathcal{X}^\sharp & \text{otherwise} \end{cases} \right)$$

Constant analysis example

$\mathcal{B}^\sharp$ has finite height, the $(\mathcal{X}_\ell^{\sharp i})$ converge in finite time.
(even though $\mathcal{B}^\sharp$ is infinite. . .)

Analysis example:

```
X:=0; Y:=10;
while X<100 do
  Y:=Y-3;
  X:=X+Y; ●
  Y:=Y+3
done
```

The constant analysis finds, at ●, the invariant: $\left\{ \begin{array}{l} \texttt{X} = \top_b^\sharp \\ \texttt{Y} = 7 \end{array} \right.$

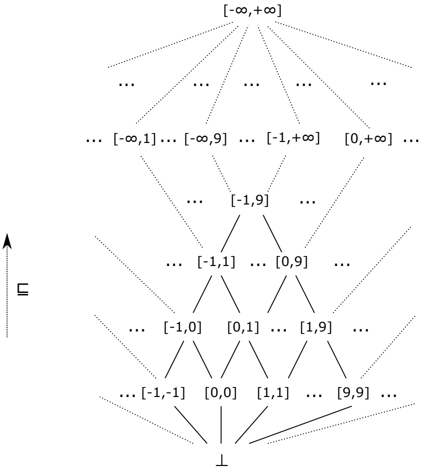
Note: the analysis can find constants that do not appear syntactically in the program.

The interval domain

The interval lattice

Introduced by [Cous76].

$$\mathcal{B}^\sharp \overset{\text{def}}{=} \{\, [a,b] \mid a \in \mathbb{I} \cup \{\,-\infty\,\},\ b \in \mathbb{I} \cup \{\,+\infty\,\},\ a \leq b \,\} \cup \{\, \perp_b^\sharp \,\}$$



Note: intervals are open at infinite bounds $+\infty$, $-\infty$.

The interval lattice (cont.)

Galois connection (α_b, γ_b):

$$\gamma_b([a,b]) \overset{\text{def}}{=} \{\, x \in \mathbb{I} \mid a \leq x \leq b \,\}$$

$$\alpha_b(\mathcal{X}) \overset{\text{def}}{=} \begin{cases} \bot^\sharp_b & \text{if } \mathcal{X} = \emptyset \\ [\min \mathcal{X}, \max \mathcal{X}] & \text{otherwise} \end{cases}$$

If $\mathbb{I} = \mathbb{Q}$, α_b is not always defined…

Partial order:

$$[a,b] \sqsubseteq_b [c,d] \overset{\text{def}}{\iff} a \geq c \text{ and } b \leq d$$

$$\top^\sharp_b \overset{\text{def}}{=}]-\infty, +\infty[$$

$$[a,b] \cup^\sharp_b [c,d] \overset{\text{def}}{=} [\min(a,c), \max(b,d)]$$

$$[a,b] \cap^\sharp_b [c,d] \overset{\text{def}}{=} \begin{cases} [\max(a,c), \min(b,d)] & \text{if } \max \leq \min \\ \bot^\sharp_b & \text{otherwise} \end{cases}$$

If $\mathbb{I} \neq \mathbb{Q}$, it is a complete lattice.

Interval abstract arithmetic operators

$$
\begin{array}{rcl}
[c,c']^\sharp_b & \overset{\text{def}}{=} & [c,c'] \\
-^\sharp_b\,[a,b] & \overset{\text{def}}{=} & [-b,-a] \\
[a,b] +^\sharp_b [c,d] & \overset{\text{def}}{=} & [a+c,b+d] \\
[a,b] -^\sharp_b [c,d] & \overset{\text{def}}{=} & [a-d,b-c] \\
[a,b] \times^\sharp_b [c,d] & \overset{\text{def}}{=} & [\min(ac,ad,bc,bd),\max(ac,ad,bc,bd)] \\
[a,b] /^\sharp_b [c,d] & \overset{\text{def}}{=} & \begin{cases} \bot^\sharp_b & \text{if } c=d=0 \\ [\min(a/c,a/d,b/c,b/d), & \text{else if } 0 \le c \\ \ \max(a/c,a/d,b/c,b/d)] & \\ [-b,-a]/^\sharp_b[-d,-c] & \text{else if } d \le 0 \\ ([a,b]/^\sharp_b[c,0]) \cup^\sharp_b ([a,b]/^\sharp_b[0,d]) & \text{otherwise} \end{cases}
\end{array}
$$

where $\left|\begin{array}{l} \pm\infty \times 0 = 0, \quad 0/0 = 0, \quad \forall x,\ x/\pm\infty = 0 \\ \forall x > 0,\ x/0 = +\infty, \quad \forall x < 0,\ x/0 = -\infty \end{array}\right.$

Operators are strict: $-^\sharp_b\bot^\sharp_b = \bot^\sharp_b$, $[a,b] +^\sharp_b \bot^\sharp_b = \bot^\sharp_b$, etc.

Exactness and optimality: Example proofs

Proof: exactness of $+^\sharp_b$

$$\begin{array}{ll} & \{\, x+y \mid x \in \gamma_b([a,b]),\ y \in \gamma_b([c,d]) \,\} \\ = & \{\, x+y \mid a \leq x \leq b \wedge c \leq y \leq d \,\} \\ = & \{\, z \mid a+c \leq z \leq b+d \,\} \\ = & \gamma_b([a+c, b+d]) \\ = & \gamma_b([a,b] \ +^\sharp_b\ [c,d]) \end{array}$$

Proof optimality of $\cup^\sharp_b$

$$\begin{array}{ll} & \alpha_b(\gamma_b([a,b]) \cup \gamma_b([c,d])) \\ = & \alpha_b(\{\, x \mid a \leq x \leq b \,\} \cup \{\, x \mid c \leq x \leq d \,\}) \\ = & \alpha_b(\{\, x \mid a \leq x \leq b \vee c \leq x \leq d \,\}) \\ = & [\min \{\, x \mid a \leq x \leq b \vee c \leq x \leq d \,\}, \max \{\, x \mid a \leq x \leq b \vee c \leq x \leq d \,\}] \\ = & [\min(a,c), \max(b,d)] \\ = & [a,b] \cup^\sharp_b [c,d] \end{array}$$

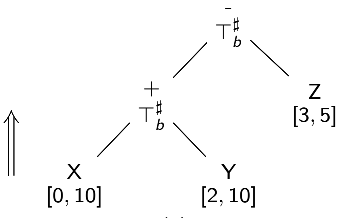
but $\cup^\sharp_b$ is not exact

...

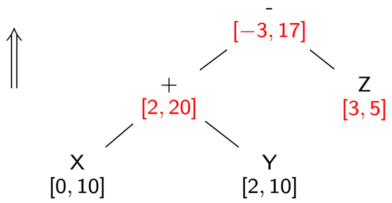
Generic interval abstract tests, step 1

Example: $C^\sharp[\![\, \mathtt{X} + \mathtt{Y} - \mathtt{Z} \leq 0 \,]\!]\, \mathcal{X}^\sharp$
with $\mathcal{X}^\sharp = \{\, \mathtt{X} \mapsto [0,10], \mathtt{Y} \mapsto [2,10], \mathtt{Z} \mapsto [3,5] \,\}$

First step: annotate the expression tree with intervals



(1)



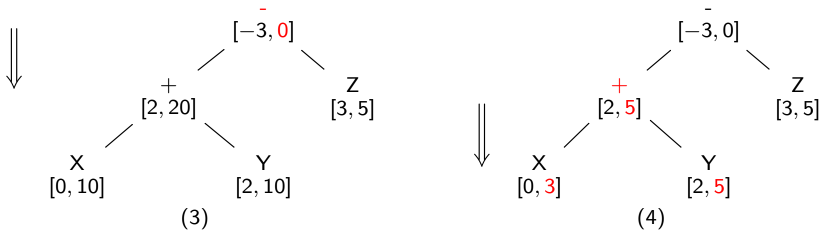
(2)

Bottom-up evaluation similar to interval expression evaluation using $+^\sharp_b$, $-^\sharp_b$, etc. but storing intervals at each node.

Generic interval abstract tests, step 2

Example: $C^\sharp[\![\, \texttt{X} + \texttt{Y} - \texttt{Z} \leq 0 \,]\!]\, \mathcal{X}^\sharp$
with $\mathcal{X}^\sharp = \{\, \texttt{X} \mapsto [0,10], \texttt{Y} \mapsto [2,10], \texttt{Z} \mapsto [3,5] \,\}$

Second step: top-down expression refinement.



- refine the root interval, knowing that the result should be negative;
- propagate refined intervals downwards;
- intervals at leaf variables provide new information to store into $\mathcal{X}^\sharp$.
  $\{\, X \mapsto [0,3], Y \mapsto [2,5], Z \mapsto [3,5] \,\}$

Backward arithmetic and comparison operators

In general, we need sound backward arithmetic and comparison operators that refine their arguments given a result.

Soundness condition: for $\overleftarrow{\leq 0}^\sharp_b$, $\overleftarrow{+}^\sharp_b$, $\overleftarrow{-}^\sharp_b$, …

$$\mathcal{X}^{\sharp\prime}_b = \overleftarrow{\leq 0}^\sharp_b(\mathcal{X}^\sharp_b) \implies$$
$$\{\, x \in \gamma_b(\mathcal{X}^\sharp_b) \,|\, x \leq 0 \,\} \subseteq \gamma_b(\mathcal{X}^{\sharp\prime}_b) \subseteq \gamma_b(\mathcal{X}^\sharp_b)$$

$$\mathcal{X}^{\sharp\prime}_b = \overleftarrow{-}^\sharp_b(\mathcal{X}^\sharp_b, \mathcal{R}^\sharp_b) \implies$$
$$\{\, x \,|\, x \in \gamma_b(\mathcal{X}^\sharp_b),\, -x \in \gamma_b(\mathcal{R}^\sharp_b) \,\} \subseteq \gamma_b(\mathcal{X}^{\sharp\prime}_b) \subseteq \gamma_b(\mathcal{X}^\sharp_b)$$

$$(\mathcal{X}^{\sharp\prime}_b, \mathcal{Y}^{\sharp\prime}_b) = \overleftarrow{+}^\sharp_b(\mathcal{X}^\sharp_b, \mathcal{Y}^\sharp_b, \mathcal{R}^\sharp_b) \implies$$
$$\{\, x \in \gamma_b(\mathcal{X}^\sharp_b) \,|\, \exists y \in \gamma_b(\mathcal{Y}^\sharp_b),\, x + y \in \gamma_b(\mathcal{R}^\sharp_b) \,\} \subseteq \gamma_b(\mathcal{X}^{\sharp\prime}_b) \subseteq \gamma_b(\mathcal{X}^\sharp_b)$$
$$\{\, y \in \gamma_b(\mathcal{Y}^\sharp_b) \,|\, \exists x \in \gamma_b(\mathcal{X}^\sharp_b),\, x + y \in \gamma_b(\mathcal{R}^\sharp_b) \,\} \subseteq \gamma_b(\mathcal{Y}^{\sharp\prime}_b) \subseteq \gamma_b(\mathcal{Y}^\sharp_b)$$
$$\vdots$$

Note: best backward operators can be designed with α_b:

e.g. for $\overleftarrow{+}^\sharp_b$: $\mathcal{X}^{\sharp\prime}_b = \alpha_b(\{\, x \in \gamma_b(\mathcal{X}^\sharp_b) \,|\, \exists y \in \gamma_b(\mathcal{Y}^\sharp_b),\, x + y \in \gamma_b(\mathcal{R}^\sharp_b) \,\})$

Generic backward operator construction

Synthesizing (non optimal) backward arithmetic operators from forward arithmetic operators.

$$\overleftarrow{\leq 0}^\sharp_b(\mathcal{X}^\sharp_b) \stackrel{\text{def}}{=} \mathcal{X}^\sharp_b \cap^\sharp_b\,]-\infty,0]^\sharp_b$$

$$\overleftarrow{-}^\sharp_b(\mathcal{X}^\sharp_b,\mathcal{R}^\sharp_b) \stackrel{\text{def}}{=} \mathcal{X}^\sharp_b \cap^\sharp_b (-^\sharp_b\, \mathcal{R}^\sharp_b)$$

$$\overleftarrow{+}^\sharp_b(\mathcal{X}^\sharp_b,\mathcal{Y}^\sharp_b,\mathcal{R}^\sharp_b) \stackrel{\text{def}}{=} (\mathcal{X}^\sharp_b \cap^\sharp_b (\mathcal{R}^\sharp_b -^\sharp_b \mathcal{Y}^\sharp_b),\ \mathcal{Y}^\sharp_b \cap^\sharp_b (\mathcal{R}^\sharp_b -^\sharp_b \mathcal{X}^\sharp_b))$$

$$\overleftarrow{-}^\sharp_b(\mathcal{X}^\sharp_b,\mathcal{Y}^\sharp_b,\mathcal{R}^\sharp_b) \stackrel{\text{def}}{=} (\mathcal{X}^\sharp_b \cap^\sharp_b (\mathcal{R}^\sharp_b +^\sharp_b \mathcal{Y}^\sharp_b),\ \mathcal{Y}^\sharp_b \cap^\sharp_b (\mathcal{X}^\sharp_b -^\sharp_b \mathcal{R}^\sharp_b))$$

$$\overleftarrow{\times}^\sharp_b(\mathcal{X}^\sharp_b,\mathcal{Y}^\sharp_b,\mathcal{R}^\sharp_b) \stackrel{\text{def}}{=} (\mathcal{X}^\sharp_b \cap^\sharp_b (\mathcal{R}^\sharp_b /^\sharp_b \mathcal{Y}^\sharp_b),\ \mathcal{Y}^\sharp_b \cap^\sharp_b (\mathcal{R}^\sharp_b /^\sharp_b \mathcal{X}^\sharp_b))$$

$$\overleftarrow{/}^\sharp_b(\mathcal{X}^\sharp_b,\mathcal{Y}^\sharp_b,\mathcal{R}^\sharp_b) \stackrel{\text{def}}{=} (\mathcal{X}^\sharp_b \cap^\sharp_b (\mathcal{S}^\sharp_b \times^\sharp_b \mathcal{Y}^\sharp_b),\ \mathcal{Y}^\sharp_b \cap^\sharp_b ((\mathcal{X}^\sharp_b /^\sharp_b \mathcal{S}^\sharp_b) \cup^\sharp_b [0,0]^\sharp_b))$$

$$\text{where } \mathcal{S}^\sharp_b = \begin{cases} \mathcal{R}^\sharp_b & \text{if } \mathbb{I} \neq \mathbb{Z} \\ \mathcal{R}^\sharp_b +^\sharp_b [-1,1]^\sharp_b & \text{if } \mathbb{I} = \mathbb{Z} \text{ (as / rounds)} \end{cases}$$

Note: $\overleftarrow{\diamond}^\sharp_b(\mathcal{X}^\sharp_b,\mathcal{Y}^\sharp_b,\mathcal{R}^\sharp_b) = (\mathcal{X}^\sharp_b,\mathcal{Y}^\sharp_b)$ is always sound (no refinement).

Interval backward operators

Applying the generic construction to the interval domain:

$$\overleftarrow{\leq 0}^\sharp_b([a,b]) \stackrel{\text{def}}{=} \begin{cases} [a, \min(b,0)] & \text{if } a \geq 0 \\ \bot^\sharp_b & \text{otherwise} \end{cases}$$

$$\overleftarrow{-}^\sharp_b([a,b],[r,s]) \stackrel{\text{def}}{=} [a,b] \cap^\sharp_b [-s,-r]$$

$$\overleftarrow{+}^\sharp_b([a,b],[c,d],[r,s]) \stackrel{\text{def}}{=} ([a,b] \cap^\sharp_b [r-d, s-c], \\ [c,d] \cap^\sharp_b [r-b, s-a])$$

...

Generic non-relational backward assignment

Abstract function: $\overleftarrow{C}^\sharp[\![\, \texttt{V} := e \,]\!]\, (\mathcal{X}^\sharp, \mathcal{R}^\sharp)$

over-approximates $\gamma(\mathcal{X}^\sharp) \cap \overleftarrow{C}[\![\, \texttt{V} := e \,]\!]\, \gamma(\mathcal{R}^\sharp)$ given:

- an abstract pre-condition $\mathcal{X}^\sharp$ to refine,
- according to a given abstract post-condition $\mathcal{R}^\sharp$.

Algorithm: similar to the abstract test

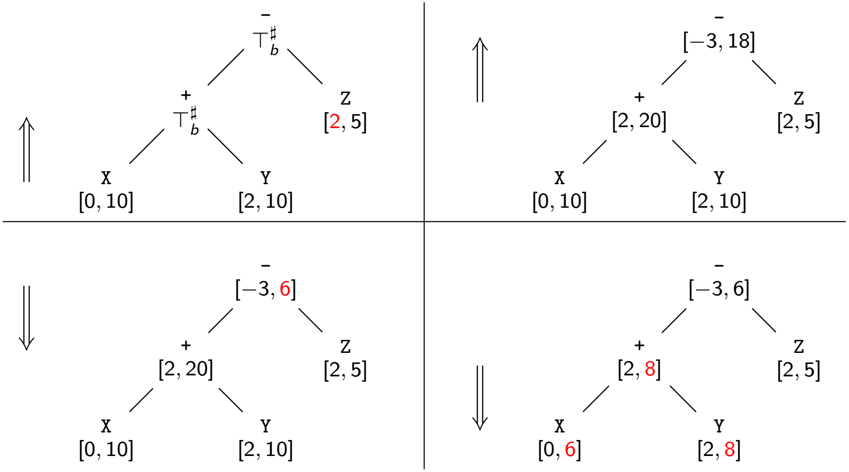
- annotate variable leaves based on $\mathcal{X}^\sharp \cap^\sharp (\mathcal{R}^\sharp[\texttt{V} \mapsto \top_b^\sharp])$;
- evaluate bottom-up using forward operators $\diamond_b^\sharp$;
- intersect the root with $\mathcal{R}^\sharp(\texttt{V})$;
- refine top-down using backward operators $\overleftarrow{\diamond}_b^\sharp$;
- return $\mathcal{X}^\sharp$ intersected with values at variable leaves.

Note:

- local iterations can also be used
- fallback: $\overleftarrow{C}^\sharp[\![\, \texttt{V} := e \,]\!]\, (\mathcal{X}^\sharp, \mathcal{R}^\sharp) = \mathcal{X}^\sharp \cap^\sharp (\mathcal{R}^\sharp[\texttt{V} \mapsto \top_b^\sharp])$

Interval backward assignment example

Example: $\overleftarrow{C}^\sharp[\![\, \texttt{X} := \texttt{X} + \texttt{Y} - \texttt{Z} \,]\!](\mathcal{X}^\sharp, \mathcal{R}^\sharp)$
with $\mathcal{X}^\sharp = \{\, \texttt{X} \mapsto [0,10], \texttt{Y} \mapsto [2,10], \texttt{Z} \mapsto [1,5] \,\}$
and $\mathcal{R}^\sharp = \{\, \texttt{X} \mapsto [-6,6], \texttt{Y} \mapsto [2,10], \texttt{Z} \mapsto [2,6] \,\}$



Interval widening

Widening on non-relational domains:

Given a value widening $\triangledown_b \colon \mathcal{B}^\sharp \times \mathcal{B}^\sharp \to \mathcal{B}^\sharp$,
we extend it point-wisely into a widening $\triangledown \colon \mathcal{D}^\sharp \times \mathcal{D}^\sharp \to \mathcal{D}^\sharp$:

$$\mathcal{X}^\sharp \triangledown \mathcal{Y}^\sharp \overset{\text{def}}{=} \lambda \mathtt{V}.(\mathcal{X}^\sharp(\mathtt{V}) \triangledown_b \mathcal{Y}^\sharp(\mathtt{V}))$$

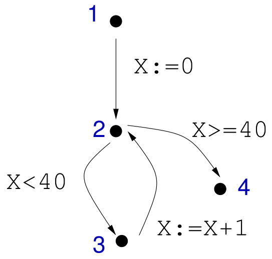
Interval widening example:

$$\begin{array}{llll}
\bot^\sharp & \triangledown_b & X^\sharp & \overset{\text{def}}{=} X^\sharp \\
[a,b] & \triangledown_b & [c,d] & \overset{\text{def}}{=} \left[\begin{cases} a & \text{if } a \leq c \\ -\infty & \text{otherwise} \end{cases}, \begin{cases} b & \text{if } b \geq d \\ +\infty & \text{otherwise} \end{cases} \right]
\end{array}$$

Unstable bounds are set to $\pm\infty$.

Analysis with widening example

Analysis example with $\mathcal{W} = \{2\}$



ℓ	$\mathcal{X}_\ell^{\sharp 0}$	$\mathcal{X}_\ell^{\sharp 1}$	$\mathcal{X}_\ell^{\sharp 2}$	$\mathcal{X}_\ell^{\sharp 3}$	$\mathcal{X}_\ell^{\sharp 4}$	$\mathcal{X}_\ell^{\sharp 5}$
1	$\top^\sharp$	$\top^\sharp$	$\top^\sharp$	$\top^\sharp$	$\top^\sharp$	$\top^\sharp$
2 $\triangledown$	$\bot^\sharp$	$= 0$	$= 0$	≥ 0	≥ 0	≥ 0
3	$\bot^\sharp$	$\bot^\sharp$	$= 0$	$= 0$	$\in [0,39]$	$\in [0,39]$
4	$\bot^\sharp$	$\bot^\sharp$	$\bot^\sharp$	$\bot^\sharp$	≥ 40	≥ 40

More precisely, at the widening point:

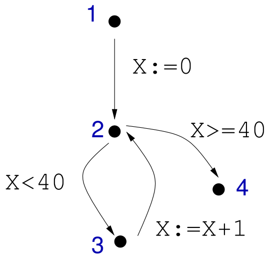
$$\begin{array}{lllllll}
\mathcal{X}_2^{\sharp 1} & = & \bot^\sharp \; \triangledown_b ([0,0] \cup_b^\sharp \bot^\sharp) & = & \bot^\sharp \; \triangledown_b [0,0] & = & [0,0] \\
\mathcal{X}_2^{\sharp 2} & = & [0,0] \; \triangledown_b ([0,0] \cup_b^\sharp \bot^\sharp) & = & [0,0] \; \triangledown_b [0,0] & = & [0,0] \\
\mathcal{X}_2^{\sharp 3} & = & [0,0] \; \triangledown_b ([0,0] \cup_b^\sharp [1,1]) & = & [0,0] \; \triangledown_b [0,1] & = & [0,+\infty[\\
\mathcal{X}_2^{\sharp 4} & = & [0,+\infty[\; \triangledown_b ([0,0] \cup_b^\sharp [1,40]) & = & [0,+\infty[\; \triangledown_b [0,40] & = & [0,+\infty[
\end{array}$$

Note that the most precise interval abstraction would be $\mathtt{X} \in [0,40]$ at 2, and $\mathtt{X} = 40$ at 4.

Influence of the widening point and iteration strategy

Changing $\mathcal{W}$ changes the analysis result

Example: The analysis is less precise for $\mathcal{W} = \{3\}$.



ℓ	$\mathcal{X}_\ell^{\sharp 1}$	$\mathcal{X}_\ell^{\sharp 2}$	$\mathcal{X}_\ell^{\sharp 3}$	$\mathcal{X}_\ell^{\sharp 4}$	$\mathcal{X}_\ell^{\sharp 5}$	$\mathcal{X}_\ell^{\sharp 6}$
1	$\top^\sharp$	$\top^\sharp$	$\top^\sharp$	$\top^\sharp$	$\top^\sharp$	$\top^\sharp$
2	$= 0$	$= 0$	$\in [0,1]$	$\in [0,1]$	≥ 0	≥ 0
3 ▽	$\perp^\sharp$	$= 0$	$= 0$	≥ 0	≥ 0	≥ 0
4	$\perp^\sharp$	$\perp^\sharp$	$\perp^\sharp$	$\perp^\sharp$	$\perp^\sharp$	≥ 40

Intuition: extrapolation to $+\infty$ is no longer contained by the tests.

Chaotic iterations

Changing the iteration order changes the analysis result in the presence of a widening.

Narrowing

Using a widening makes the analysis less precise.

Some precision can be retrieved by using a narrowing $\triangle$.

Definition: narrowing $\triangle$

Binary operator $\mathcal{D}^\sharp \times \mathcal{D}^\sharp \to \mathcal{D}^\sharp$ such that:

- $(\mathcal{X}^\sharp \cap^\sharp \mathcal{Y}^\sharp) \sqsubseteq (\mathcal{X}^\sharp \triangle \mathcal{Y}^\sharp) \sqsubseteq \mathcal{X}^\sharp$,
- for all sequences $(\mathcal{X}_i^\sharp)$, the decreasing sequence $(\mathcal{Y}_i^\sharp)$ defined by $\begin{cases} \mathcal{Y}_0^\sharp & \stackrel{\text{def}}{=} & \mathcal{X}_0^\sharp \\ \mathcal{Y}_{i+1}^\sharp & \stackrel{\text{def}}{=} & \mathcal{Y}_i^\sharp \triangle \mathcal{X}_{i+1}^\sharp \end{cases}$ is stationary.

This is not the dual of a widening!

Narrowing examples

Trivial narrowing:

$\mathcal{X}^\sharp \mathbin{\triangle} \mathcal{Y}^\sharp \stackrel{\text{def}}{=} \mathcal{X}^\sharp$ is a correct narrowing.

Finite-time intersection narrowing:

$$\mathcal{X}^{\sharp i} \mathbin{\triangle} \mathcal{Y}^\sharp \stackrel{\text{def}}{=} \begin{cases} \mathcal{X}^{\sharp i} \cap^\sharp \mathcal{Y}^\sharp & \text{if } i \leq N \\ \mathcal{X}^{\sharp i} & \text{if } i > N \end{cases}$$

Interval narrowing:

$$[a, b] \mathbin{\triangle_b} [c, d] \stackrel{\text{def}}{=} \left[\begin{cases} c & \text{if } a = -\infty \\ a & \text{otherwise} \end{cases}, \begin{cases} d & \text{if } b = +\infty \\ b & \text{otherwise} \end{cases} \right]$$

(refine only infinite bounds)

Point-wise extension to $\mathcal{D}^\sharp$: $\mathcal{X}^\sharp \mathbin{\triangle} \mathcal{Y}^\sharp \stackrel{\text{def}}{=} \lambda \mathtt{V}.(\mathcal{X}^\sharp(\mathtt{V}) \mathbin{\triangle_b} \mathcal{Y}^\sharp(\mathtt{V}))$

Iterations with narrowing

Let $\mathcal{X}_\ell^{\sharp\delta}$ be the result after widening stabilisation, *i.e.*:

$$\mathcal{X}_\ell^{\sharp\delta} \sqsupseteq \begin{cases} \top^\sharp & \text{if } \ell = e \\ \displaystyle\bigcup^\sharp_{(\ell',c,\ell)\in A} \mathsf{C}^\sharp[\![\, c \,]\!]\, \mathcal{X}_{\ell'}^{\sharp\delta} & \text{if } \ell \neq e \end{cases}$$

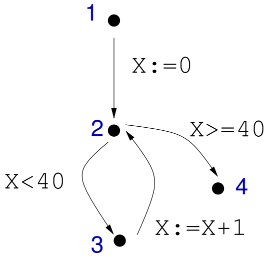
The following sequence is computed:

$$\mathcal{Y}_\ell^{\sharp 0} \overset{\text{def}}{=} \mathcal{X}_\ell^{\sharp\delta} \qquad \mathcal{Y}_\ell^{\sharp i+1} \overset{\text{def}}{=} \begin{cases} \top^\sharp & \text{if } \ell = e \\ \displaystyle\bigcup^\sharp_{(\ell',c,\ell)\in A} \mathsf{C}^\sharp[\![\, c \,]\!]\, \mathcal{Y}_{\ell'}^{\sharp i} & \text{if } \ell \notin \mathcal{W} \\ \mathcal{Y}_\ell^{\sharp i} \mathbin{\triangle} \displaystyle\bigcup^\sharp_{(\ell',c,\ell)\in A} \mathsf{C}^\sharp[\![\, c \,]\!]\, \mathcal{Y}_{\ell'}^{\sharp i} & \text{if } \ell \in \mathcal{W} \end{cases}$$

- the sequence $(\mathcal{Y}_\ell^{\sharp i})$ is decreasing and converges in finite time,
- all $(\mathcal{Y}_\ell^{\sharp i})$ are solutions of the abstract semantic system.

Analysis with narrowing example

Example with $\mathcal{W} = \{2\}$



ℓ	$\mathcal{Y}^{\sharp 0}_\ell$	$\mathcal{Y}^{\sharp 1}_\ell$	$\mathcal{Y}^{\sharp 2}_\ell$	$\mathcal{Y}^{\sharp 3}_\ell$
1	$\top^\sharp$	$\top^\sharp$	$\top^\sharp$	$\top^\sharp$
2 $\triangle$	≥ 0	$\in [0,40]$	$\in [0,40]$	$\in [0,40]$
3	$\in [0,39]$	$\in [0,39]$	$\in [0,39]$	$\in [0,39]$
4	≥ 40	≥ 40	$= 40$	$= 40$

Narrowing at 2 gives:

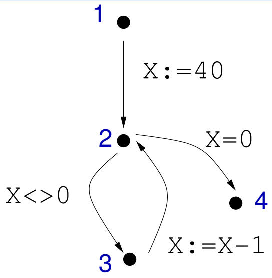
$$\begin{array}{lclclcl} \mathcal{Y}^{\sharp 1}_2 & = & [0,+\infty[\, \triangle_b ([0,0] \cup^\sharp_b [1,40]) & = & [0,+\infty[\, \triangle_b [0,40] & = & [0,40] \\ \mathcal{Y}^{\sharp 2}_2 & = & [0,40] \quad \triangle_b ([0,0] \cup^\sharp_b [1,40]) & = & [0,40] \quad \triangle_b [0,40] & = & [0,40] \end{array}$$

Then $\mathcal{Y}^{\sharp 2}_2 : \mathtt{X} \in [0,40]$ gives $\mathcal{Y}^{\sharp 3}_4 : \mathtt{X} = 40$.

We found the most precise invariants!

Improving the widening

Example of imprecise analysis



ℓ	intervals with $\triangledown_b$	extended signs	intervals with $\triangledown'_b$
1	$\top^\sharp$	$\top^\sharp$	$\top^\sharp$
2 $\triangledown$	$\mathtt{X} \leq 40$	$\mathtt{X} \geq 0$	$\mathtt{X} \in [0, 40]$
3	$\mathtt{X} \leq 40$	$\mathtt{X} > 0$	$\mathtt{X} \in [0, 40]$
4	$\mathtt{X} = 0$	$\mathtt{X} = 0$	$\mathtt{X} = 0$

The interval domain cannot prove that $\mathtt{X} \geq 0$ at 2, while the (less powerful) sign domain can!

Solution: improve the interval widening

$$[a,b] \ \triangledown'_b \ [c,d] \overset{\text{def}}{=} \left[\begin{cases} a & \text{if } a \leq c \\ 0 & \text{if } 0 \leq c < a \\ -\infty & \text{otherwise} \end{cases}, \begin{cases} b & \text{if } b \geq d \\ 0 & \text{if } 0 \geq b > d \\ +\infty & \text{otherwise} \end{cases} \right]$$

($\triangledown'_b$ checks the stability of 0)

Widening with thresholds

Analysis problem:

```
X:=0;
while ● 1=1 do
  if [0,1]=0 then
    X:=X+1;
    if X>40 then X:=0 fi
  fi
done
```

We wish to prove that $X \in [0,40]$ at ●.

- Widening at ● finds the loop invariant $\mathtt{X} \in [0,+\infty[$.
  $\mathcal{X}^\sharp_\bullet = [0,0] \mathbin{\triangledown_b} ([0,0] \cup^\sharp [0,1]) = [0,0] \mathbin{\triangledown_b} [0,1] = [0,+\infty[$
- Narrowing is unable to refine the invariant:
  $\mathcal{Y}^\sharp_\bullet = [0,+\infty[\mathbin{\triangle_b} ([0,0] \cup^\sharp [0,+\infty[) = [0,+\infty[$

  (the code that limits X is not executed at every loop iteration)

Widening with thresholds (cont.)

Solution:

Choose a finite set T of thresholds containing $+\infty$ and $-\infty$.

Definition: widening with thresholds $\triangledown_b^T$

$$[a,b] \triangledown_b^T [c,d] \overset{\text{def}}{=} \left[\begin{cases} a & \text{if } a \leq c \\ \max\{x \in T \mid x \leq c\} & \text{otherwise} \end{cases}, \begin{cases} b & \text{if } b \geq d \\ \min\{x \in T \mid x \geq d\} & \text{otherwise} \end{cases} \right]$$

The widening tests and stops at the first stable bound in T.

Widening with thresholds (cont.)

Applications:

- On the previous example, we find:
  $X \in [\,0,\ \min\{x \in T \mid x \geq 40\}\,]$.

- Useful when it is easy to find a 'good' set T.

  Example: array bound-checking

- Useful if an over-approximation of the bound is sufficient.

  Example: arithmetic overflow checking

Limitations: only works if some non-∞ bound in T is stable.

Example: with $T = \{\,5, 15\,\}$

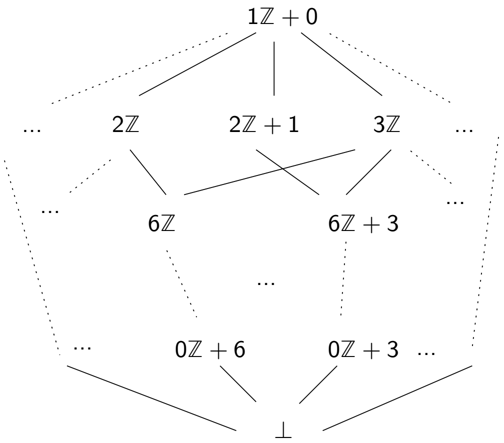
`while 1=1 do`
`  X:=X+1;`
`  if X>10 then X=0 fi`
`done`
	`while 1=1 do`
	`  X:=X+1;`
	`  if X<>10 then X=0 fi`
	`done`

15 is stable	no stable bound

The congruence domain

The congruence lattice

$$\mathcal{B}^\sharp \overset{\text{def}}{=} \{\, (a\mathbb{Z}+b) \mid a \in \mathbb{N},\ b \in \mathbb{Z}\,\} \cup \{\, \bot_b^\sharp \,\}$$



Introduced by Granger [Gran89].
We take $\mathbb{I} = \mathbb{Z}$.

The congruence lattice (cont.)

Concretization:

$$\gamma_b(\mathcal{X}_b^\sharp) \stackrel{\text{def}}{=} \begin{cases} \{\, ak+b \mid k \in \mathbb{Z} \,\} & \text{if } \mathcal{X}_b^\sharp = (a\mathbb{Z}+b) \\ \emptyset & \text{if } \mathcal{X}_b^\sharp = \bot_b^\sharp \end{cases}$$

Note that $\gamma(0\mathbb{Z}+b) = \{b\}$.
γ_b is not injective: $\gamma_b(2\mathbb{Z}+1) = \gamma_b(2\mathbb{Z}+3)$.

Definitions:

Given $x, x' \in \mathbb{Z}$, $y, y' \in \mathbb{N}$, we define:

- $y/y' \stackrel{\text{def}}{\iff} y$ divides y' $(\exists k \in \mathbb{N},\ y' = ky)$ (note that $\forall y$: $y/0$)
- $x \equiv x'\ [y] \stackrel{\text{def}}{\iff} y/|x-x'|$ (in particular, $x \equiv x'\ [0] \iff x = x'$)
- $\vee$ is the LCM, extended with $y \vee 0 \stackrel{\text{def}}{=} 0 \vee y \stackrel{\text{def}}{=} 0$
- $\wedge$ is the GCD, extended with $y \wedge 0 \stackrel{\text{def}}{=} 0 \wedge y \stackrel{\text{def}}{=} y$

$(\mathbb{N}, /, \vee, \wedge, 1, 0)$ is a complete distributive lattice.

Abstract congruence operators

Complete lattice structure on $\mathcal{B}^\sharp$:

- $(a\mathbb{Z}+b) \sqsubseteq_b (a'\mathbb{Z}+b') \overset{\text{def}}{\iff} a'/a$ and $b \equiv b'\ [a']$
- $\top_b^\sharp \overset{\text{def}}{=} (1\mathbb{Z}+0)$
- $(a\mathbb{Z}+b) \cup_b^\sharp (a'\mathbb{Z}+b') \overset{\text{def}}{=} (a \wedge a' \wedge |b-b'|)\mathbb{Z}+b$
- $(a\mathbb{Z}+b) \cap_b^\sharp (a'\mathbb{Z}+b') \overset{\text{def}}{=} \begin{cases} (a \vee a')\mathbb{Z}+b'' & \text{if } b \equiv b'\ [a \wedge a'] \\ \bot_b^\sharp & \text{otherwise} \end{cases}$

  b'' such that $b'' \equiv b\ [a \vee a'] \equiv b'\ [a \vee a']$ is given by Bezout's Theorem.

Galois connection: $\alpha_b(\mathcal{X}) = \bigcup_{c \in \mathcal{X}}{}_b^\sharp\ (0\mathbb{Z}+c)$

(up to equivalence $a\mathbb{Z}+b \equiv a'\mathbb{Z}+b' \overset{\text{def}}{\iff} a = a' \wedge b \equiv b'\ [a]$)

Abstract congruence operators (cont.)

Arithmetic operators:

$$[c, c']^\sharp_b \overset{\text{def}}{=} \begin{cases} 0\mathbb{Z} + c & \text{if } c = c' \\ \top^\sharp_b & \text{otherwise} \end{cases}$$

$$-^\sharp_b \,(a\mathbb{Z} + b) \overset{\text{def}}{=} a\mathbb{Z} + (-b)$$

$$(a\mathbb{Z} + b) +^\sharp_b (a'\mathbb{Z} + b') \overset{\text{def}}{=} (a \wedge a')\mathbb{Z} + (b + b')$$

$$(a\mathbb{Z} + b) -^\sharp_b (a'\mathbb{Z} + b') \overset{\text{def}}{=} (a \wedge a')\mathbb{Z} + (b - b')$$

$$(a\mathbb{Z} + b) \times^\sharp_b (a'\mathbb{Z} + b') \overset{\text{def}}{=} (aa' \wedge ab' \wedge a'b)\mathbb{Z} + bb'$$

$$(a\mathbb{Z} + b) /^\sharp_b (a'\mathbb{Z} + b') \overset{\text{def}}{=}$$

$$\begin{cases} \bot^\sharp_b & \text{if } a'\mathbb{Z} + b' = 0\mathbb{Z} + 0 \\ (a/|b'|)\mathbb{Z} + (b/b') & \text{if } a' = 0,\ b' \neq 0,\ b'|a,\ \text{ and } b'|b \\ \top^\sharp_b & \text{otherwise (not optimal)} \end{cases}$$

Abstract congruence operators (cont.)

Test operators:

$$\overleftarrow{\leq 0}^{\sharp}_{b}\,(a\mathbb{Z}+b) \overset{\text{def}}{=} \begin{cases} \bot^{\sharp}_{b} & \text{if } a=0,\ b>0 \\ a\mathbb{Z}+b & \text{otherwise} \end{cases}$$

$$\vdots$$

Note: better than the generic $\overleftarrow{\leq 0}^{\sharp}_{b}\,(\mathcal{X}^{\sharp}_{b}) \overset{\text{def}}{=} \mathcal{X}^{\sharp}_{b} \cap^{\sharp}_{b} \,]-\infty,0]^{\sharp}_{b} = \mathcal{X}^{\sharp}_{b}$

Extrapolation operators:

- no infinite increasing chain $\Longrightarrow$ no need for $\triangledown$
- infinite decreasing chains $\Longrightarrow$ $\vartriangle$ needed

$$(a\mathbb{Z}+b)\ \vartriangle_{b}\ (a'\mathbb{Z}+b') \overset{\text{def}}{=} \begin{cases} a'\mathbb{Z}+b' & \text{if } a=1 \\ a\mathbb{Z}+b & \text{otherwise} \end{cases}$$

Note: $\mathcal{X}^{\sharp} \vartriangle \mathcal{Y}^{\sharp} \overset{\text{def}}{=} \mathcal{X}^{\sharp}$ is always a narrowing.

Reduced products of domains

Non-reduced product of domains

Product representation:

Cartesian product $\mathcal{D}^\sharp_{1\times 2}$ of $\mathcal{D}^\sharp_1$ and $\mathcal{D}^\sharp_2$:

- $\mathcal{D}^\sharp_{1\times 2} \overset{\text{def}}{=} \mathcal{D}^\sharp_1 \times \mathcal{D}^\sharp_2$
- $\gamma_{1\times 2}(\mathcal{X}^\sharp_1, \mathcal{X}^\sharp_2) \overset{\text{def}}{=} \gamma_1(\mathcal{X}^\sharp_1) \cap \gamma_2(\mathcal{X}^\sharp_2)$
- $\alpha_{1\times 2}(\mathcal{X}) \overset{\text{def}}{=} (\alpha_1(\mathcal{X}), \alpha_2(\mathcal{X}))$
- $(\mathcal{X}^\sharp_1, \mathcal{X}^\sharp_2) \sqsubseteq_{1\times 2} (\mathcal{Y}^\sharp_1, \mathcal{Y}^\sharp_2) \overset{\text{def}}{\iff} \mathcal{X}^\sharp_1 \sqsubseteq_1 \mathcal{Y}^\sharp_1$ and $\mathcal{X}^\sharp_2 \sqsubseteq_2 \mathcal{Y}^\sharp_2$

Abstract operators: performed in parallel on both components:

- $(\mathcal{X}^\sharp_1, \mathcal{X}^\sharp_2) \cup^\sharp_{1\times 2} (\mathcal{Y}^\sharp_1, \mathcal{Y}^\sharp_2) \overset{\text{def}}{=} (\mathcal{X}^\sharp_1 \cup^\sharp_1 \mathcal{Y}^\sharp_1, \mathcal{X}^\sharp_2 \cup^\sharp_2 \mathcal{Y}^\sharp_2)$
- $(\mathcal{X}^\sharp_1, \mathcal{X}^\sharp_2) \triangledown_{1\times 2} (\mathcal{Y}^\sharp_1, \mathcal{Y}^\sharp_2) \overset{\text{def}}{=} (\mathcal{X}^\sharp_1 \triangledown_1 \mathcal{Y}^\sharp_1, \mathcal{X}^\sharp_2 \triangledown_2 \mathcal{Y}^\sharp_2)$
- $\mathsf{C}^\sharp[\![c]\!]_{1\times 2}(\mathcal{X}^\sharp_1, \mathcal{X}^\sharp_2) \overset{\text{def}}{=} (\mathsf{C}^\sharp[\![c]\!]_1(\mathcal{X}^\sharp_1), \mathsf{C}^\sharp[\![c]\!]_2(\mathcal{X}^\sharp_2))$

Non-reduced product example

The product analysis is no more precise than two separate analyses.

Example: interval–congruence product:

```
 X:=1;
 while X-10<=0 do
   X:=X+2
 done;
●if X-12>=0 then♦ X:=0★ fi
```

	interval	congruence	product
●	$X \in [11, 12]$	$X \equiv 1\ [2]$	$X = 11$
♦	$X = 12$	$X \equiv 1\ [2]$	$\emptyset$
★	$X = 0$	$X = 0$	$X = 0$

We cannot prove that the `if` branch is never taken!

Fully-reduced product

Definition:

Given the Galois connections (α_1, γ_1) and (α_2, γ_2) on $\mathcal{D}_1^\sharp$ and $\mathcal{D}_2^\sharp$ we define the reduction operator ρ as:

$$\rho : \mathcal{D}_{1\times 2}^\sharp \to \mathcal{D}_{1\times 2}^\sharp$$
$$\rho(\mathcal{X}_1^\sharp, \mathcal{X}_2^\sharp) \stackrel{\text{def}}{=} (\alpha_1(\gamma_1(\mathcal{X}_1^\sharp) \cap \gamma_2(\mathcal{X}_2^\sharp)), \alpha_2(\gamma_1(\mathcal{X}_1^\sharp) \cap \gamma_2(\mathcal{X}_2^\sharp)))$$

ρ propagates information between domains.

Application:

We can reduce the result of each abstract operator, except $\triangledown$:

- $(\mathcal{X}_1^\sharp, \mathcal{X}_2^\sharp) \cup_{1\times 2}^\sharp (\mathcal{Y}_1^\sharp, \mathcal{Y}_2^\sharp) \stackrel{\text{def}}{=} \rho(\mathcal{X}_1^\sharp \cup_1^\sharp \mathcal{Y}_1^\sharp, \mathcal{X}_2^\sharp \cup_2^\sharp \mathcal{Y}_2^\sharp)$,
- $C^\sharp[\![c]\!]_{1\times 2}(\mathcal{X}_1^\sharp, \mathcal{X}_2^\sharp) \stackrel{\text{def}}{=} \rho(C^\sharp[\![c]\!]_1(\mathcal{X}_1^\sharp), C^\sharp[\![c]\!]_2(\mathcal{X}_2^\sharp))$.

We refrain from reducing after a widening $\triangledown$,
this may jeopardize the convergence (octagon domain example).

Fully-reduced product example

Reduction example: between the interval and congruence domains:

Noting: $$a' \overset{\text{def}}{=} \min \{ x \ge a \mid x \equiv d\ [c] \}$$
$$b' \overset{\text{def}}{=} \max \{ x \le b \mid x \equiv d\ [c] \}$$

We get:

$$\rho_b([a,b], c\mathbb{Z}+d) \overset{\text{def}}{=} \begin{cases} (\bot_b^\sharp, \bot_b^\sharp) & \text{if } a' > b' \\ ([a',a'], 0\mathbb{Z}+a') & \text{if } a' = b' \\ ([a',b'], c\mathbb{Z}+d) & \text{if } a' < b' \end{cases}$$

extended point-wisely to ρ on $\mathcal{D}^\sharp$.

Application:

- $\rho_b([10,11],\ 2\mathbb{Z}+1) = ([11,11],\ 0\mathbb{Z}+11)$
  (proves that the branch is never taken on our example)
- $\rho_b([1,3],\ 4\mathbb{Z}) = (\bot_b^\sharp,\ \bot_b^\sharp)$

Partially-reduced product

Definition: of a partial reduction:

any function $\rho : \mathcal{D}^\sharp_{1\times 2} \to \mathcal{D}^\sharp_{1\times 2}$ such that:

$$(\mathcal{Y}^\sharp_1, \mathcal{Y}^\sharp_2) = \rho(\mathcal{X}^\sharp_1, \mathcal{X}^\sharp_2) \Longrightarrow \begin{cases} \gamma_{1\times 2}(\mathcal{Y}^\sharp_1, \mathcal{Y}^\sharp_2) = \gamma_{1\times 2}(\mathcal{X}^\sharp_1, \mathcal{X}^\sharp_2) \\ \gamma_1(\mathcal{Y}^\sharp_1) \subseteq \gamma_1(\mathcal{X}^\sharp_1) \\ \gamma_2(\mathcal{Y}^\sharp_2) \subseteq \gamma_2(\mathcal{X}^\sharp_2) \end{cases}$$

Useful when:

- there is no Galois connection, or
- a full reduction exists but is expensive to compute.

Partial reduction example:

$$\rho(\mathcal{X}^\sharp_1, \mathcal{X}^\sharp_2) \stackrel{\text{def}}{=} \begin{cases} (\bot^\sharp, \bot^\sharp) & \text{if } \mathcal{X}^\sharp_1 = \bot^\sharp \text{ or } \mathcal{X}^\sharp_2 = \bot^\sharp \\ (\mathcal{X}^\sharp_1, \mathcal{X}^\sharp_2) & \text{otherwise} \end{cases}$$

(works on all domains)

For more complex examples, see [Blan03].

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