

# Order Theory

MPRI 2–6: Abstract Interpretation,  
application to verification and static analysis

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Course 1  
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- Partially ordered structures
  - (complete) partial orders
  - (complete) lattices
- Fixpoints
- Abstractions
  - Galois connections, upper closure operators  
(first-class citizens)
  - Concretization-only framework
  - Operator abstraction
  - Fixpoint abstraction

# Partial orders

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# Partial orders

Given a set  $X$ , a relation  $\sqsubseteq \in X \times X$  is a **partial order** if it is:

- 1 reflexive:  $\forall x \in X, x \sqsubseteq x$
- 2 antisymmetric:  $\forall x, y \in X, (x \sqsubseteq y) \wedge (y \sqsubseteq x) \implies x = y$
- 3 transitive:  $\forall x, y, z \in X, (x \sqsubseteq y) \wedge (y \sqsubseteq z) \implies x \sqsubseteq z$

$(X, \sqsubseteq)$  is a **poset** (partially ordered set).

If we drop antisymmetry, we have a **preorder** instead.

# Examples: partial orders

## Partial orders:

- $(\mathbb{Z}, \leq)$   
(completely ordered)
- $(\mathcal{P}(X), \subseteq)$   
(not completely ordered:  $\{1\} \not\subseteq \{2\}$ ,  $\{2\} \not\subseteq \{1\}$ )
- $(S, =)$  is a poset for any  $S$
- $(\mathbb{Z}^2, \sqsubseteq)$ , where  $(a, b) \sqsubseteq (a', b') \iff (a \geq a') \wedge (b \leq b')$   
(ordering of interval bounds that implies inclusion)

# Examples: preorders

## Preorders:

- $(\mathcal{P}(X), \sqsubseteq)$ , where  $a \sqsubseteq b \iff |a| \leq |b|$

(ordered by cardinal)

- $(\mathbb{Z}^2, \sqsubseteq)$ , where  $(a, b) \sqsubseteq (a', b') \iff \{x \mid a \leq x \leq b\} \subseteq \{x \mid a' \leq x \leq b'\}$

(inclusion of intervals represented by pairs of bounds)

not antisymmetric:  $[1, 0] \neq [2, 0]$  but  $[1, 0] \sqsubseteq [2, 0] \sqsubseteq [1, 0]$

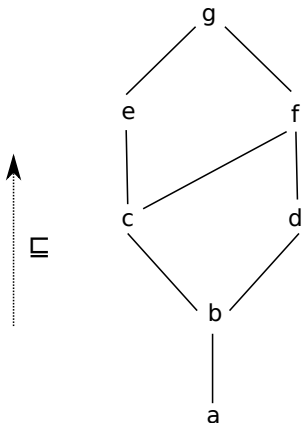
## Equivalence: $\equiv$

$$X \equiv Y \iff (X \sqsubseteq Y) \wedge (Y \sqsubseteq X)$$

We obtain a partial order by **quotienting** by  $\equiv$ .

# Examples of posets (cont.)

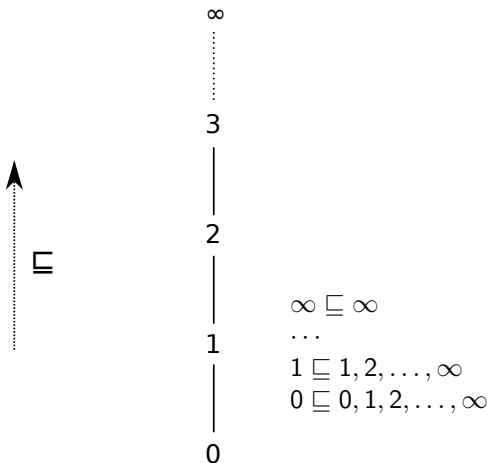
- Given by a **Hasse diagram**, e.g.:



$$\begin{aligned}
 g &\subseteq g \\
 f &\subseteq f, g \\
 e &\subseteq e, g \\
 d &\subseteq d, f, g \\
 c &\subseteq c, e, f, g \\
 b &\subseteq b, c, d, e, f, g \\
 a &\subseteq a, b, c, d, e, f, g
 \end{aligned}$$

# Examples of posets (cont.)

- Infinite Hasse diagram for  $(\mathbb{N} \cup \{\infty\}, \leq)$ :





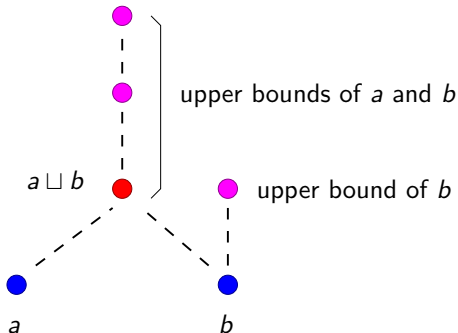
# Use of posets (informally)

Posets are a very useful notion to discuss:

- **logic**: formulas ordered by implication  $\implies$
- **program verification**: program semantics  $\sqsubseteq$  specification  
(e.g.: behaviors of program  $\subseteq$  accepted behaviors)
- **approximation**:  $\sqsubseteq$  is an information order  
(“ $a \sqsubseteq b$ ” means: “ $a$  carries more information than  $b$ ”)
- **iteration**: fixpoint computation  
(e.g., a computation is directed, with a limit:  $X_1 \sqsubseteq X_2 \sqsubseteq \dots \sqsubseteq X_n$ )

# (Least) Upper bounds

- $c$  is an **upper bound** of  $a$  and  $b$  if:  $a \sqsubseteq c$  and  $b \sqsubseteq c$
- $c$  is a **least upper bound** (**lub** or **join**) of  $a$  and  $b$  if
  - $c$  is an upper bound of  $a$  and  $b$
  - for every upper bound  $d$  of  $a$  and  $b$ ,  $c \sqsubseteq d$



# (Least) Upper bounds

If it exists, the lub of  $a$  and  $b$  is **unique**, and denoted as  $a \sqcup b$ .

(proof: assume that  $c$  and  $d$  are both lubs of  $a$  and  $b$ ; by definition of lubs,  $c \sqsubseteq d$  and  $d \sqsubseteq c$ ; by antisymmetry of  $\sqsubseteq$ ,  $c = d$ )

Generalized to upper bounds of arbitrary (even infinite) sets  $\sqcup Y$ ,  $Y \subseteq X$

(well-defined, as  $\sqcup$  is commutative and associative).

Similarly, we define **greatest lower bounds** (**glb**, **meet**)  $a \sqcap b$ ,  $\sqcap Y$ .

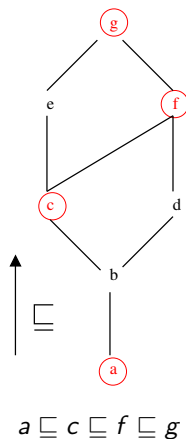
$(a \sqcap b \sqsubseteq a) \wedge (a \sqcap b \sqsubseteq b)$  and  $\forall c, (c \sqsubseteq a) \wedge (c \sqsubseteq b) \implies (c \sqsubseteq a \sqcap b)$

Note: not all posets have lubs, glbs

(e.g.:  $a \sqcup b$  not defined on  $(\{a, b\}, =)$ )

# Chains

$C \subseteq X$  is a **chain** in  $(X, \sqsubseteq)$  if it is totally ordered by  $\sqsubseteq$ :  
 $\forall x, y \in C, (x \sqsubseteq y) \vee (y \sqsubseteq x)$ .



# Complete partial orders (CPO)

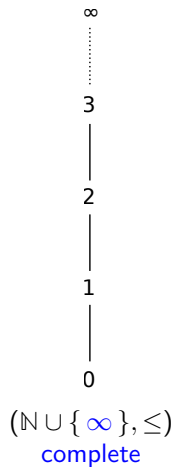
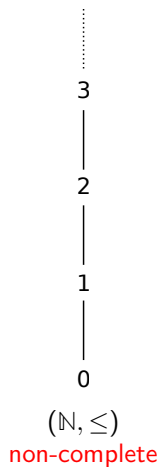
A poset  $(X, \sqsubseteq)$  is a **complete** partial order (**CPO**) if every chain  $C$  (including  $\emptyset$ ) has a least upper bound  $\sqcup C$ .

A CPO has a **least element**  $\sqcup \emptyset$ , denoted  $\perp$ .

## Examples, Counter-examples:

- $(\mathbb{N}, \leq)$  is not complete, but  $(\mathbb{N} \cup \{\infty\}, \leq)$  is complete.
- $(\{x \in \mathbb{Q} \mid 0 \leq x \leq 1\}, \leq)$  is not complete, but  $(\{x \in \mathbb{R} \mid 0 \leq x \leq 1\}, \leq)$  is complete.
- $(\mathcal{P}(Y), \subseteq)$  is complete for any  $Y$ .
- $(X, \sqsubseteq)$  is complete if  $X$  is finite.

# Complete partial order examples



# Lattices

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# Lattices

A **lattice**  $(X, \sqsubseteq, \sqcup, \sqcap)$  is a poset with

- 1 a lub  $a \sqcup b$  for every pair of elements  $a$  and  $b$ ;
- 2 a glb  $a \sqcap b$  for every pair of elements  $a$  and  $b$ .

Examples:

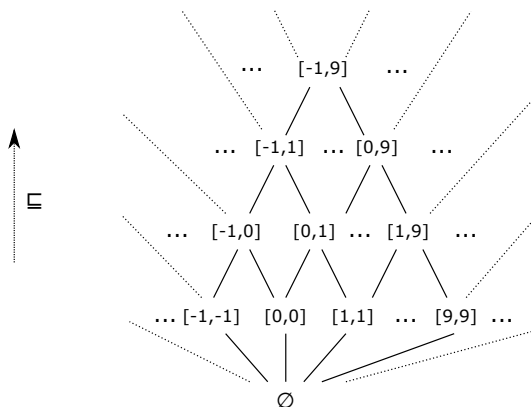
- integers  $(\mathbb{Z}, \leq, \max, \min)$
- integer intervals (next slide)
- divisibility (in two slides)

If we drop one condition, we have a (join or meet) **semilattice**.

Reference on lattices: Birkhoff [\[Birk76\]](#).



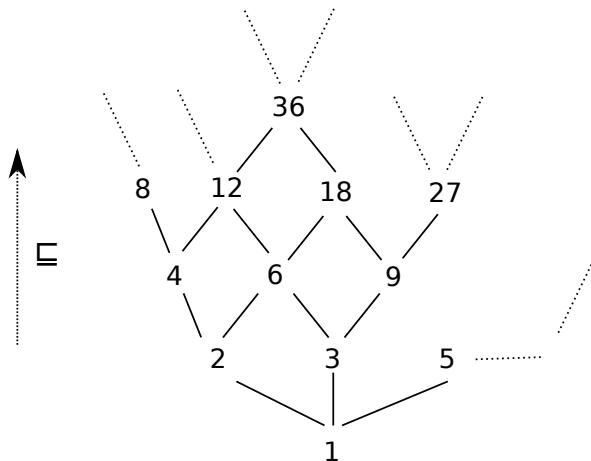
# Example: the interval lattice



Integer intervals:  $(\{ [a, b] \mid a, b \in \mathbb{Z}, a \leq b \} \cup \{ \emptyset \}, \subseteq, \sqcup, \cap)$

where  $[a, b] \sqcup [a', b'] \stackrel{\text{def}}{=} [\min(a, a'), \max(b, b')]$ .

# Example: the divisibility lattice



**Divisibility** ( $\mathbb{N}^*$ ,  $|$ , lcm, gcd) where  $x|y \stackrel{\text{def}}{\iff} \exists k \in \mathbb{N}, kx = y$

# Example: the divisibility lattice (cont.)

Let  $P \stackrel{\text{def}}{=} \{p_1, p_2, \dots\}$  be the (infinite) set of **prime numbers**.

We have a correspondence  $\iota$  between  $\mathbb{N}^*$  and  $P \rightarrow \mathbb{N}$ :

- $\alpha = \iota(x)$  is the (unique) decomposition of  $x$  into prime factors
- $\iota^{-1}(\alpha) \stackrel{\text{def}}{=} \prod_{a \in P} a^{\alpha(a)} = x$
- $\iota$  is **one-to-one** on functions  $P \rightarrow \mathbb{N}$  with finite support  
( $\alpha(a) = 0$  except for finitely many factors  $a$ )

We have a correspondence between  $(\mathbb{N}^*, |, \text{lcm}, \text{gcd})$   
and  $(\mathbb{N}, \leq, \text{max}, \text{min})$ .

Assume that  $\alpha = \iota(x)$  and  $\beta = \iota(y)$  are the decompositions of  $x$  and  $y$ , then:

- $\prod_{a \in P} a^{\max(\alpha(a), \beta(a))} = \text{lcm}(\prod_{a \in P} a^{\alpha(a)}, \prod_{a \in P} a^{\beta(a)}) = \text{lcm}(x, y)$
- $\prod_{a \in P} a^{\min(\alpha(a), \beta(a))} = \text{gcd}(\prod_{a \in P} a^{\alpha(a)}, \prod_{a \in P} a^{\beta(a)}) = \text{gcd}(x, y)$
- $(\forall a: \alpha(a) \leq \beta(a)) \iff (\prod_{a \in P} a^{\alpha(a)} \mid \prod_{a \in P} a^{\beta(a)}) \iff x \mid y$

# Complete lattices

A **complete lattice**  $(X, \sqsubseteq, \sqcup, \sqcap, \perp, \top)$  is a poset with

- 1 a lub  $\sqcup S$  for every set  $S \subseteq X$
- 2 a glb  $\sqcap S$  for every set  $S \subseteq X$
- 3 a least element  $\perp$
- 4 a greatest element  $\top$

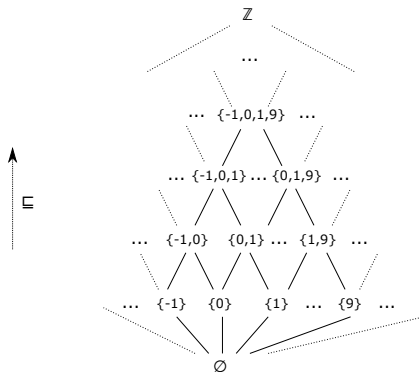
## Notes:

- 1 implies 2 as  $\sqcap S = \sqcup \{y \mid \forall x \in S, y \sqsubseteq x\}$   
(and 2 implies 1 as well),
- 1 and 2 imply 3 and 4:  $\perp = \sqcup \emptyset = \sqcap X, \top = \sqcap \emptyset = \sqcup X$ ,
- a complete lattice is also a CPO.

# Complete lattice examples

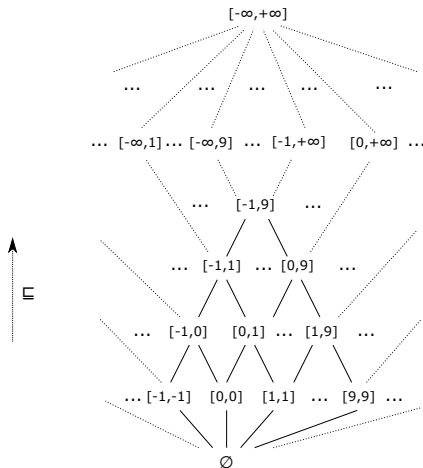
- **real segment**  $[0, 1]$ :  $(\{x \in \mathbb{R} \mid 0 \leq x \leq 1\}, \leq, \max, \min, 0, 1)$
- **powersets**  $(\mathcal{P}(S), \subseteq, \cup, \cap, \emptyset, S)$   
(next slide)
- **any finite lattice**  
( $\sqcup Y$  and  $\sqcap Y$  for finite  $Y \subseteq X$  are always defined)
- **integer intervals** with finite and **infinite** bounds:  
 $(\{[a, b] \mid a \in \mathbb{Z} \cup \{-\infty\}, b \in \mathbb{Z} \cup \{+\infty\}, a \leq b\} \cup \{\emptyset\},$   
 $\subseteq, \sqcup, \cap, \emptyset, [-\infty, +\infty])$   
 with  $\sqcup_{i \in I} [a_i, b_i] \stackrel{\text{def}}{=} [\min_{i \in I} a_i, \max_{i \in I} b_i]$ .  
 (in two slides)

# Example: the powerset complete lattice



Example:  $(\mathcal{P}(\mathbb{Z}), \subseteq, \cup, \cap, \emptyset, \mathbb{Z})$

# Example: the intervals complete lattice



The **integer intervals** with finite and **infinite** bounds:

$$(\{ [a, b] \mid a \in \mathbb{Z} \cup \{-\infty\}, b \in \mathbb{Z} \cup \{+\infty\}, a \leq b \} \cup \{\emptyset\}, \\ \subseteq, \sqcup, \cap, \emptyset, [-\infty, +\infty])$$

# Derivation

Given a (complete) lattice or partial order  $(X, \sqsubseteq, \sqcup, \sqcap, \perp, \top)$  we can derive new (complete) lattices or partial orders by:

## ■ duality

$(X, \supseteq, \sqcap, \sqcup, \top, \perp)$

- $\sqsubseteq$  is reversed
- $\sqcup$  and  $\sqcap$  are switched
- $\perp$  and  $\top$  are switched

## ■ lifting (adding a smallest element)

$(X \cup \{\perp'\}, \sqsubseteq', \sqcup', \sqcap', \perp', \top)$

- $a \sqsubseteq' b \iff a = \perp' \vee a \sqsubseteq b$
- $\perp' \sqcup' a = a \sqcup' \perp' = a$ , and  $a \sqcup' b = a \sqcup b$  if  $a, b \neq \perp'$
- $\perp' \sqcap' a = a \sqcap' \perp' = \perp'$ , and  $a \sqcap' b = a \sqcap b$  if  $a, b \neq \perp'$
- $\perp'$  replaces  $\perp$
- $\top$  is unchanged



# Derivation (cont.)

Given (complete) lattices or partial orders:

$(X_1, \sqsubseteq_1, \sqcup_1, \sqcap_1, \perp_1, \top_1)$  and  $(X_2, \sqsubseteq_2, \sqcup_2, \sqcap_2, \perp_2, \top_2)$

We can combine them by:

- **product**

$(X_1 \times X_2, \sqsubseteq, \sqcup, \sqcap, \perp, \top)$  where

- $(x, y) \sqsubseteq (x', y') \iff x \sqsubseteq_1 x' \wedge y \sqsubseteq_2 y'$
- $(x, y) \sqcup (x', y') \stackrel{\text{def}}{=} (x \sqcup_1 x', y \sqcup_2 y')$
- $(x, y) \sqcap (x', y') \stackrel{\text{def}}{=} (x \sqcap_1 x', y \sqcap_2 y')$
- $\perp \stackrel{\text{def}}{=} (\perp_1, \perp_2)$
- $\top \stackrel{\text{def}}{=} (\top_1, \top_2)$

- **smashed product** (coalescent product, merging  $\perp_1$  and  $\perp_2$ )

$((X_1 \setminus \{\perp_1\}) \times (X_2 \setminus \{\perp_2\})) \cup \{\perp\}, \sqsubseteq, \sqcup, \sqcap, \perp, \top)$

(as  $X_1 \times X_2$ , but all elements of the form  $(\perp_1, y)$  and  $(x, \perp_2)$  are identified to a unique  $\perp$  element)

# Derivation (cont.)

Given a (complete) lattice or partial order  $(X, \sqsubseteq, \sqcup, \sqcap, \perp, \top)$  and a set  $S$ :

- **point-wise lifting** (functions from  $S$  to  $X$ )

$(S \rightarrow X, \sqsubseteq', \sqcup', \sqcap', \perp', \top')$  where

- $x \sqsubseteq' y \iff \forall s \in S: x(s) \sqsubseteq y(s)$
- $\forall s \in S: (x \sqcup' y)(s) \stackrel{\text{def}}{=} x(s) \sqcup y(s)$
- $\forall s \in S: (x \sqcap' y)(s) \stackrel{\text{def}}{=} x(s) \sqcap y(s)$
- $\forall s \in S: \perp'(s) = \perp$
- $\forall s \in S: \top'(s) = \top$

- **smashed point-wise lifting**

$((S \rightarrow (X \setminus \{\perp\})) \cup \{\perp'\}, \sqsubseteq', \sqcup', \sqcap', \perp', \top')$

as  $S \rightarrow X$ , but identify to  $\perp'$  any map  $x$  where  $\exists s \in S: x(s) = \perp$

(e.g. map each program variable in  $S$  to an interval in  $X$ )

# Distributivity

A lattice  $(X, \sqsubseteq, \sqcup, \sqcap)$  is **distributive** if:

- $a \sqcup (b \sqcap c) = (a \sqcup b) \sqcap (a \sqcup c)$  and
- $a \sqcap (b \sqcup c) = (a \sqcap b) \sqcup (a \sqcap c)$

## Examples, Counter-examples:

- $(\mathcal{P}(X), \subseteq, \cup, \cap)$  is distributive
- intervals are **not** distributive  
 $[0, 0] \sqcup [2, 2]) \sqcap [1, 1] = [0, 2] \sqcap [1, 1] = [1, 1]$  but  
 $([0, 0] \sqcap [1, 1]) \sqcup ([2, 2] \sqcap [1, 1]) = \emptyset \sqcup \emptyset = \emptyset$

common cause of precision loss in static analyses:  
 merging abstract information early, at control-flow joins  
 vs. merging executions paths late, at the end of the program

# Sublattice

Given a lattice  $(X, \sqsubseteq, \sqcup, \sqcap)$  and  $X' \subseteq X$   
 $(X', \sqsubseteq, \sqcup, \sqcap)$  is a **sublattice** of  $X$  if  $X'$  is **closed** under  $\sqcup$  and  $\sqcap$

Example, Counter-examples:

- if  $Y \subseteq X$ ,  $(\mathcal{P}(Y), \subseteq, \cup, \cap, \emptyset, Y)$  is a sublattice of  $(\mathcal{P}(X), \subseteq, \cup, \cap, \emptyset, X)$
- integer intervals are **not** a sublattice of  $(\mathcal{P}(\mathbb{Z}), \subseteq, \cup, \cap, \emptyset, \mathbb{Z})$   
 $[\min(a, a'), \max(b, b')] \neq [a, b] \cup [a', b']$

another common cause of precision loss in static analyses:

$\sqcup$  cannot represent the exact union, and loses precision

# Functions and Fixpoints

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# Functions

A function  $f : (X_1, \sqsubseteq_1, \sqcup_1, \perp_1) \rightarrow (X_2, \sqsubseteq_2, \sqcup_2, \perp_2)$  is

- **monotonic** if

$$\forall x, x', x \sqsubseteq_1 x' \implies f(x) \sqsubseteq_2 f(x')$$

(aka: increasing, isotone, order-preserving, morphism)

- **strict** if  $f(\perp_1) = \perp_2$

- **continuous** between CPO if

$$\forall C \text{ chain } \subseteq X_1, \{f(c) \mid c \in C\} \text{ is a chain in } X_2 \\ \text{and } f(\sqcup_1 C) = \sqcup_2 \{f(c) \mid c \in C\}$$

- a (complete)  **$\sqcup$ -morphism** between (complete) lattices  
if  $\forall S \subseteq X_1, f(\sqcup_1 S) = \sqcup_2 \{f(s) \mid s \in S\}$

- **extensive** if  $X_1 = X_2$  and  $\forall x, x \sqsubseteq_1 f(x)$

- **reductive** if  $X_1 = X_2$  and  $\forall x, f(x) \sqsubseteq_1 x$

# Fixpoints

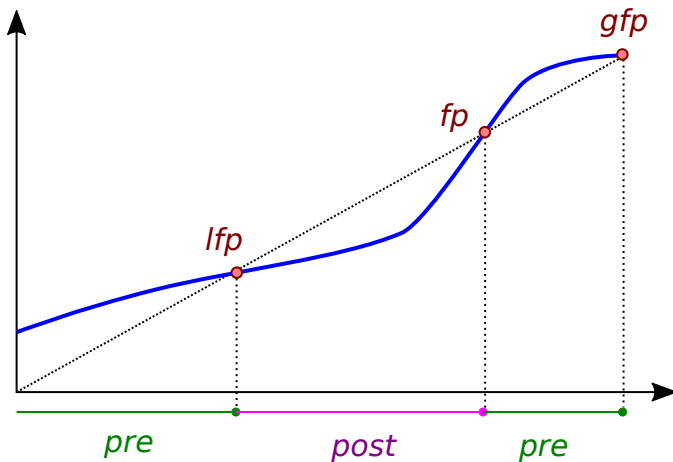
Given  $f : (X, \sqsubseteq) \rightarrow (X, \sqsubseteq)$

- $x$  is a **fixpoint** of  $f$  if  $f(x) = x$
- $x$  is a **pre-fixpoint** of  $f$  if  $x \sqsubseteq f(x)$
- $x$  is a **post-fixpoint** of  $f$  if  $f(x) \sqsubseteq x$

We may have several fixpoints (or none)

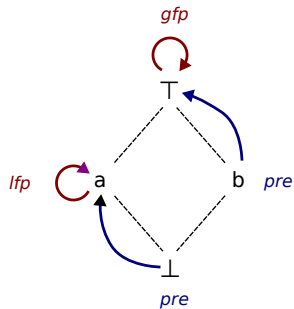
- $\text{fp}(f) \stackrel{\text{def}}{=} \{x \in X \mid f(x) = x\}$
- $\text{lfp}_x f \stackrel{\text{def}}{=} \min_{\sqsubseteq} \{y \in \text{fp}(f) \mid x \sqsubseteq y\}$  if it exists  
(least fixpoint greater than  $x$ )
- $\text{lfp} f \stackrel{\text{def}}{=} \text{lfp}_{\perp} f$   
(least fixpoint)
- dually:  $\text{gfp}_x f \stackrel{\text{def}}{=} \max_{\sqsubseteq} \{y \in \text{fp}(f) \mid y \sqsubseteq x\}$ ,  $\text{gfp} f \stackrel{\text{def}}{=} \text{gfp}_{\top} f$   
(greatest fixpoints)

# Fixpoints: illustration



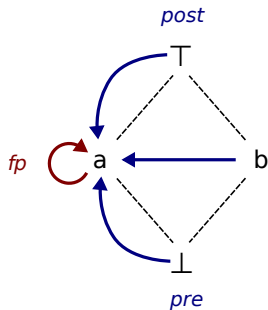


# Fixpoints: example



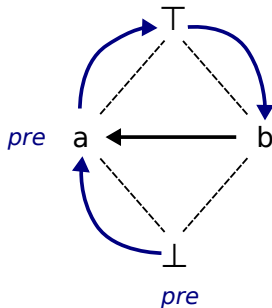
Monotonic function with two distinct fixpoints

# Fixpoints: example



Monotonic function with a **unique fixpoint**

# Fixpoints: example



Non-monotonic function with no fixpoint

# Uses of fixpoints: examples

## ■ Express solutions of mutually recursive equation systems

Example:

The solutions of  $\begin{cases} x_1 = f(x_1, x_2) \\ x_2 = g(x_1, x_2) \end{cases}$  with  $x_1, x_2$  in lattice  $X$

are exactly the fixpoint of  $\vec{F}$  in lattice  $X \times X$ , where

$$\vec{F} \left( \begin{array}{c} x_1, \\ x_2 \end{array} \right) = \left( \begin{array}{c} f(x_1, x_2), \\ g(x_1, x_2) \end{array} \right)$$

The least solution of the system is  $\text{lfp } \vec{F}$ .

# Uses of fixpoints: examples

- Close (complete) sets to satisfy a given property

Example:

$r \subseteq X \times X$  is **transitive** if:

$$(a, b) \in r \wedge (b, c) \in r \implies (a, c) \in r$$

The **transitive closure** of  $r$  is the smallest transitive relation containing  $r$ .

Let  $f(s) = r \cup \{ (a, c) \mid (a, b) \in s \wedge (b, c) \in s \}$ , then  $\text{lfp } f$ :

- $\text{lfp } f$  contains  $r$
- $\text{lfp } f$  is transitive
- $\text{lfp } f$  is minimal

$\implies \text{lfp } f$  is the transitive closure of  $r$ .

# Tarski's fixpoint theorem

## Tarski's theorem

If  $f : X \rightarrow X$  is **monotonic** in a **complete lattice**  $X$  then  $\text{fp}(f)$  is a complete lattice.

Proved by Knaster and Tarski [Tars55].

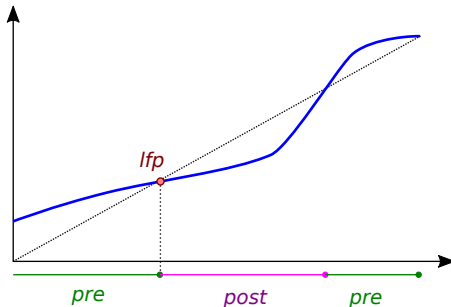
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We prove  $\text{lfp } f = \sqcap \{x \mid f(x) \sqsubseteq x\}$  (meet of post-fixpoints).



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### Proof:

We prove  $\text{lfp } f = \sqcap \{x \mid f(x) \sqsubseteq x\}$  (meet of post-fixpoints). Let

$f^* = \{x \mid f(x) \sqsubseteq x\}$  and  $a = \sqcap f^*$ .

$\forall x \in f^*, a \sqsubseteq x$  (by definition of  $\sqcap$ )

so  $f(a) \sqsubseteq f(x)$  (as  $f$  is monotonic)

so  $f(a) \sqsubseteq x$  (as  $x$  is a post-fixpoint).

We deduce that  $f(a) \sqsubseteq \sqcap f^*$ , i.e.  $f(a) \sqsubseteq a$ .



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$$f(a) \sqsubseteq a$$

$$\text{so } f(f(a)) \sqsubseteq f(a) \quad (\text{as } f \text{ is monotonic})$$

$$\text{so } f(a) \in f^* \quad (\text{by definition of } f^*)$$

$$\text{so } a \sqsubseteq f(a).$$

We deduce that  $f(a) = a$ , so  $a \in \text{fp}(f)$ .

Note that  $y \in \text{fp}(f)$  implies  $y \in f^*$ .

As  $a = \sqcap f^*$ ,  $a \sqsubseteq y$ , and we deduce  $a = \text{lfp } f$ .

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If  $f : X \rightarrow X$  is **monotonic** in a **complete lattice**  $X$  then  $\text{fp}(f)$  is a complete lattice.

### Proof:

Given  $S \subseteq \text{fp}(f)$ , we prove that  $\text{lfp}_{\sqcup S} f$  exists.

Consider  $X' = \{x \in X \mid \sqcup S \sqsubseteq x\}$ .

$X'$  is a complete lattice.

Moreover  $\forall x' \in X', f(x') \in X'$ .

$f$  can be restricted to a monotonic function  $f'$  on  $X'$ .

We apply the preceding result, so that  $\text{lfp } f' = \text{lfp}_{\sqcup S} f$  exists.

By definition,  $\text{lfp}_{\sqcup S} f \in \text{fp}(f)$  and is smaller than any fixpoint larger than all  $s \in S$ .

# Tarski's fixpoint theorem

## Tarski's theorem

If  $f : X \rightarrow X$  is **monotonic** in a **complete lattice**  $X$  then  $\text{fp}(f)$  is a complete lattice.

### Proof:

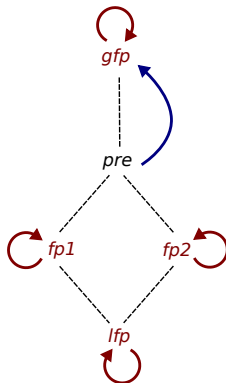
By duality, we construct  $\text{gfp } f$  and  $\text{gfp}_{\sqcap S} f$ .

The complete lattice of fixpoints is:

$(\text{fp}(f), \sqsubseteq, \lambda S. \text{lfp}_{\sqcup S} f, \lambda S. \text{gfp}_{\sqcap S} f, \text{lfp } f, \text{gfp } f)$ .

Not necessarily a sublattice of  $(X, \sqsubseteq, \sqcup, \sqcap, \perp, \top)$ !

# Tarski's fixpoint theorem: example



**Lattice:**  $(\{ \text{lfp}, \text{fp1}, \text{fp2}, \text{pre}, \text{gfp} \}, \sqcup, \sqcap, \text{lfp}, \text{gfp})$

**Fixpoint lattice:**  $(\{ \text{lfp}, \text{fp1}, \text{fp2}, \text{gfp} \}, \sqcup', \sqcap', \text{lfp}, \text{gfp})$

(not a sublattice as  $\text{fp1} \sqcup' \text{fp2} = \text{gfp}$  while  $\text{fp1} \sqcup \text{fp2} = \text{pre}$ ,  
but **gfp** is the smallest fixpoint greater than **pre**)

# “Kleene” fixpoint theorem

## “Kleene” fixpoint theorem

If  $f : X \rightarrow X$  is **continuous** in a **CPO**  $X$  and  $a \sqsubseteq f(a)$  then  $\text{lfp}_a f$  exists.

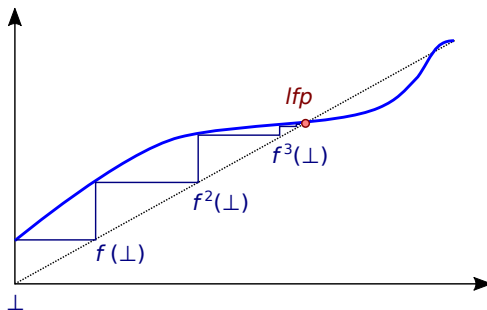
Inspired by Kleene [Klee52].

# “Kleene” fixpoint theorem

## “Kleene” fixpoint theorem

If  $f : X \rightarrow X$  is **continuous** in a **CPO**  $X$  and  $a \sqsubseteq f(a)$  then  $\text{lfp}_a f$  exists.

We prove that  $\{f^n(a) \mid n \in \mathbb{N}\}$  is a chain and  
 $\text{lfp}_a f = \sqcup \{f^n(a) \mid n \in \mathbb{N}\}$ .



# “Kleene” fixpoint theorem

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We prove that  $\{f^n(a) \mid n \in \mathbb{N}\}$  is a chain and  
 $\text{lfp}_a f = \sqcup \{f^n(a) \mid n \in \mathbb{N}\}$ .

$a \sqsubseteq f(a)$  by hypothesis.

$f(a) \sqsubseteq f(f(a))$  by monotony of  $f$ .

(Note that any continuous function is monotonic.

Indeed,  $x \sqsubseteq y \implies x \sqcup y = y \implies f(x \sqcup y) = f(y)$ ;

by continuity  $f(x) \sqcup f(y) = f(x \sqcup y) = f(y)$ , which implies  $f(x) \sqsubseteq f(y)$ .)

By recurrence  $\forall n, f^n(a) \sqsubseteq f^{n+1}(a)$ .

Thus,  $\{f^n(a) \mid n \in \mathbb{N}\}$  is a chain and  $\sqcup \{f^n(a) \mid n \in \mathbb{N}\}$  exists.

# “Kleene” fixpoint theorem

## “Kleene” fixpoint theorem

If  $f : X \rightarrow X$  is **continuous** in a **CPO**  $X$  and  $a \sqsubseteq f(a)$  then  $\text{lfp}_a f$  exists.

$$\begin{aligned}
 & f(\sqcup \{ f^n(a) \mid n \in \mathbb{N} \}) \\
 &= \sqcup \{ f^{n+1}(a) \mid n \in \mathbb{N} \} \quad (\text{by continuity}) \\
 &= a \sqcup (\sqcup \{ f^{n+1}(a) \mid n \in \mathbb{N} \}) \quad (\text{as all } f^{n+1}(a) \text{ are greater than } a) \\
 &= \sqcup \{ f^n(a) \mid n \in \mathbb{N} \}.
 \end{aligned}$$

$$\text{So, } \sqcup \{ f^n(a) \mid n \in \mathbb{N} \} \in \text{fp}(f)$$

Moreover, any fixpoint greater than  $a$  must also be greater than all  $f^n(a)$ ,  $n \in \mathbb{N}$ .

$$\text{So, } \sqcup \{ f^n(a) \mid n \in \mathbb{N} \} = \text{lfp}_a f.$$



# Well-ordered sets

$(S, \sqsubseteq)$  is a **well-ordered set** if:

- $\sqsubseteq$  is a **total order** on  $S$
- every  $X \subseteq S$  such that  $X \neq \emptyset$  has a **least element**  $\sqcap X \in X$

Consequences:

- any element  $x \in S$  has a **successor**  $x + 1 \stackrel{\text{def}}{=} \sqcap \{y \mid x \sqsubset y\}$   
(except the greatest element, if it exists)
- if  $\nexists y, x = y + 1$ ,  $x$  is a **limit** and  $x = \sqcup \{y \mid y \sqsubset x\}$   
(every bounded subset  $X \subseteq S$  has a lub  $\sqcup X = \sqcap \{y \mid \forall x \in X, x \sqsubseteq y\}$ )

Examples:

- $(\mathbb{N}, \leq)$  and  $(\mathbb{N} \cup \{\infty\}, \leq)$  are well-ordered
- $(\mathbb{Z}, \leq)$ ,  $(\mathbb{R}, \leq)$ ,  $(\mathbb{R}^+, \leq)$  are **not** well-ordered
- **ordinals**  $0, 1, 2, \dots, \omega, \omega + 1, \dots$  are well-ordered ( $\omega$  is a limit)  
**well-ordered sets are ordinals** up to order-isomorphism  
(i.e., bijective functions  $f$  such that  $f$  and  $f^{-1}$  are monotonic)

# Constructive Tarski theorem by transfinite iterations

Given a function  $f : X \rightarrow X$  and  $a \in X$ ,  
the **transfinite iterates** of  $f$  from  $a$  are:

$$\begin{cases} x_0 \stackrel{\text{def}}{=} a \\ x_n \stackrel{\text{def}}{=} f(x_{n-1}) & \text{if } n \text{ is a successor ordinal} \\ x_n \stackrel{\text{def}}{=} \sqcup \{x_m \mid m < n\} & \text{if } n \text{ is a limit ordinal} \end{cases}$$

## Constructive Tarski theorem

If  $f : X \rightarrow X$  is **monotonic** in a **CPO**  $X$  and  $a \sqsubseteq f(a)$ , then  $\text{lfp}_a f = x_\delta$  for some ordinal  $\delta$ .

Generalisation of “Kleene” fixpoint theorem, from [Cous79].

# Proof

$f$  is monotonic in a CPO  $X$ ,

$$\begin{cases} x_0 \stackrel{\text{def}}{=} a \sqsubseteq f(a) \\ x_n \stackrel{\text{def}}{=} f(x_{n-1}) & \text{if } n \text{ is a successor ordinal} \\ x_n \stackrel{\text{def}}{=} \sqcup \{x_m \mid m < n\} & \text{if } n \text{ is a limit ordinal} \end{cases}$$

Proof:

We prove that  $\exists \delta, x_\delta = x_{\delta+1}$ .

We note that  $m \leq n \implies x_m \sqsubseteq x_n$ .

Assume by **contradiction** that  $\nexists \delta, x_\delta = x_{\delta+1}$ .

If  $n$  is a successor ordinal, then  $x_{n-1} \sqsubset x_n$ .

If  $n$  is a limit ordinal, then  $\forall m < n, x_m \sqsubset x_n$ .

Thus, all the  $x_n$  are distinct.

By choosing  $n > |X|$ , we arrive at a contradiction.

Thus  **$\delta$  exists**.

# Proof

$f$  is monotonic in a CPO  $X$ ,

$$\begin{cases} x_0 \stackrel{\text{def}}{=} a \sqsubseteq f(a) \\ x_n \stackrel{\text{def}}{=} f(x_{n-1}) & \text{if } n \text{ is a successor ordinal} \\ x_n \stackrel{\text{def}}{=} \sqcup \{x_m \mid m < n\} & \text{if } n \text{ is a limit ordinal} \end{cases}$$

Proof:

Given  $\delta$  such that  $x_{\delta+1} = x_\delta$ , we prove that  $x_\delta = \text{lfp}_a f$ .

$f(x_\delta) = x_{\delta+1} = x_\delta$ , so  $x_\delta \in \text{fp}(f)$ .

Given any  $y \in \text{fp}(f)$ ,  $y \sqsupseteq a$ , we prove by **transfinite induction** that  $\forall n, x_n \sqsubseteq y$ .

By definition  $x_0 = a \sqsubseteq y$ .

If  $n$  is a successor ordinal, by monotony,

$x_{n-1} \sqsubseteq y \implies f(x_{n-1}) \sqsubseteq f(y)$ , i.e.,  $x_n \sqsubseteq y$ .

If  $n$  is a limit ordinal,  $\forall m < n$ ,  $x_m \sqsubseteq y$  implies

$x_n = \sqcup \{x_m \mid m < n\} \sqsubseteq y$ .

Hence,  $x_\delta \sqsubseteq y$  and  $x_\delta = \text{lfp}_a f$ .

# Ascending chain condition (ACC)

An **ascending chain**  $C$  in  $(X, \sqsubseteq)$  is a sequence  $c_i \in X$  such that  $i \leq j \implies c_i \sqsubseteq c_j$ .

A poset  $(X, \sqsubseteq)$  satisfies the **ascending chain condition (ACC)** iff for every ascending chain  $C$ ,  $\exists i \in \mathbb{N}, \forall j \geq i, c_i = c_j$ .

Similarly, we can define the **descending chain condition (DCC)**.

## Examples:

- the **powerset poset**  $(\mathcal{P}(X), \subseteq)$  is ACC when  $X$  is finite
- the **pointed integer poset**  $(\mathbb{Z} \cup \{\perp\}, \sqsubseteq)$  where  $x \sqsubseteq y \iff x = \perp \vee x = y$  is ACC and DCC
- the **divisibility poset**  $(\mathbb{N}^*, |)$  is DCC but not ACC.

# Kleene fixpoints in ACC posets

## “Kleene” finite fixpoint theorem

If  $f : X \rightarrow X$  is **monotonic** in an **ACC poset**  $X$  and  $a \sqsubseteq f(a)$  then  $\text{lfp}_a f$  exists.

Proof:

We prove  $\exists n \in \mathbb{N}, \text{lfp}_a f = f^n(a)$ .

By monotony of  $f$ , the sequence  $x_n = f^n(a)$  is an **increasing chain**.

By definition of ACC,  $\exists n \in \mathbb{N}, x_n = x_{n+1} = f(x_n)$ .

Thus,  $x_n \in \text{fp}(f)$ .

Obviously,  $a = x_0 \sqsubseteq f(x_n)$ .

Moreover, if  $y \in \text{fp}(f)$  and  $y \sqsupseteq a$ , then  $\forall i, y \sqsupseteq f^i(a) = x_i$ .

Hence,  $y \sqsupseteq x_n$  and  $x_n = \text{lfp}_a(f)$ .

# Comparison of fixpoint theorems

theorem	function	domain	fixpoint	method
Tarski	monotonic	complete lattice	$\text{fp}(f)$	meet of post-fixpoints
Kleene	continuous	CPO	$\text{lfp}_a(f)$	countable iterations
constructive Tarski	monotonic	CPO	$\text{lfp}_a(f)$	transfinite iteration
ACC Kleene	monotonic	poset	$\text{lfp}_a(f)$	finite iteration

# Galois connections

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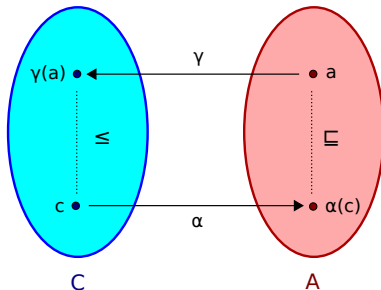


# Galois connections

Given two posets  $(C, \leq)$  and  $(A, \sqsubseteq)$ , the pair  $(\alpha : C \rightarrow A, \gamma : A \rightarrow C)$  is a **Galois connection** iff:

$$\forall a \in A, c \in C, \alpha(c) \sqsubseteq a \iff c \leq \gamma(a)$$

which is noted  $(C, \leq) \xleftrightarrow[\alpha]{\gamma} (A, \sqsubseteq)$ .



- $\alpha$  is the **upper adjoint** or **abstraction**;  $A$  is the abstract domain.
- $\gamma$  is the **lower adjoint** or **concretization**;  $C$  is the concrete domain.

# Galois connection example

Abstract domain of **intervals of integers**  $\mathbb{Z}$   
 represented as **pairs of bounds**  $(a, b)$ .

We have:  $(\mathcal{P}(\mathbb{Z}), \subseteq) \xrightleftharpoons[\alpha]{\gamma} (I, \sqsubseteq)$

- $I \stackrel{\text{def}}{=} (\mathbb{Z} \cup \{-\infty\}) \times (\mathbb{Z} \cup \{+\infty\})$
- $(a, b) \sqsubseteq (a', b') \iff (a \geq a') \wedge (b \leq b')$
- $\gamma(a, b) \stackrel{\text{def}}{=} \{x \in \mathbb{Z} \mid a \leq x \leq b\}$
- $\alpha(X) \stackrel{\text{def}}{=} (\min X, \max X)$

proof:

# Galois connection example

Abstract domain of **intervals of integers**  $\mathbb{Z}$   
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- $\gamma(a, b) \stackrel{\text{def}}{=} \{x \in \mathbb{Z} \mid a \leq x \leq b\}$
- $\alpha(X) \stackrel{\text{def}}{=} (\min X, \max X)$

proof:

$$\begin{aligned}
 \alpha(X) &\sqsubseteq (a, b) \\
 \iff \min X &\geq a \wedge \max X \leq b \\
 \iff \forall x \in X: &a \leq x \leq b \\
 \iff \forall x \in X: x &\in \{y \mid a \leq y \leq b\} \\
 \iff \forall x \in X: x &\in \gamma(a, b) \\
 \iff X &\subseteq \gamma(a, b)
 \end{aligned}$$

# Properties of Galois connections

Assuming  $\forall a, c, \alpha(c) \sqsubseteq a \iff c \leq \gamma(a)$ , we have:

1  $\gamma \circ \alpha$  is extensive:  $\forall c, c \leq \gamma(\alpha(c))$

proof:  $\alpha(c) \sqsubseteq \alpha(c) \implies c \leq \gamma(\alpha(c))$

2  $\alpha \circ \gamma$  is reductive:  $\forall a, \alpha(\gamma(a)) \sqsubseteq a$

3  $\alpha$  is monotonic

proof:  $c \leq c' \implies c \leq \gamma(\alpha(c')) \implies \alpha(c) \sqsubseteq \alpha(c')$

4  $\gamma$  is monotonic

5  $\gamma \circ \alpha \circ \gamma = \gamma$

proof:  $\alpha(\gamma(a)) \sqsubseteq \alpha(\gamma(a)) \implies \gamma(a) \leq \gamma(\alpha(\gamma(a)))$  and  $a \sqsupseteq \alpha(\gamma(a)) \implies \gamma(a) \geq \gamma(\alpha(\gamma(a)))$

6  $\alpha \circ \gamma \circ \alpha = \alpha$

7  $\alpha \circ \gamma$  is idempotent:  $\alpha \circ \gamma \circ \alpha \circ \gamma = \alpha \circ \gamma$

8  $\gamma \circ \alpha$  is idempotent

# Alternate characterization

If the pair  $(\alpha : C \rightarrow A, \gamma : A \rightarrow C)$  satisfies:

- 1  $\gamma$  is monotonic
- 2  $\alpha$  is monotonic
- 3  $\gamma \circ \alpha$  is extensive
- 4  $\alpha \circ \gamma$  is reductive

then  $(\alpha, \gamma)$  is a Galois connection.

(proof left as exercise)

# Uniqueness of the adjoint

Given  $(C, \leq) \xrightleftharpoons[\alpha]{\gamma} (A, \sqsubseteq)$ ,

each adjoint can be **uniquely defined** in term of the other:

$$1 \quad \alpha(c) = \sqcap \{ a \mid c \leq \gamma(a) \}$$

$$2 \quad \gamma(a) = \sqcup \{ c \mid \alpha(c) \sqsubseteq a \}$$

Proof: of 1

$$\forall a, c \leq \gamma(a) \implies \alpha(c) \sqsubseteq a.$$

Hence,  $\alpha(c)$  is a lower bound of  $\{ a \mid c \leq \gamma(a) \}$ .

Assume that  $a'$  is another lower bound.

$$\text{Then, } \forall a, c \leq \gamma(a) \implies a' \sqsubseteq a.$$

By Galois connection, we have then  $\forall a, \alpha(c) \sqsubseteq a \implies a' \sqsubseteq a$ .

This implies  $a' \sqsubseteq \alpha(c)$ .

Hence, the greatest lower bound of  $\{ a \mid c \leq \gamma(a) \}$  exists, and equals  $\alpha(c)$ .

The proof of 2 is similar (by duality).

# Properties of Galois connections (cont.)

If  $(\alpha : C \rightarrow A, \gamma : A \rightarrow C)$ , then:

1  $\forall X \subseteq C$ , if  $\vee X$  exists, then  $\alpha(\vee X) = \sqcup \{ \alpha(x) \mid x \in X \}$

2  $\forall X \subseteq A$ , if  $\sqcap X$  exists, then  $\gamma(\sqcap X) = \wedge \{ \gamma(x) \mid x \in X \}$

Proof: of 1

By definition of lubs,  $\forall x \in X, x \leq \vee X$ .

By monotony,  $\forall x \in X, \alpha(x) \sqsubseteq \alpha(\vee X)$ .

Hence,  $\alpha(\vee X)$  is an upper bound of  $\{ \alpha(x) \mid x \in X \}$ .

Assume that  $y$  is another upper bound of  $\{ \alpha(x) \mid x \in X \}$ .

Then,  $\forall x \in X, \alpha(x) \sqsubseteq y$ .

By Galois connection  $\forall x \in X, x \leq \gamma(y)$ .

By definition of lubs,  $\vee X \leq \gamma(y)$ .

By Galois connection,  $\alpha(\vee X) \sqsubseteq y$ .

Hence,  $\{ \alpha(x) \mid x \in X \}$  has a lub, which equals  $\alpha(\vee X)$ .

The proof of 2 is similar (by duality).

# Deriving Galois connections

Given  $(C, \leq) \xleftrightarrow[\alpha]{\gamma} (A, \sqsubseteq)$ , we have:

■ **duality:**  $(A, \sqsupseteq) \xleftrightarrow[\gamma]{\alpha} (C, \geq)$

$$(\alpha(c) \sqsubseteq a \iff c \leq \gamma(a) \text{ is exactly } \gamma(a) \geq c \iff a \sqsupseteq \alpha(c))$$

■ **point-wise lifting** by some set  $S$ :  $(S \rightarrow C, \dot{\leq}) \xleftrightarrow[\dot{\alpha}]{\dot{\gamma}} (S \rightarrow A, \dot{\sqsubseteq})$  where

$$f \dot{\leq} f' \iff \forall s, f(s) \leq f'(s), \quad (\dot{\gamma}(f))(s) = \gamma(f(s)),$$

$$f \dot{\sqsubseteq} f' \iff \forall s, f(s) \sqsubseteq f'(s), \quad (\dot{\alpha}(f))(s) = \alpha(f(s)).$$

Given  $(X_1, \sqsubseteq_1) \xleftrightarrow[\alpha_1]{\gamma_1} (X_2, \sqsubseteq_2) \xleftrightarrow[\alpha_2]{\gamma_2} (X_3, \sqsubseteq_3)$ :

■ **composition:**  $(X_1, \sqsubseteq_1) \xleftrightarrow[\alpha_2 \circ \alpha_1]{\gamma_1 \circ \gamma_2} (X_3, \sqsubseteq_3)$

$$((\alpha_2 \circ \alpha_1)(c) \sqsubseteq_3 a \iff \alpha_1(c) \sqsubseteq_2 \gamma_2(a) \iff c \sqsubseteq_1 (\gamma_1 \circ \gamma_2)(a))$$



# Galois embeddings

If  $(C, \leq) \xleftrightarrow[\alpha]{\gamma} (A, \sqsubseteq)$ , the following properties are equivalent:

1  $\alpha$  is surjective

$$(\forall a \in A, \exists c \in C, \alpha(c) = a)$$

2  $\gamma$  is injective

$$(\forall a, a' \in A, \gamma(a) = \gamma(a') \implies a = a')$$

3  $\alpha \circ \gamma = id$

$$(\forall a \in A, id(a) = a)$$

Such  $(\alpha, \gamma)$  is called a **Galois embedding**, which is noted  $(C, \leq) \xleftrightarrow[\alpha]{\gamma} (A, \sqsubseteq)$

Proof:

# Galois embeddings

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Proof:  $1 \implies 2$

Assume that  $\gamma(a) = \gamma(a')$ .

By surjectivity, take  $c, c'$  such that  $a = \alpha(c)$ ,  $a' = \alpha(c')$ .

Then  $\gamma(\alpha(c)) = \gamma(\alpha(c'))$ .

And  $\alpha(\gamma(\alpha(c))) = \alpha(\gamma(\alpha(c')))$ .

As  $\alpha \circ \gamma \circ \alpha = \alpha$ ,  $\alpha(c) = \alpha(c')$ .

Hence  $a = a'$ .

# Galois embeddings

If  $(C, \leq) \xleftrightarrow[\alpha]{\gamma} (A, \sqsubseteq)$ , the following properties are equivalent:

1  $\alpha$  is surjective

$$(\forall a \in A, \exists c \in C, \alpha(c) = a)$$

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$$(\forall a, a' \in A, \gamma(a) = \gamma(a') \implies a = a')$$

3  $\alpha \circ \gamma = id$

$$(\forall a \in A, id(a) = a)$$

Such  $(\alpha, \gamma)$  is called a **Galois embedding**, which is noted  $(C, \leq) \xleftrightarrow[\alpha]{\gamma} (A, \sqsubseteq)$

Proof:  $2 \implies 3$

Given  $a \in A$ , we know that  $\gamma(\alpha(\gamma(a))) = \gamma(a)$ .

By injectivity of  $\gamma$ ,  $\alpha(\gamma(a)) = a$ .

# Galois embeddings

If  $(C, \leq) \xleftrightarrow[\alpha]{\gamma} (A, \sqsubseteq)$ , the following properties are equivalent:

1  $\alpha$  is surjective

$$(\forall a \in A, \exists c \in C, \alpha(c) = a)$$

2  $\gamma$  is injective

$$(\forall a, a' \in A, \gamma(a) = \gamma(a') \implies a = a')$$

3  $\alpha \circ \gamma = id$

$$(\forall a \in A, id(a) = a)$$

Such  $(\alpha, \gamma)$  is called a **Galois embedding**, which is noted  $(C, \leq) \xleftrightarrow[\alpha]{\gamma} (A, \sqsubseteq)$

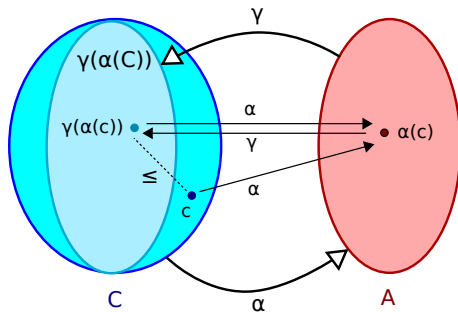
Proof:  $3 \implies 1$

Given  $a \in A$ , we have  $\alpha(\gamma(a)) = a$ .

Hence,  $\exists c \in C, \alpha(c) = a$ , using  $c = \gamma(a)$ .

# Galois embeddings (cont.)

$$(C, \leq) \xleftarrow[\alpha]{\gamma} (A, \sqsubseteq)$$



A Galois connection can be made into an embedding by **quotienting**  $A$  by the equivalence relation  $a \equiv a' \iff \gamma(a) = \gamma(a')$ .

# Galois embedding example

Abstract domain of **intervals of integers**  $\mathbb{Z}$   
 represented as **pairs of ordered bounds**  $(a, b)$  or  $\perp$ .

We have:  $(\mathcal{P}(\mathbb{Z}), \subseteq) \xrightleftharpoons[\alpha]{\gamma} (I, \sqsubseteq)$

- $I \stackrel{\text{def}}{=} \{(a, b) \mid a \in \mathbb{Z} \cup \{-\infty\}, b \in \mathbb{Z} \cup \{+\infty\}, a \leq b\} \cup \{\perp\}$
- $(a, b) \sqsubseteq (a', b') \iff (a \geq a') \wedge (b \leq b'), \quad \forall x: \perp \sqsubseteq x$
- $\gamma(a, b) \stackrel{\text{def}}{=} \{x \in \mathbb{Z} \mid a \leq x \leq b\}, \quad \gamma(\perp) = \emptyset$
- $\alpha(X) \stackrel{\text{def}}{=} (\min X, \max X), \text{ or } \perp \text{ if } X = \emptyset$

proof:

# Galois embedding example

Abstract domain of **intervals of integers**  $\mathbb{Z}$   
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We have:  $(\mathcal{P}(\mathbb{Z}), \subseteq) \xrightleftharpoons[\alpha]{\gamma} (I, \sqsubseteq)$

- $I \stackrel{\text{def}}{=} \{(a, b) \mid a \in \mathbb{Z} \cup \{-\infty\}, b \in \mathbb{Z} \cup \{+\infty\}, a \leq b\} \cup \{\perp\}$
- $(a, b) \sqsubseteq (a', b') \iff (a \geq a') \wedge (b \leq b'), \quad \forall x: \perp \sqsubseteq x$
- $\gamma(a, b) \stackrel{\text{def}}{=} \{x \in \mathbb{Z} \mid a \leq x \leq b\}, \quad \gamma(\perp) = \emptyset$
- $\alpha(X) \stackrel{\text{def}}{=} (\min X, \max X), \text{ or } \perp \text{ if } X = \emptyset$

proof:

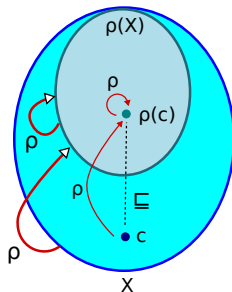
Quotient of the “pair of bounds” domain  $(\mathbb{Z} \cup \{-\infty\}) \times (\mathbb{Z} \cup \{+\infty\})$  by the relation  
 $(a, b) \equiv (a', b') \iff \gamma(a, b) = \gamma(a', b')$

i.e.,  $(a \leq b \wedge a = a' \wedge b = b') \vee (a > b \wedge a' > b')$ .

# Upper closures

$\rho : X \rightarrow X$  is an **upper closure** in the poset  $(X, \sqsubseteq)$  if it is:

- 1 **monotonic**:  $x \sqsubseteq x' \implies \rho(x) \sqsubseteq \rho(x')$ ,
- 2 **extensive**:  $x \sqsubseteq \rho(x)$ , and
- 3 **idempotent**:  $\rho \circ \rho = \rho$ .





# Upper closures and Galois connections

Given  $(C, \leq) \xrightleftharpoons[\alpha]{\gamma} (A, \sqsubseteq)$ ,

$\gamma \circ \alpha$  is an upper closure on  $(C, \leq)$ .

Given an upper closure  $\rho$  on  $(X, \sqsubseteq)$ , we have a Galois embedding:

$$(X, \sqsubseteq) \xrightleftharpoons[\rho]{id} (\rho(X), \sqsubseteq)$$

$\implies$  we can rephrase abstract interpretation using upper closures instead of Galois connections, but we lose:

- the notion of abstract representation  
(a data-structure  $A$  representing elements in  $\rho(X)$ )
- the ability to have several distinct abstract representations for a single concrete object  
(non-necessarily injective  $\gamma$  versus  $id$ )

# Operator approximations

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# Abstractions in the concretization framework

Given a concrete  $(C, \leq)$  and an abstract  $(A, \sqsubseteq)$  poset and a **monotonic concretization**  $\gamma : A \rightarrow C$

( $\gamma(a)$  is the “meaning” of  $a$  in  $C$ ; we use intervals in our examples)

- $a \in A$  is a **sound abstraction** of  $c \in C$  if  $c \leq \gamma(a)$ .

(e.g.:  $[0, 10]$  is a sound abstraction of  $\{0, 1, 2, 5\}$  in the integer interval domain)

- $g : A \rightarrow A$  is a **sound abstraction** of  $f : C \rightarrow C$  if  $\forall a \in A: (f \circ \gamma)(a) \leq (\gamma \circ g)(a)$ .

(e.g.:  $\lambda([a, b]).[-\infty, +\infty]$  is a sound abstraction of  $\lambda X. \{x + 1 \mid x \in X\}$  in the interval domain)

- $g : A \rightarrow A$  is an **exact abstraction** of  $f : C \rightarrow C$  if  $f \circ \gamma = \gamma \circ g$ .

(e.g.:  $\lambda([a, b]).[a + 1, b + 1]$  is an exact abstraction of  $\lambda X. \{x + 1 \mid x \in X\}$  in the interval domain)

# Abstractions in the Galois connection framework

Assume now that  $(C, \leq) \xleftrightarrow[\alpha]{\gamma} (A, \sqsubseteq)$ .

## ■ sound abstractions

- $c \leq \gamma(a)$  is equivalent to  $\alpha(c) \sqsubseteq a$ .
- $(f \circ \gamma)(a) \leq (\gamma \circ g)(a)$  is equivalent to  $(\alpha \circ f \circ \gamma)(a) \sqsubseteq g(a)$ .

## ■ Given $c \in C$ , its **best abstraction** is $\alpha(c)$ .

(proof: recall that  $\alpha(c) = \sqcap \{ a \mid c \leq \gamma(a) \}$ , so,  $\alpha(c)$  is the smallest sound abstraction of  $c$ )

(e.g.:  $\alpha(\{0, 1, 2, 5\}) = [0, 5]$  in the interval domain)

## ■ Given $f : C \rightarrow C$ , its **best abstraction** is $\alpha \circ f \circ \gamma$

(proof:  $g$  sound  $\iff \forall a, (\alpha \circ f \circ \gamma)(a) \sqsubseteq g(a)$ , so  $\alpha \circ f \circ \gamma$  is the smallest sound abstraction of  $f$ )

(e.g.:  $g([a, b]) = [2a, 2b]$  is the best abstraction in the interval domain of  $f(X) = \{ 2x \mid x \in X \}$ ; it is not an exact abstraction as  $\gamma(g([0, 1])) = \{0, 1, 2\} \not\sqsupseteq \{0, 2\} = f(\gamma([0, 1]))$ )

# Composition of sound, best, and exact abstractions

If  $g$  and  $g'$  soundly abstract respectively  $f$  and  $f'$  then:

- if  $f$  is monotonic,  
then  $g \circ g'$  is a sound abstraction of  $f \circ f'$ ,

(proof:  $\forall a, (f \circ f' \circ \gamma)(a) \leq (f \circ \gamma \circ g')(a) \leq (\gamma \circ g \circ g')(a)$ )

- if  $g, g'$  are exact abstractions of  $f$  and  $f'$ ,  
then  $g \circ g'$  is an exact abstraction,

(proof:  $f \circ f' \circ \gamma = f \circ \gamma \circ g' = \gamma \circ g \circ g'$ )

- if  $g$  and  $g'$  are the best abstractions of  $f$  and  $f'$ ,  
then  $g \circ g'$  is not always the best abstraction!

(e.g.:  $g([a, b]) = [a, \min(b, 1)]$  and  $g'([a, b]) = [2a, 2b]$  are the best abstractions of  $f(X) = \{x \in X \mid x \leq 1\}$  and  $f'(X) = \{2x \mid x \in X\}$  in the interval domain, but  $g \circ g'$  is not the best abstraction of  $f \circ f'$  as  $(g \circ g')([0, 1]) = [0, 1]$  while  $(\alpha \circ f \circ f' \circ \gamma)([0, 1]) = [0, 0]$ )

# Fixpoint approximations

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# Fixpoint transfer

If we have:

- a **Galois connection**  $(C, \leq) \xleftrightarrow[\alpha]{\gamma} (A, \sqsubseteq)$  between **CPOs**
- **monotonic concrete** and **abstract** functions  
 $f : C \rightarrow C$ ,  $f^\# : A \rightarrow A$
- a **commutation condition**  $\alpha \circ f = f^\# \circ \alpha$
- a prefixpoint  $a$  of  $f$  ( $a \leq f(a)$ ) and its **abstraction**  $a^\# = \alpha(a)$

then  $\alpha(\text{lfp}_a f) = \text{lfp}_{a^\#} f^\#$ .

(proof on next slide)

# Fixpoint transfer (proof)

## Proof:

By the constructive Tarski theorem,  $\text{lfp}_a f$  is the limit of transfinite iterations:  $a_0 \stackrel{\text{def}}{=} a$ ,  $a_{n+1} \stackrel{\text{def}}{=} f(a_n)$ , and  $a_n \stackrel{\text{def}}{=} \bigvee \{ a_m \mid m < n \}$  for limit ordinals  $n$ .

Likewise,  $\text{lfp}_{a^\#} f^\#$  is the limit of a transfinite iteration  $a_n^\#$ .

We prove by transfinite induction that  $a_n^\# = \alpha(a_n)$  for all ordinals  $n$ :

- $a_0^\# = \alpha(a_0)$ , by definition;
- $a_{n+1}^\# = f^\#(a_n^\#) = f^\#(\alpha(a_n)) = \alpha(f(a_n)) = \alpha(a_{n+1})$  for successor ordinals, by commutation;
- $a_n^\# = \bigsqcup \{ a_m^\# \mid m < n \} = \bigsqcup \{ \alpha(a_m) \mid m < n \} = \alpha(\bigvee \{ a_m \mid m < n \}) = \alpha(a_n)$  for limit ordinals, because  $\alpha$  is always continuous in Galois connections.

Hence,  $\text{lfp}_{a^\#} f^\# = \alpha(\text{lfp}_a f)$ .



# Fixpoint approximation

If we have:

- a complete lattice  $(C, \leq, \vee, \wedge, \perp, \top)$
- a monotonic concrete function  $f$
- a sound abstraction  $f^\sharp : A \rightarrow A$  of  $f$   
 $(\forall x^\sharp : (f \circ \gamma)(x^\sharp) \leq (\gamma \circ f^\sharp)(x^\sharp))$
- a post-fixpoint  $a^\sharp$  of  $f^\sharp$  ( $f^\sharp(a^\sharp) \sqsubseteq a^\sharp$ )

then  $a^\sharp$  is a sound abstraction of  $\text{lfp } f$ :  $\text{lfp } f \leq \gamma(a^\sharp)$ .

Proof:

By definition,  $f^\sharp(a^\sharp) \sqsubseteq a^\sharp$ .

By monotony,  $\gamma(f^\sharp(a^\sharp)) \leq \gamma(a^\sharp)$ .

By soundness,  $f(\gamma(a^\sharp)) \leq \gamma(a^\sharp)$ .

By Tarski's theorem  $\text{lfp } f = \bigwedge \{x \mid f(x) \leq x\}$ .

Hence,  $\text{lfp } f \leq \gamma(a^\sharp)$ .

Other fixpoint transfer / approximation theorems can be constructed...

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