A decision procedure for unitary quantum cellular automata

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Vocabulary

$\mathcal{C}$: countable set of configurations

probability $p \in [0,1]$	amplitude $\alpha \in C$
distribution $u \in [0,1]^{\mathcal{C}}$	superposition $u \in C^{\mathcal{C}}$
$\lvert u \rvert = \sum_{c \in \mathcal{C}} u_c = 1$	$\lVert u \rVert = \sqrt{\sum_{c \in \mathcal{C}} \lvert u_c \rvert^2} = 1$

Probability of observation

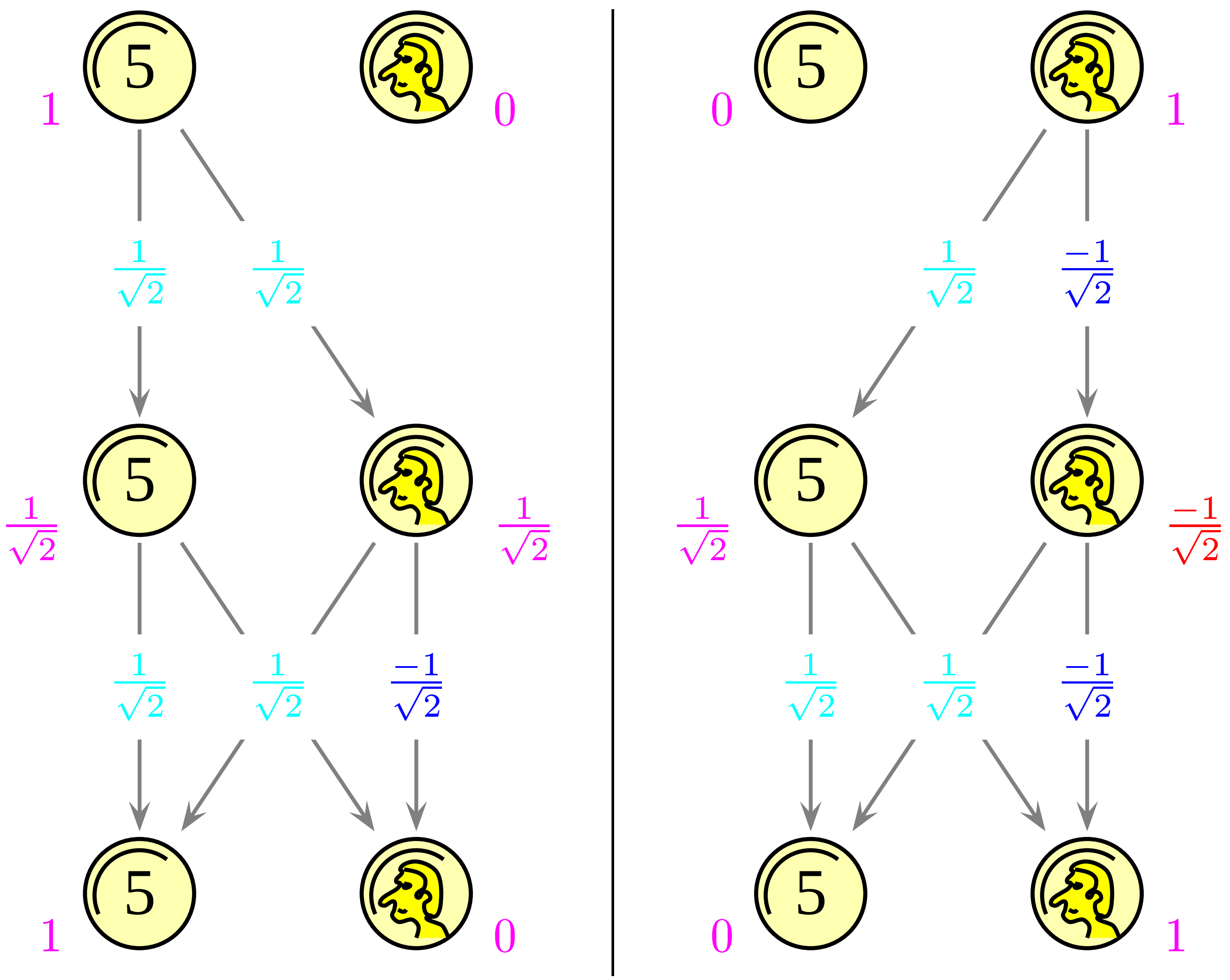
u_c	$\lvert u_c \rvert^2$

Linear evolution operator

stochastic matrix	unitary matrix
(preserves norm $\lvert \cdot \rvert$)	(preserves norm $\lVert \cdot \rVert$ and is surjective)

Quantum Flip

$$M = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$



Unitary matrices

A matrix is unitary if and only if

1. is is isometric (preserves the norm $\|\cdot\|$)

2. and it is surjective

- the column vectors have norm 1
- and the column vectors are mutually orthogonal.

- the line vectors have norm 1.

Example

of an isometric matrix which is not unitary.

$$\begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \\ \vdots & & & \ddots \end{pmatrix}$$

Our contribution

A cubic algorithm for

Problem "valid finite QCA"

Input: description of a finite QCA

Question: Is the evolution operator unitary?

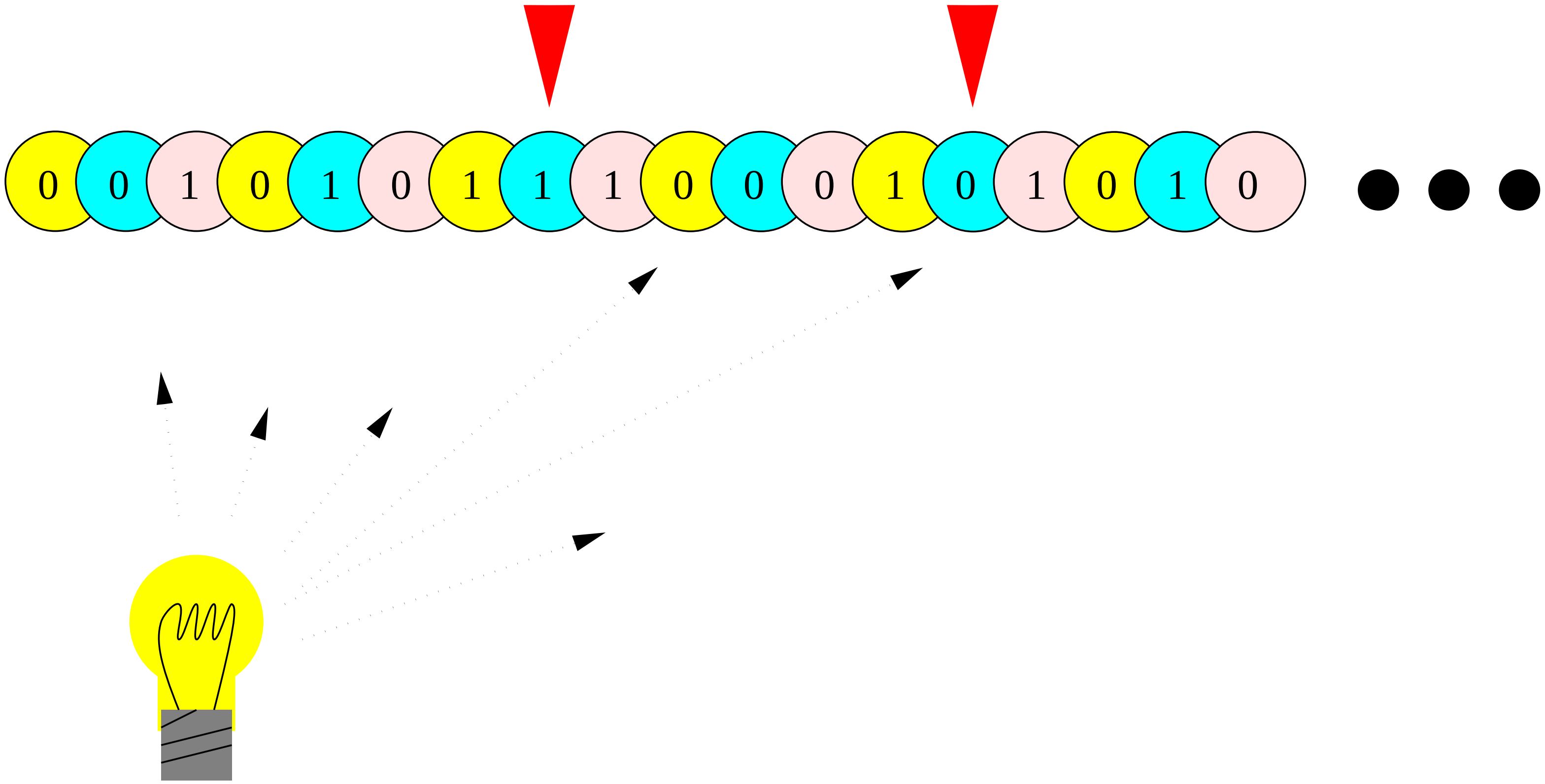
- Our algorithm checks whether

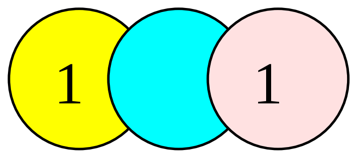
  1. the column vectors have norm 1
  2. the column vectors are mutually orthogonal

3. the line vectors have norm 1

A possible realization of a QCA

[1993 Lloyd] based on [1988 Oberteich, Teich, Mahler]

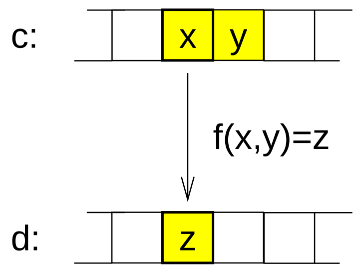


Flash light changes all atoms matching 

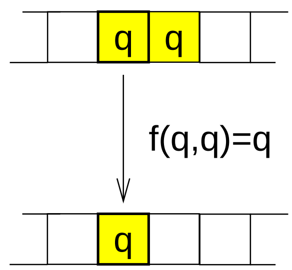
Finite cellular automaton

Q : finite state set

$f : Q^2 \to Q$: local transition function



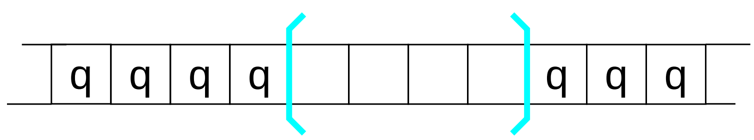
q : quiescent state, such that



Global transition function

Set of configurations

$$\mathcal{C} = \{c \in Q^{Z} : |\{i : c_i \neq q\}| < \infty\}$$



$F : \mathcal{C} \to \mathcal{C}$ induced by f

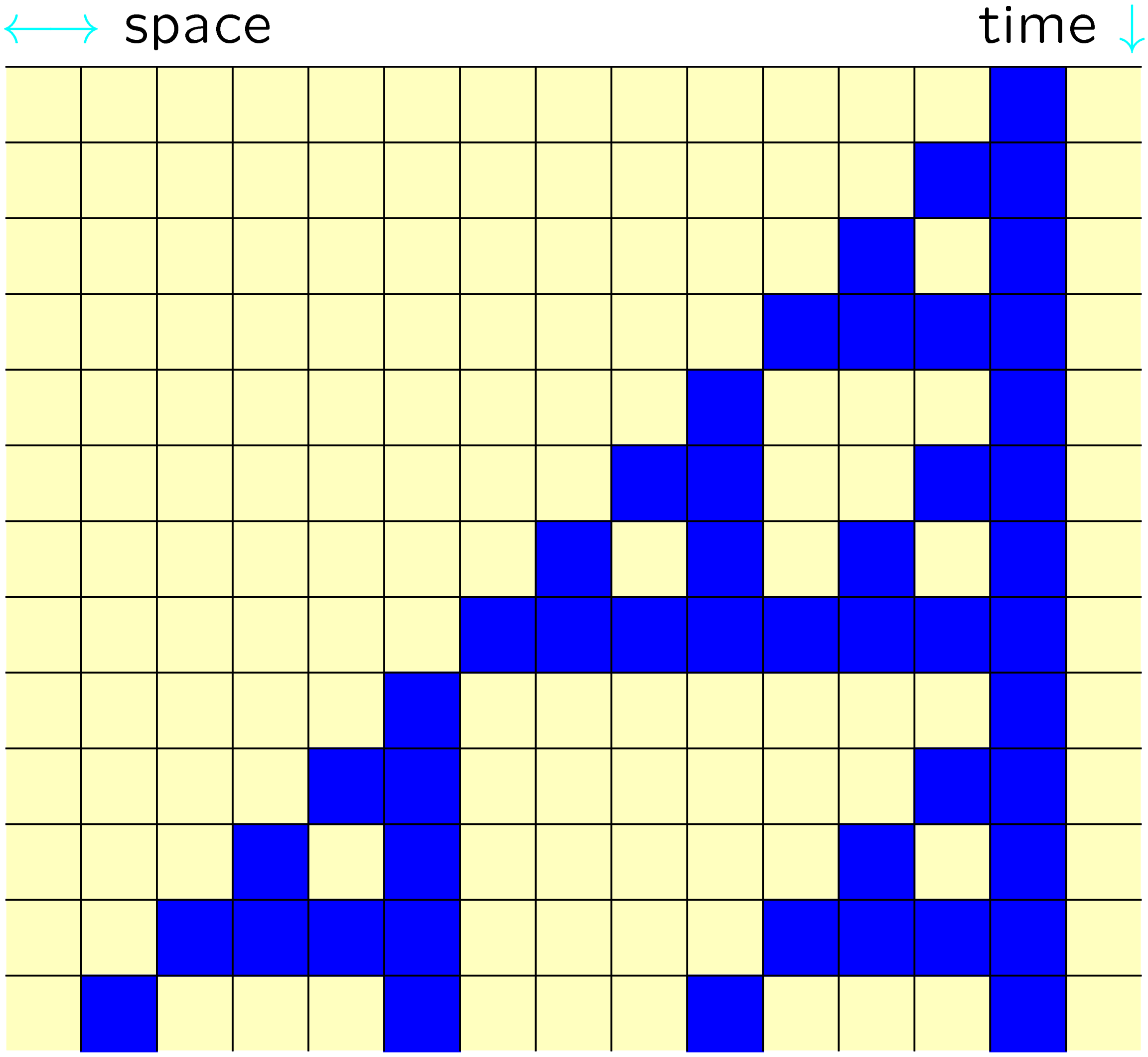
$F : c \mapsto d$

Example of a finite CA

exclusive OR

$Q = \{\square, \blacksquare\},\ q = \square$

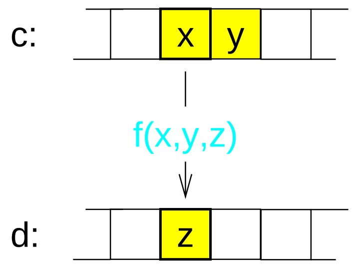
$$f(x,y) = \begin{cases} \square & \text{if } y = x, \\ \blacksquare & \text{if } y \neq x. \end{cases}$$



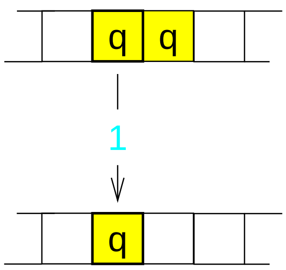
Finite quantum cellular automaton

Q : finite state set

$f : Q^2 \times Q \to C$: determines the amplitude of a local transition



q : quiescent state, such that



$$f(q \ldots q,\, x) = \begin{cases} 1 & \text{if } x = q, \\ 0 & \text{otherwise.} \end{cases}$$

Global evolution operator

$U : \mathcal{C} \times \mathcal{C} \to C$

$$U(d, c) = \prod_{i \in Z} f(c_i, c_{i+1},\, d_i)$$

Example

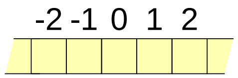
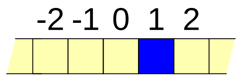
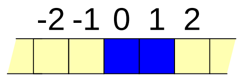
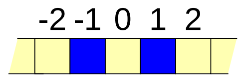
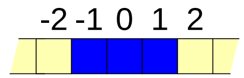
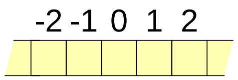
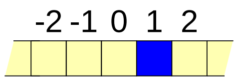
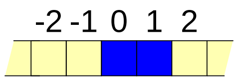
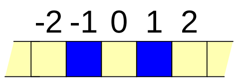
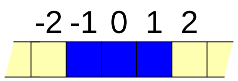
Controlled Quantum Flip

$Q = \{□, ■\}$

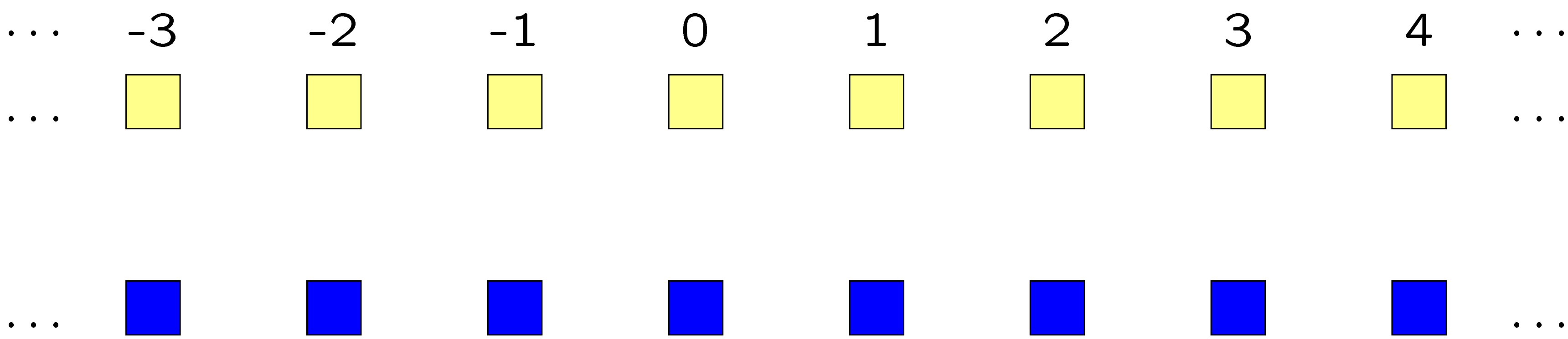
Local function f :

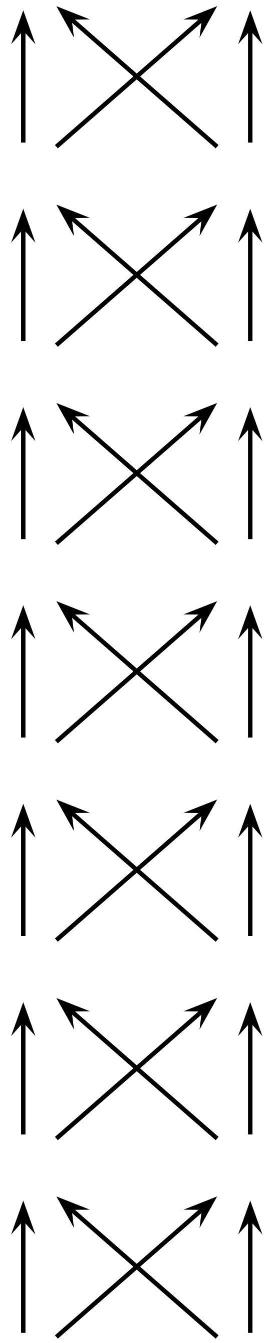
	□	■
□□	1	0
■□	0	1
□■	$1/\sqrt{2}$	$1/\sqrt{2}$
■■	$1/\sqrt{2}$	$-1/\sqrt{2}$

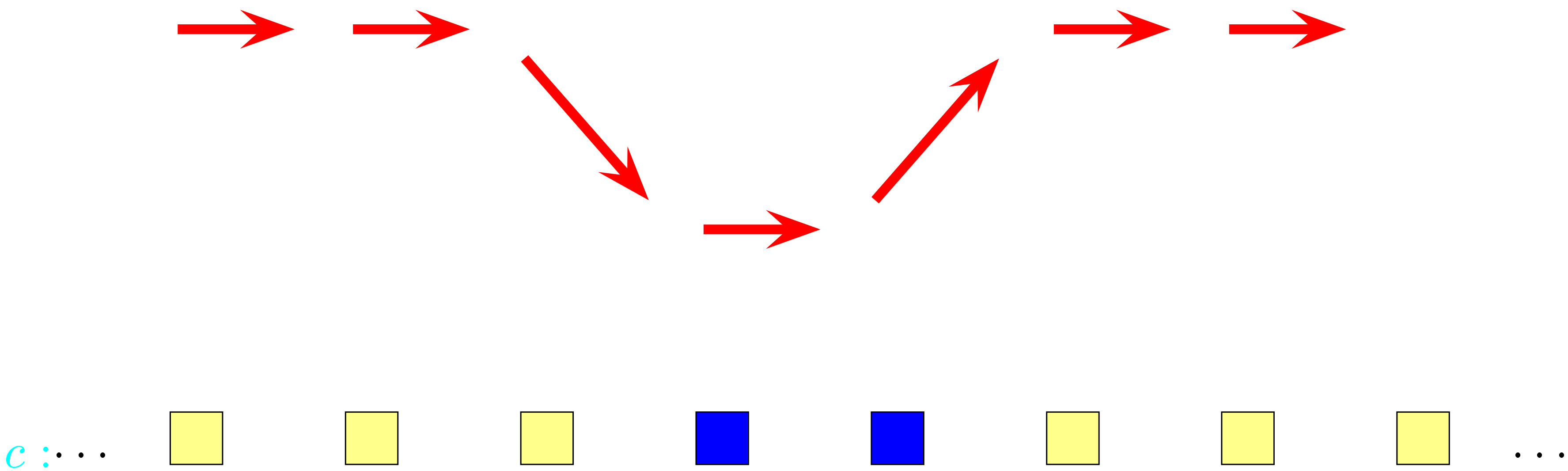
Evolution operator

						$\cdots$
	1	0	0	0	0	$\cdots$
	0	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$\left(\frac{1}{\sqrt{2}}\right)^3$	$\cdots$
	0	$\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	0	$-\left(\frac{1}{\sqrt{2}}\right)^3$	$\cdots$
	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\left(\frac{1}{\sqrt{2}}\right)^3$	$\cdots$
	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\left(\frac{1}{\sqrt{2}}\right)^3$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

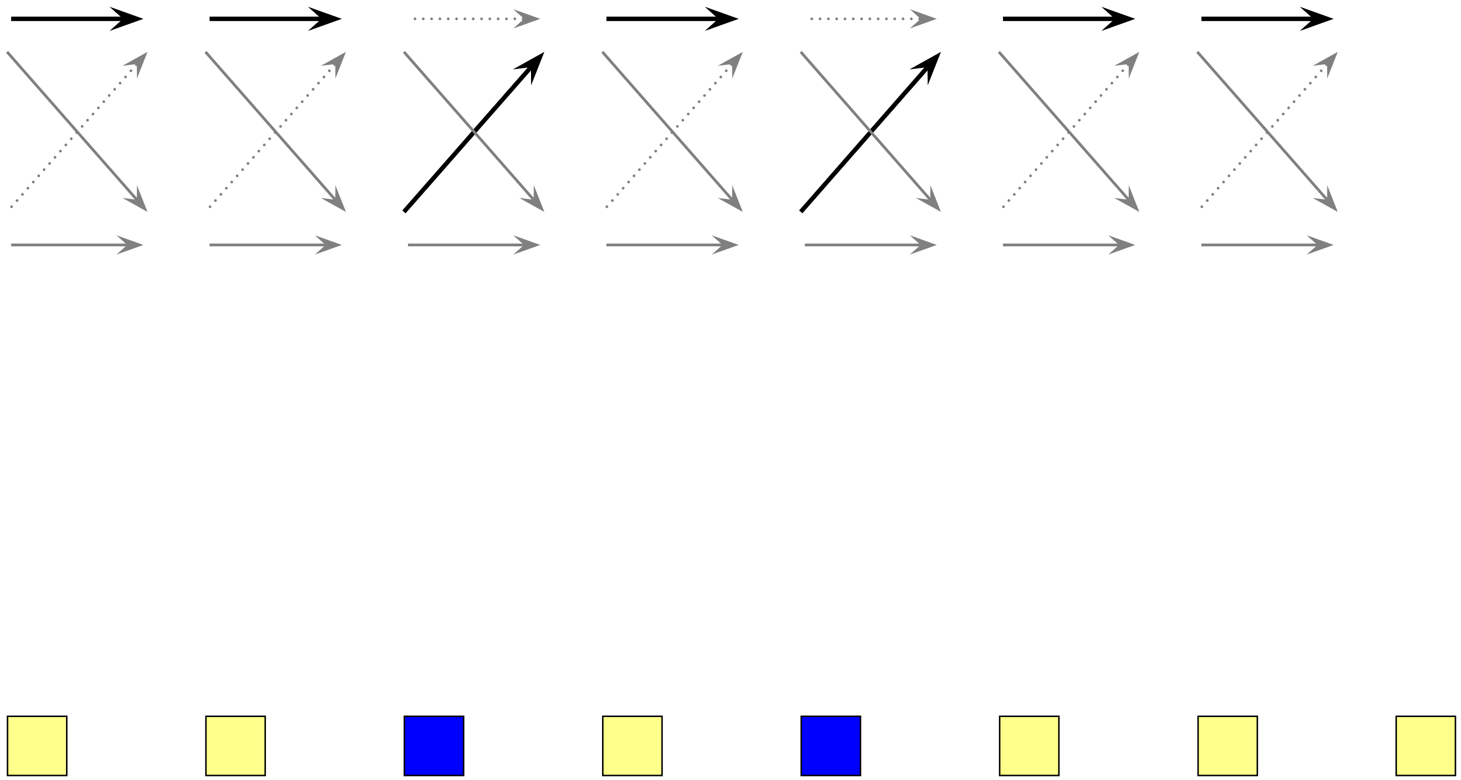
Configuration graph






c : . . .
. . .

Weighted G_d



$d : \dots$ $\dots$

1	1
1	0

$$M_{\square} = \begin{pmatrix} 1 & 0 \\ 1/2 & 1/2 \end{pmatrix} \qquad M_{\blacksquare} = \begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix}$$

Problems

Vector space $R^{|Q|}$

Border vectors

Compute $\vec{l}$ and $\vec{r}$

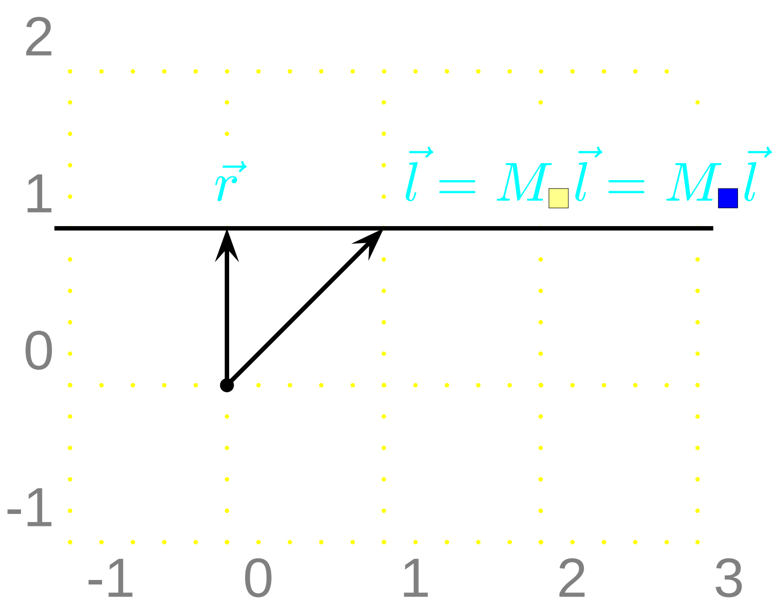
Closed affine hyperplane

Input: vectors $\vec{l}, \vec{r}$

linear applications $M_{□}, M_{■}$

Question: Do we have for every finite sequence $x, \ldots, z \in \{□, ■\}$

$$\langle M_x \ldots M_z \vec{l} | \vec{r} \rangle = 1?$$



“Quantum cellular automata”

The name QCA has been used for different models:

Classical circuits made out of ‘quantum dots’
[1993 Lent, Tougaw, Porod, Bernstein]

Hamiltonian operator (QCA described by a)
[1994 Biafore, Margolus]

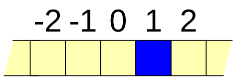
Evolution operator (QCA described by an)
[1995 Watrous, Dürr, LêThanh, Santha, van Dam]

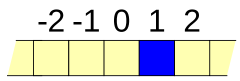
Inverse the computation

[1972 Richardson] or [Hedlund 69]

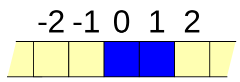
The inverse of a bijective (finite) CA is also a (finite) CA.

Counterexample

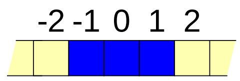
$U^{-1}|$ 

$\rangle = \frac{1}{\sqrt{2}} \; |$ 

$\rangle$

$+ \left(\frac{1}{\sqrt{2}}\right)^2 \; |$ 

$\rangle$

$+ \left(\frac{1}{\sqrt{2}}\right)^3 \; |$ 

$\rangle$

$+ \quad \ldots$

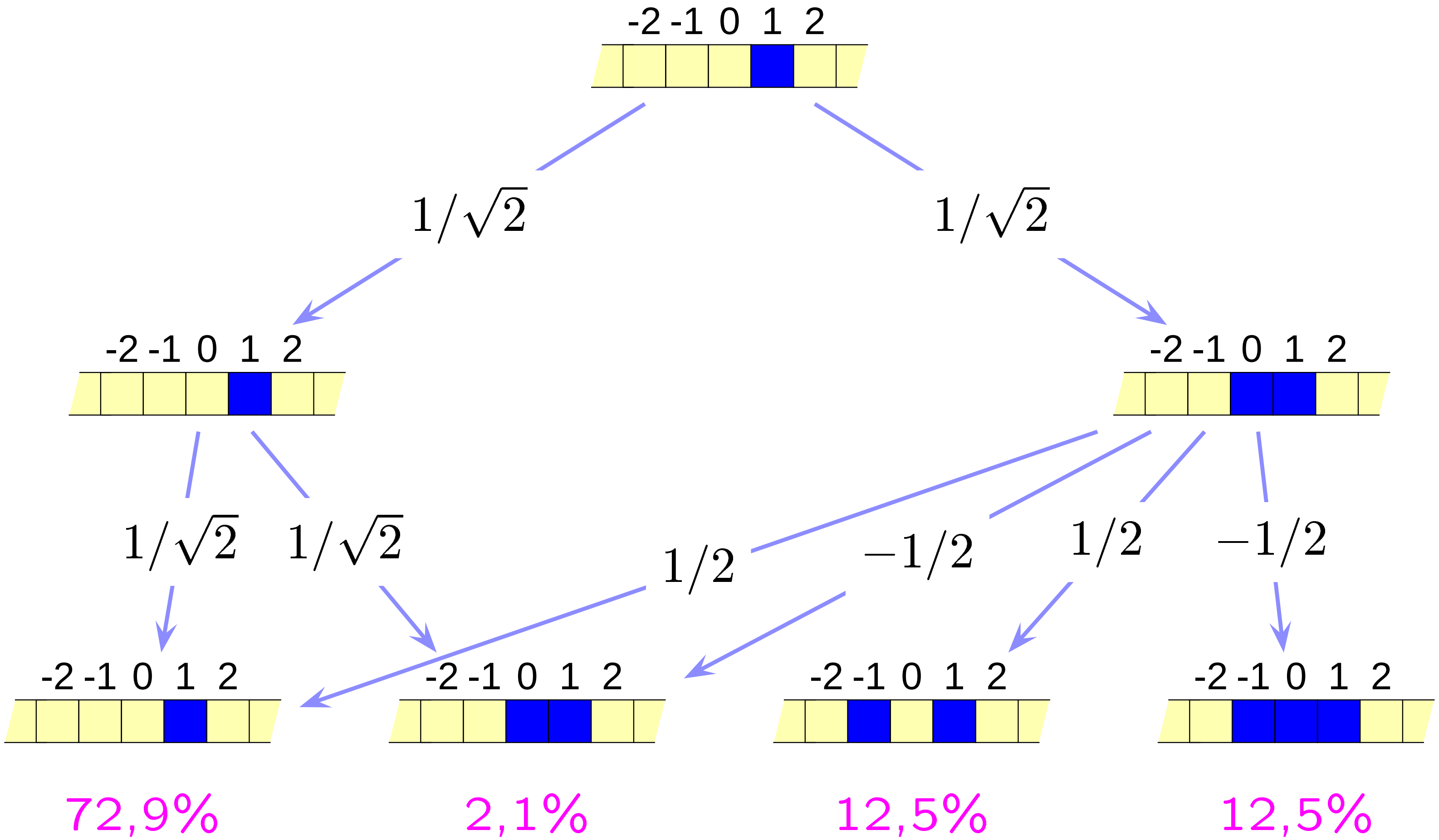
Define a QCA without restrictions on the configurations

- $\mathcal{C}$ is not countable, so U cannot be written in matrix form.
- $U(d, c)$ can vanish to 0.

Solution

Use measure theory.

Evolution



Rounded probabilities of observation