Quantum query complexity of some graph problems

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Single Source Shortest Paths

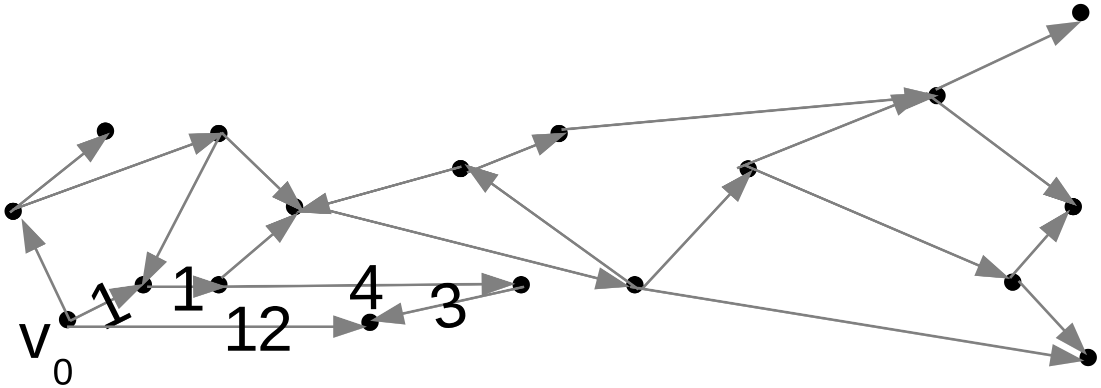
Given a directed graph G(V,E), with non-negative edge weights and a source vertex v_0

find the shortest paths to all vertices v

How many queries of the type "what is the weight of the edge (u,v)?" are necessary to solve the problem with bounded error?

Classical $\theta(n^2)$

Quantum $\omega(n^{3/2})$, $O(n^{3/2}\log^{3/2} n)$



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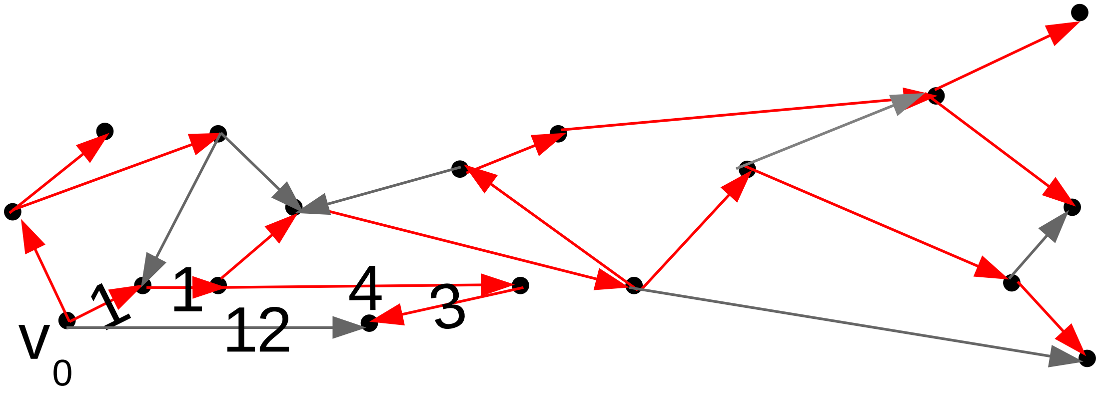


General algorithm

Tree $T=\{v_0\}$ covering vertices $S=\{v_0\}$

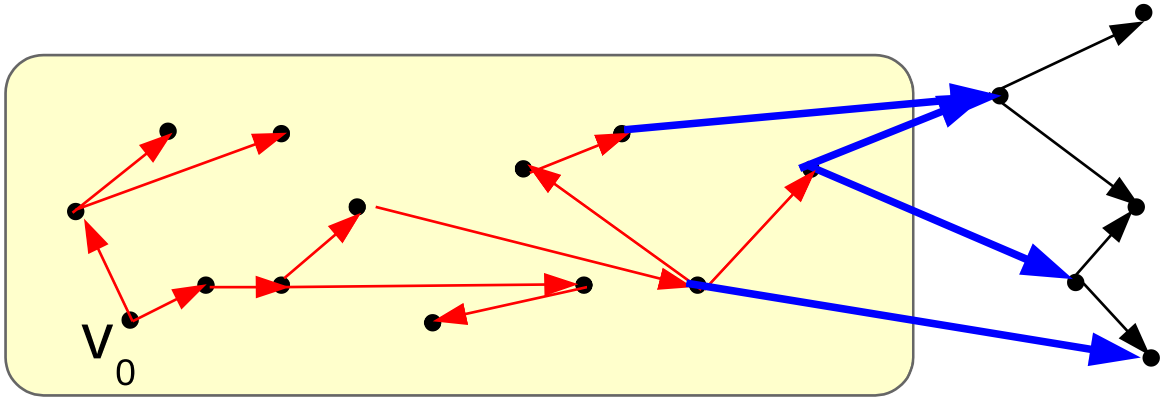
while $|S|<n$

add cheapest border edge $(u,v) \in E \cap S x(V \setminus S)$ to A

add v to S

Definition cost of edge (u,v)

=shortest path weight(v_0,u) + edge weight(u,v)

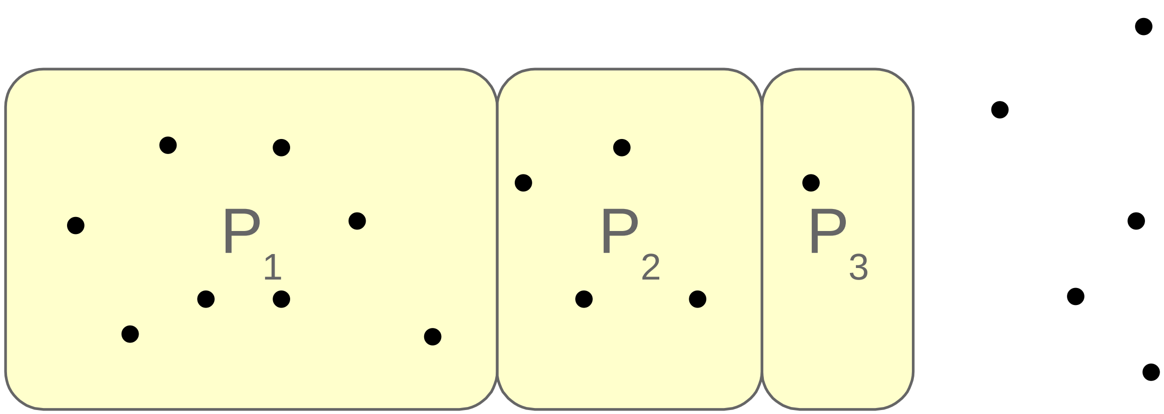


Quantum procedure for finding cheapest border edge

Consider the decomposition of |S| into powers of 2

Decompose S into $P_1 \cup \ldots \cup P_k$ s.t.

- $|P_1| > \ldots > |P_k|$
- and each $|P_i|$ is a power of 2



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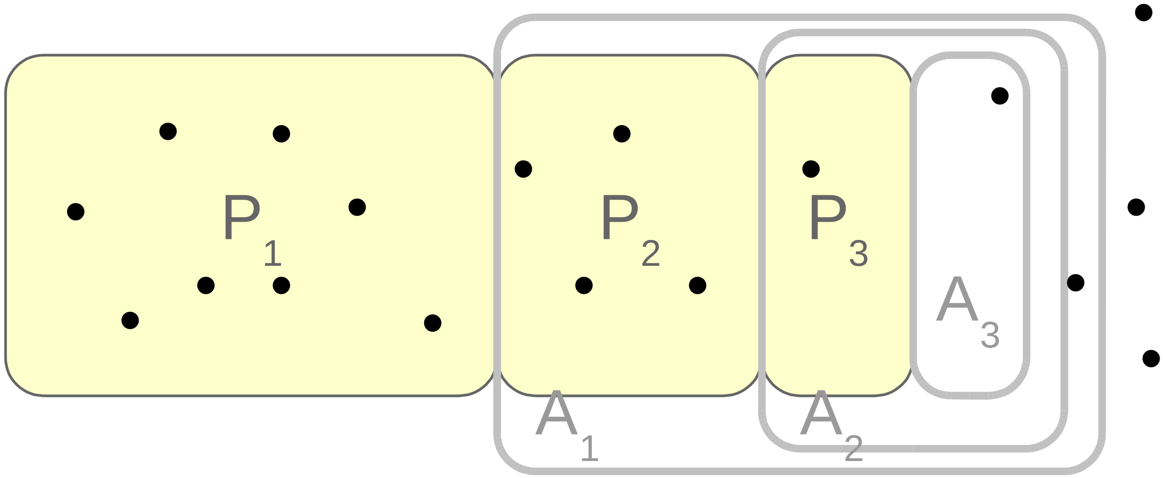
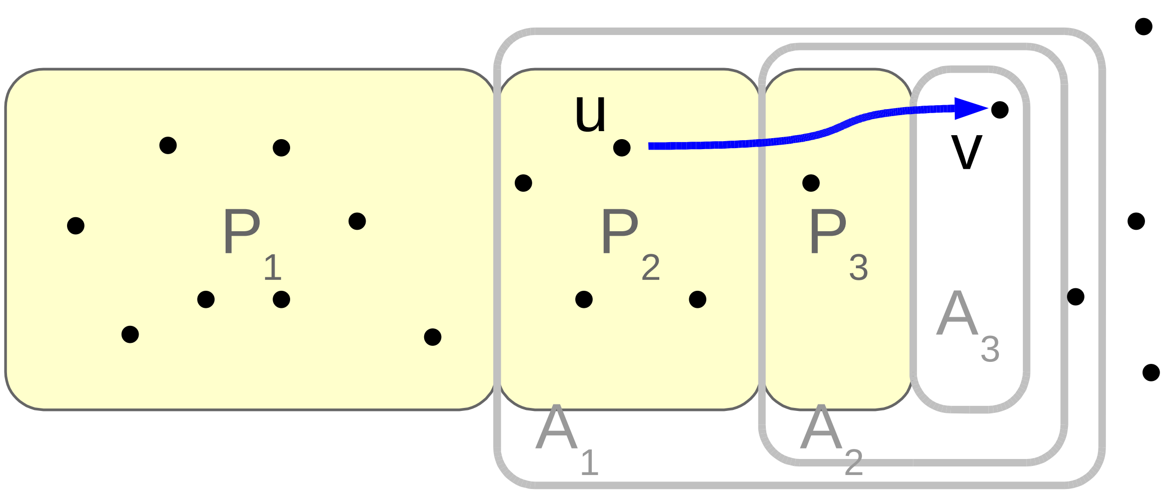
- $|P_1| > .. > |P_k|$
- and each $|P_i|$ is a power of 2
- Suppose for every P_i we computed A_i : the $|P_i|$ cheapest border edges of P_i with distinct targets (for edges with source $\in P_i$ and target $\notin P_1 \cup .. \cup P_i$)



Observations

- $A_i \cap S x(V \setminus S)$ (restricted to targets$\notin S$) is non empty for every i
- The cheapest border edge of S (u,v) has its source $u \in P_i$ for some i, and therefore $v \in A_i$
- Thus $(A_1 \cup .. \cup A_k) \cap S x(V \setminus S)$ contains the cheapest border edge of S



Computing A_k using a minimum search procedure

Input

matrix $\mathbb{N}^{a \times b}$

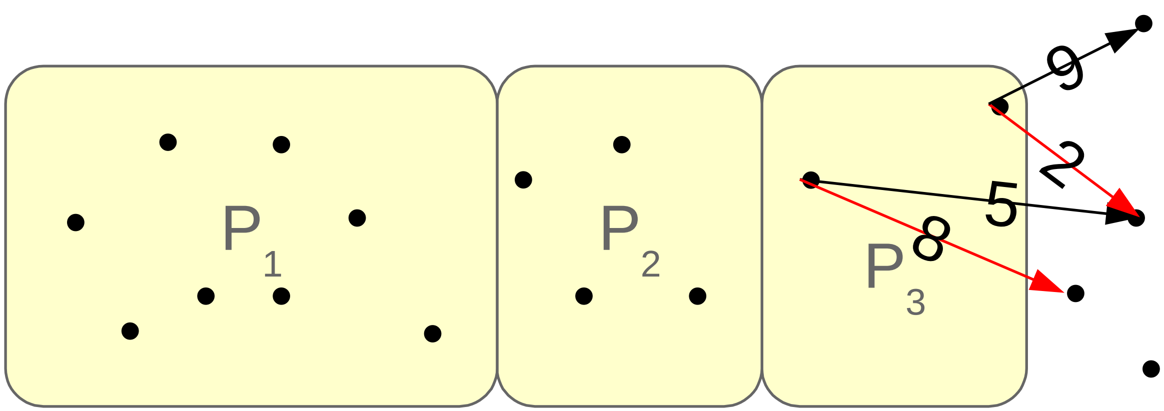
Output

a column disjoint minimal entries

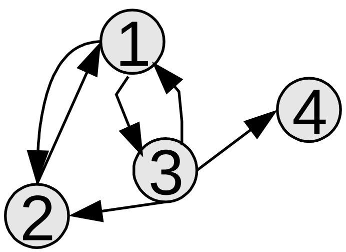
Bounded error quantum query complexity

$\theta(a\sqrt{b})$

8	5	∞	∞
∞	2	9	∞



Bounded error quantum query complexity



Adjacency matrix model

	1:	2:	3:	4:
1:	0	1	1	0
2:	1	0	0	0
3:	1	1	0	1
4:	0	0	0	0

Adjacency array model

1: 2 3
2: 1
3: 1 4 2
4:

Bounded error (classical) quantum query complexity

	Adjacency matrix (classical)	Adjacency matrix (quantum)	Adjacency array (classical)	Adjacency array (quantum)
Single source shortest paths	$\theta(n^2)$	$\omega(n^{3/2})$, $O(n^{3/2}\log^2 n)$	$\theta(m)$	$\omega(\sqrt{(nm)})$, $O(\sqrt{(nm)}\log^2 n)$
Minimum weight spanning tree	$\theta(n^2)$	$\theta(n^{3/2})$	$\theta(m)$	$\theta(\sqrt{(nm)})$
Connectivity (undirected graph)	$\theta(n^2)$	$\theta(n^{3/2})$	$\theta(m)$	$\theta(n)$
Strong Connectivity (directed graph)	$\theta(n^2)$	$\theta(n^{3/2})$	$\theta(m)$	$\omega(\sqrt{(nm)})$, $O(\sqrt{(nmlogn)})$