Reconstructing hv-Convex Polyominoes from Orthogonal Projections

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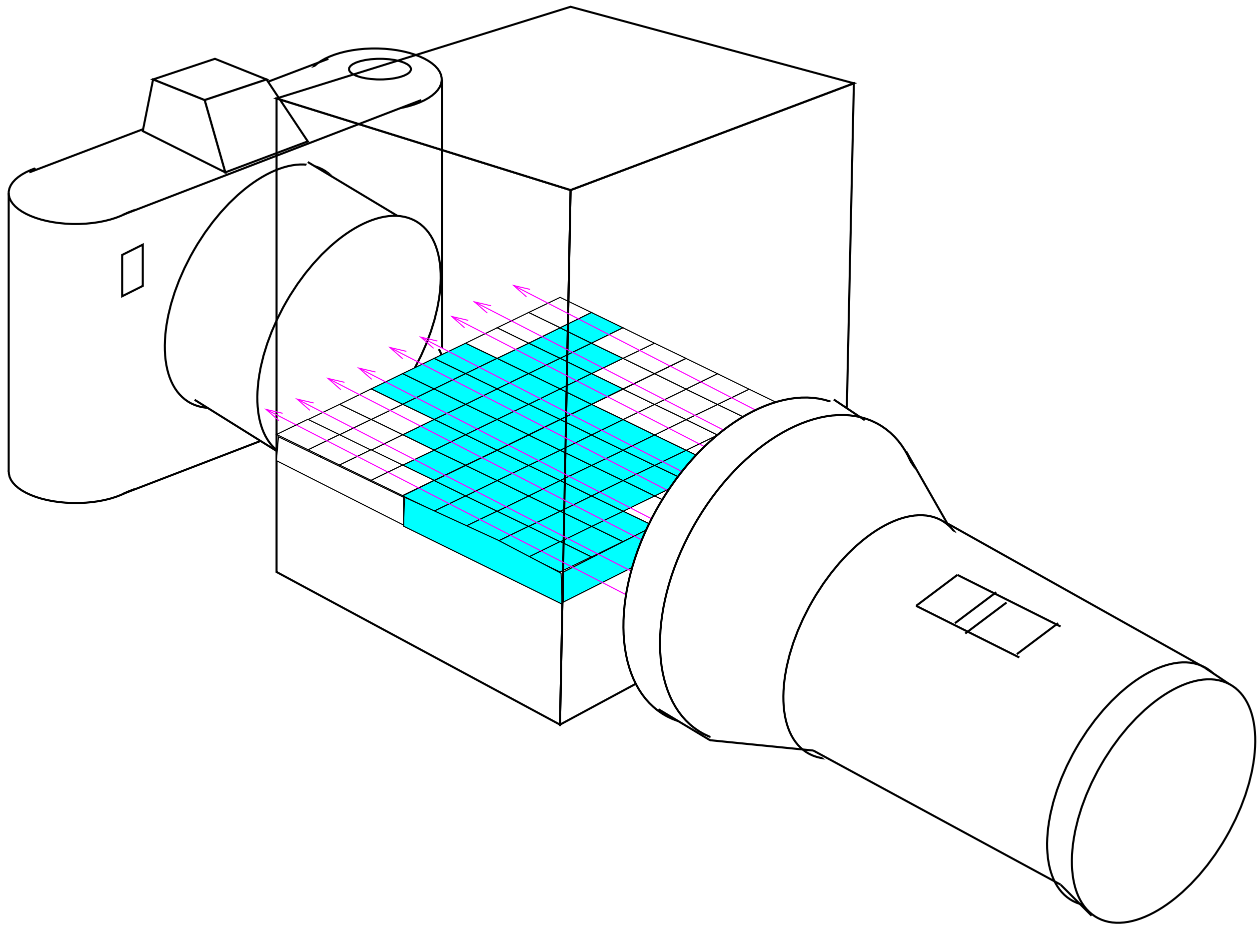
Discrete Tomography

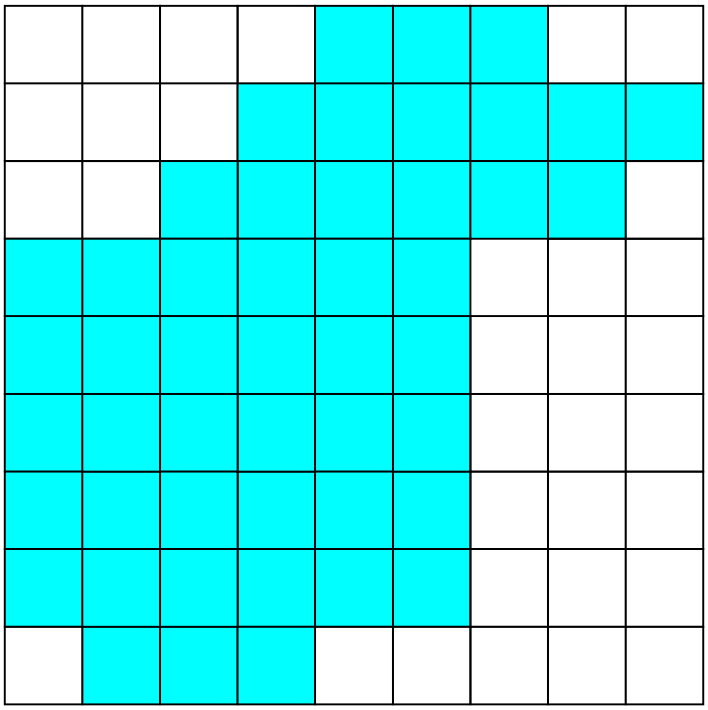
High Resolution Transmission Electron Microscope..



..counts # of atoms of each type on each projection line

The problem

Output



Input

	5	6	7	8	8	8	3	2	1
3									
6									
6									
6									
6									
6									
6									
6									
3									

Formal

$T \in \{0,1\}^{m \times n}$, $rowsum_i(T) = \sum_j T_{i,j}$ and $colsum_j(T) = \sum_i T_{i,j}$

Input

vectors $r \in N^m$ and $c \in N^n$

Assume w.l.o.g.

$r_i \leq n$, $c_j \leq m$, and $\sum_i r_i = \sum_j c_j$

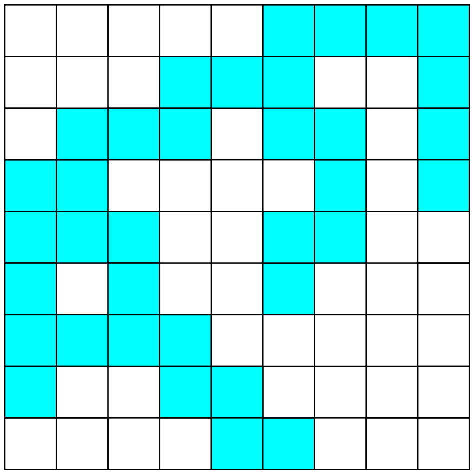
Output

T such that

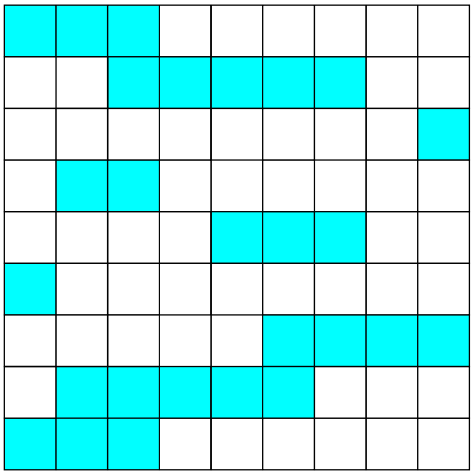
$$r_i = rowsum_i(T) \text{ and } c_j = colsum_j(T)$$

Properties

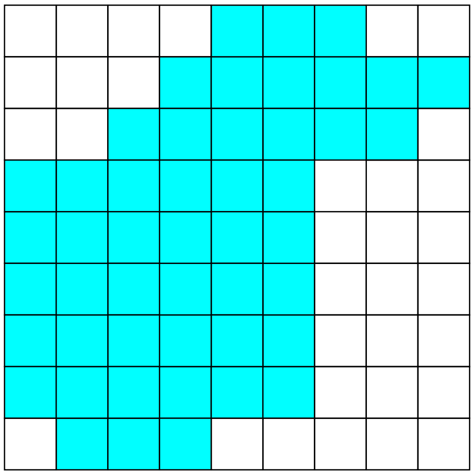
polyomino



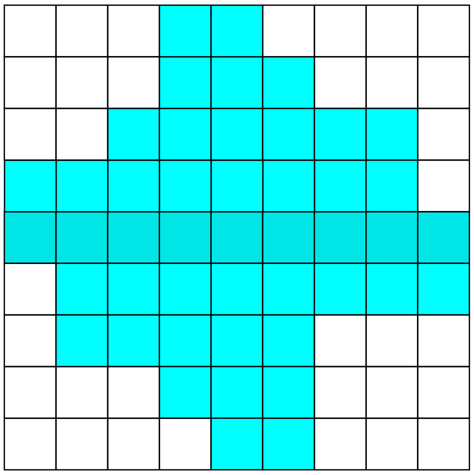
h-convex



hv-convex polyomino



centered hv-convex polyomino



Complexity of reconstruction problem

for a $m \times n$ grid

	-	polyomino
-	$O(m+n)$	NP-complete
	[Ryser63]	[Woeginger96]
h-convex	NP-complete	NP-complete
	[Barcucci..96]	[Barcucci..96]
hv-convex	NP-complete	
	[Woeginger96]	

(previous work)

$O(n^4m^4)$

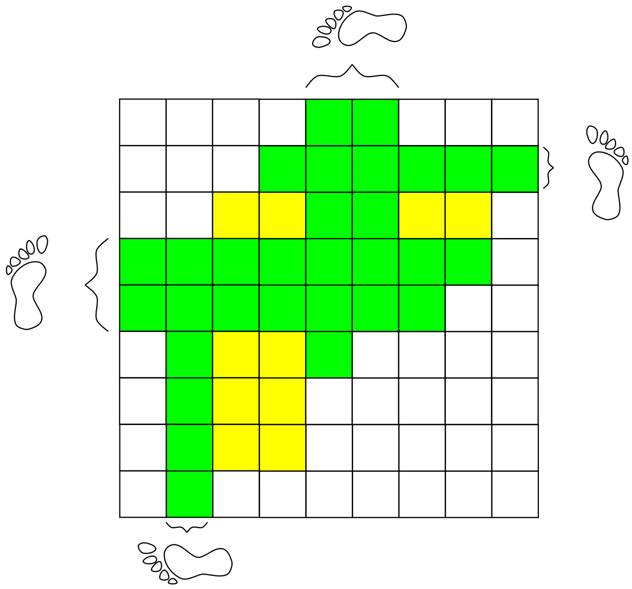
[Barcucci..96]]

(our contribution)

	$O(mn\min(n^2, m^2))$
	[*]
centered	$O(m+n)$
hv-convex	[*]

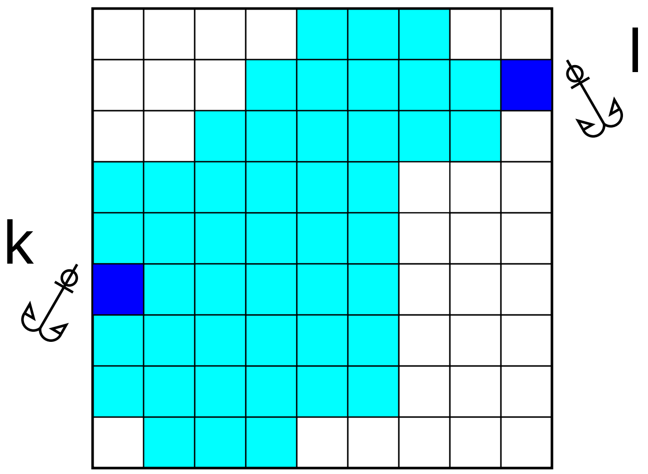
Previous algorithm

for reconstructing hv-convex polyominoes was $O(n^4m^4)$ [Barcucci..96].



Guess 4 feet positions,
apply filling operations (augments kernel)
and reduction operations (shrinks shell)

Our algorithm



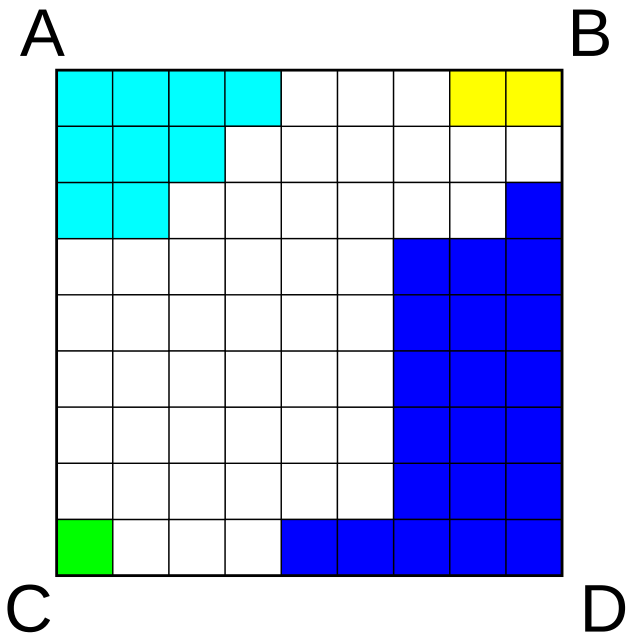
2SAT Formula $F_{k,l}$

is true if and only if there is a realization anchored at k, l.

Algorithm

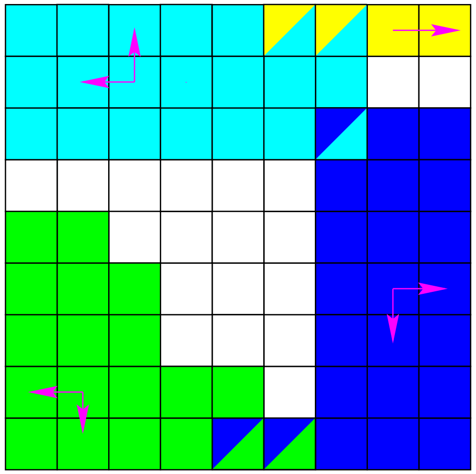
for every k,l: verify if $F_{k,l}$ is satisfiable

Decomposition into corner regions



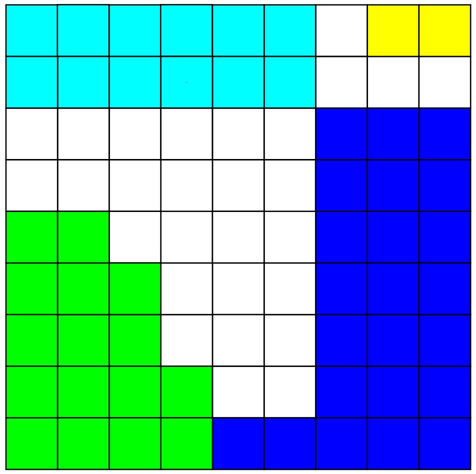
Corners

$$Cor = \bigwedge_{i,j} \left\{ \begin{array}{ll} A_{i,j} \Rightarrow A_{i-1,j} & A_{i,j} \Rightarrow A_{i,j-1} \\ B_{i,j} \Rightarrow B_{i-1,j} & B_{i,j} \Rightarrow B_{i,j+1} \\ C_{i,j} \Rightarrow C_{i+1,j} & C_{i,j} \Rightarrow C_{i,j-1} \\ D_{i,j} \Rightarrow D_{i+1,j} & D_{i,j} \Rightarrow D_{i,j+1} \end{array} \right\}$$



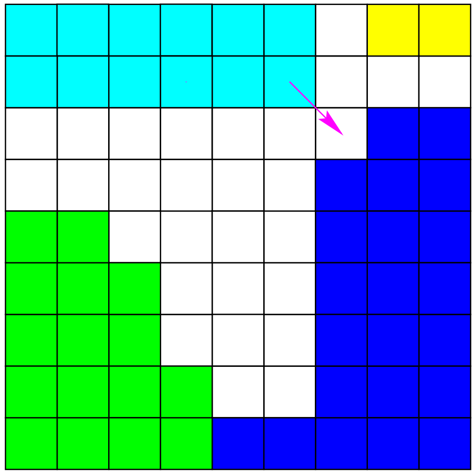
Disjointness

$$Dis = \bigwedge_{i,j} \{X_{i,j} \Rightarrow \overline{Y}_{i,j} : \text{ for distinct symbols } X, Y \in \{A, B, C, D\}\}$$



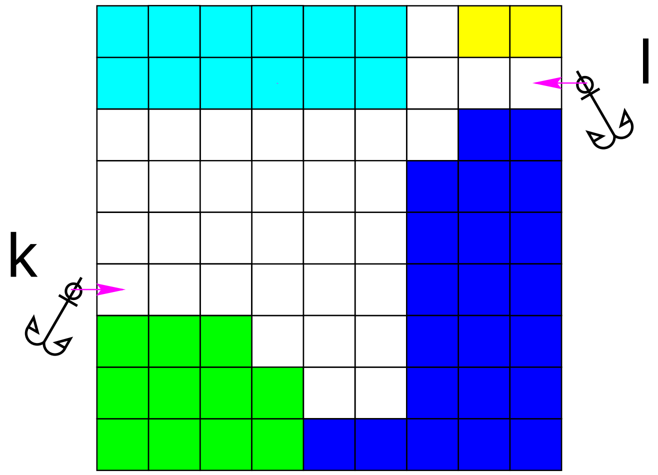
Connectivity

$$Con = \bigwedge_{i,j} \left\{ A_{i,j} \Rightarrow \overline{D}_{i+1,j+1} \quad B_{i,j} \Rightarrow \overline{C}_{i+1,j-1} \right\}$$



Anchors

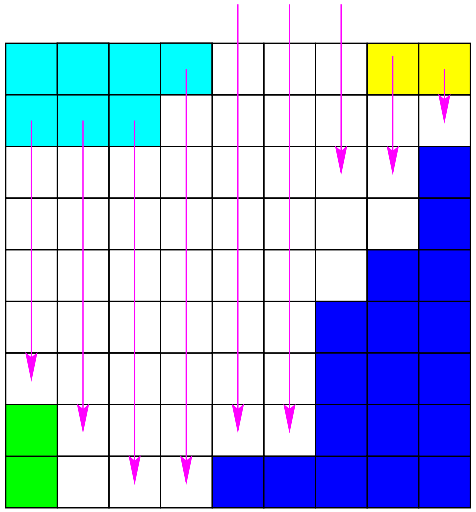
$$Anc = \left(\overline{A}_{k,1} \wedge \overline{B}_{k,1} \wedge \overline{C}_{k,1} \wedge \overline{D}_{k,1}\right) \wedge \left(\overline{A}_{l,n} \wedge \overline{B}_{l,n} \wedge \overline{C}_{l,n} \wedge \overline{D}_{l,n}\right)$$



Lower bound on column sums

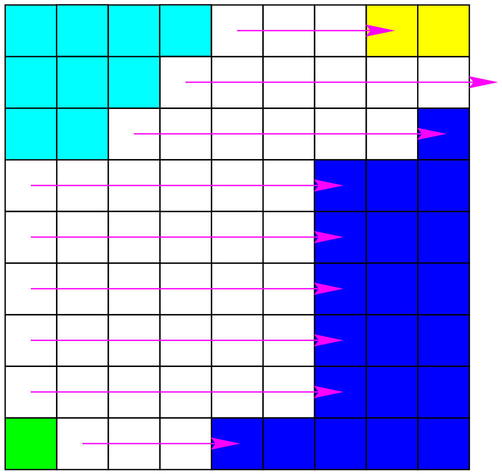
$$LBC = \bigwedge_{i,j} \{(A_{i,j} \vee B_{i,j}) \Rightarrow (\overline{C}_{i+c_j,j} \wedge \overline{D}_{i+c_j,j})\} \wedge \bigwedge_{j} \{\overline{C}_{c_j,j} \quad \overline{D}_{c_j,j}\}$$

$$= \bigwedge_{i,j} \left\{ \begin{array}{ll} A_{i,j} \Rightarrow \overline{C}_{i+c_j,j} & A_{i,j} \Rightarrow \overline{D}_{i+c_j,j} \\ B_{i,j} \Rightarrow \overline{C}_{i+c_j,j} & B_{i,j} \Rightarrow \overline{D}_{i+c_j,j} \end{array} \right\} \wedge \bigwedge_{j} \{\overline{C}_{c_j,j} \quad \overline{D}_{c_j,j}\}$$



Upper bound on row sums

$$\begin{aligned} UBR &= \bigwedge_{i,j} \{(\overline{A}_{i,j} \wedge \overline{C}_{i,j}) \Rightarrow (B_{i,j+r_i} \vee D_{i,j+r_i})\} \\ &= \bigwedge_{j} \left\{ \begin{array}{cc} \bigwedge_{i\le\min\{k,l\}} \overline{A}_{i,j} \Rightarrow B_{i,j+r_i} & \bigwedge_{k\le i\le l} \overline{C}_{i,j} \Rightarrow B_{i,j+r_i} \\ \bigwedge_{l\le i\le k} \overline{A}_{i,j} \Rightarrow D_{i,j+r_i} & \bigwedge_{\max\{k,l\}\le i} \overline{C}_{i,j} \Rightarrow D_{i,j+r_i} \end{array} \right\} \end{aligned}$$



Reduction

$$F_{k,l} = Cor \wedge Dis \wedge Con \wedge Anc \wedge LBC \wedge UBR$$

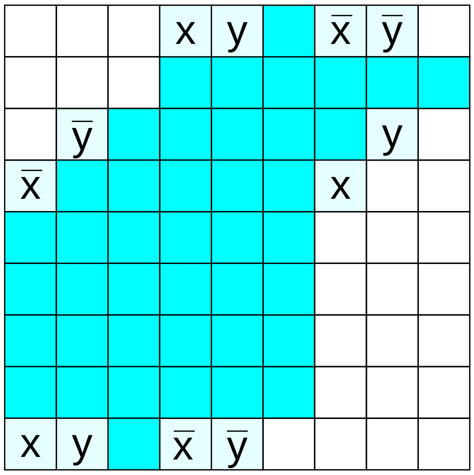
Thm There is a solution anchored at k, l if and only if $F_{k,l}$ is satisfied.

Idea Suppose $F_{k,l}$ is satisfied. Set $T = \overline{A \vee B \vee C \vee D}$, then

$$\sum_i r_i \geq \sum_i rowsum_i(T) = \sum_j colsum_j(T) \geq \sum_j c_j.$$

Since we assumed $\sum_i r_i = \sum_j c_j$, T is a solution.

Multiple solutions



$$x \Rightarrow y$$

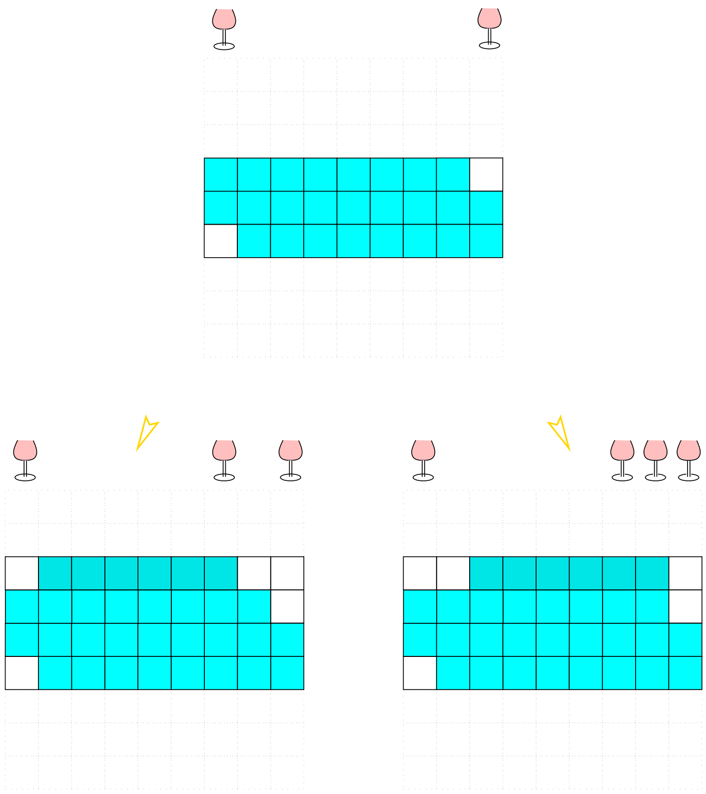
Improvements

- Choose k to be a multiple of c_1 (l a multiple of c_n)
- dynamic 2SAT:
  - Given a set of very similar formulas,
  - decide if at least one is satisfiable

Partial realization

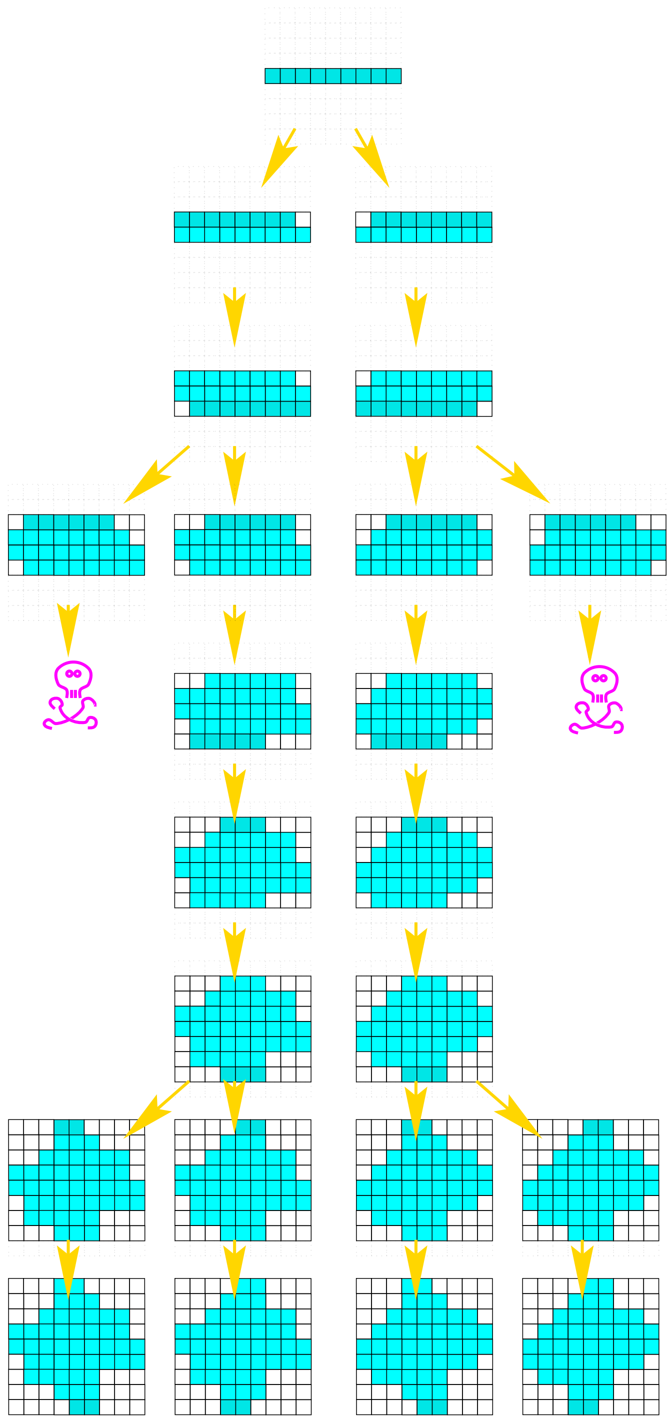
Column j is saturated ( ) in P if $colsum_j(P) = c_j$.

Add rows in order of decreasing sums:



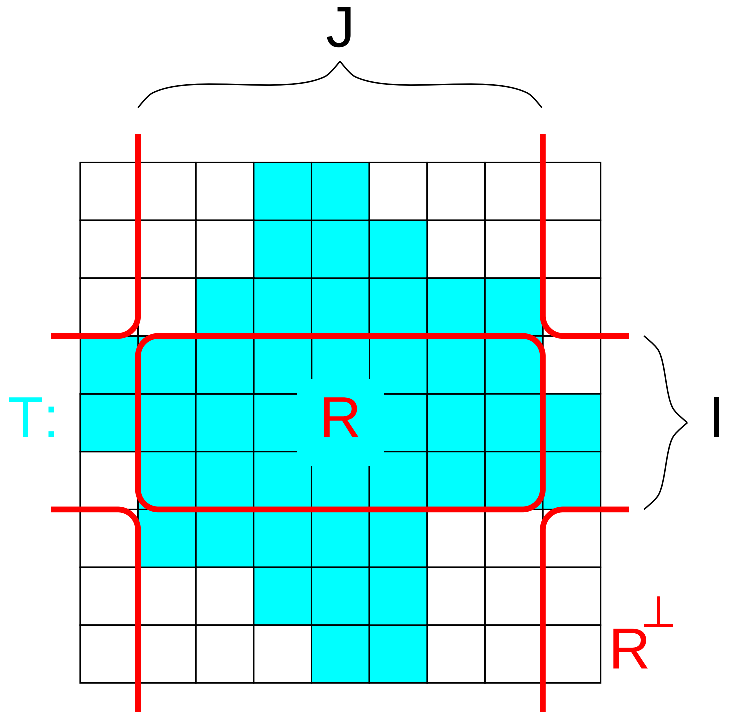
Lemma there are at most two possible positions

Explore the tree



Rectangles

$R = I \times J$ and $\sum_{i \in I} r_i = \sum_{j \notin J} c_j + |I| \cdot |J|$.

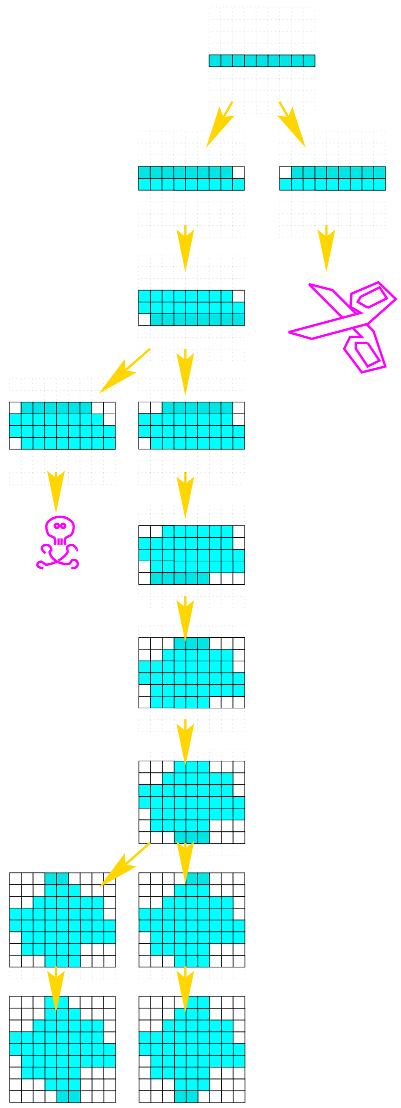


Lemma then every realization T of (r, c) satisfies

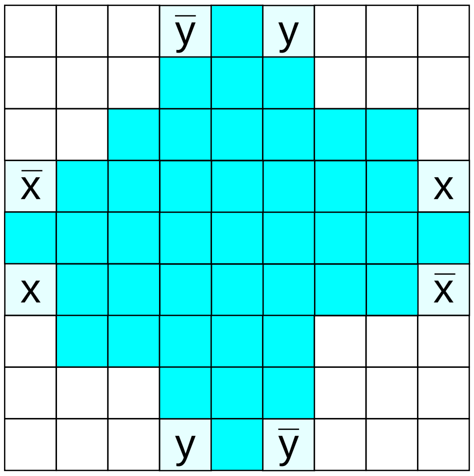
$$R \subseteq T \text{ and } R^{\perp} \cap T = \emptyset$$

Prune the tree

There are two possibilities for a row iff the partial realization contains a rectangle. Thus we can discard all other nodes in the tree. Thus at every level we have at most two nodes.



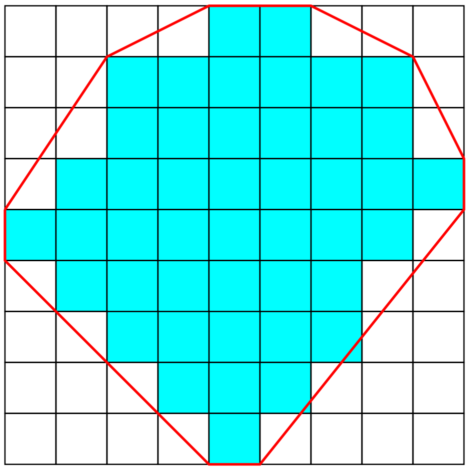
Multiple solutions



Number of solutions is always a power of 2

The real problem

Reconstructing truly convex polyominoes



is still open.