

TADI: Wavelets
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Content

Part 1: Fourier Transform, Short Time Fourier Transform

Recall: vector space spaces and important properties to know

Fourier transform

Short time Fourier Transform

Part 2: Wavelets

Part 3: discrete wavelet transform for images, applications

Vector space (1)

- ▶ **Field:** $(\mathbb{K}, +, \cdot)$ a set with two operations (internal composition laws, denoted $+$ and $\cdot$)
In general and in this lecture $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and such as $+$ is commutative ($\forall \lambda, \mu \in \mathbb{K}, \lambda + \mu = \mu + \lambda$), 0 is the neutral element for $+$ and 1 for $\cdot$.
 - ▶ internal law: $\forall x, y \in \mathbb{K}, x + y \in \mathbb{K}$
 - ▶ neutral element: $\forall x \in \mathbb{K}, x + 0 = x$
- ▶ **Vector space:** $(E, +, \cdot)$ is a vector space over the field $\mathbb{K}$ if:
 - ▶ $\mathbb{K}$ is a field (two internal composition laws also denoted $+$ and $\cdot$ by abuse of language)
 - ▶ $+$ is an internal commutative law on E : $E \times E \rightarrow E$ (vector addition)
 - ▶ $\cdot$ is an external law (left multiplication): $\mathbb{K} \times E \rightarrow E$ (also called multiplication by a scalar) such as:
 - ▶ $\cdot$ is distributive over $+$: $\forall \lambda \in \mathbb{K}, \forall v, w \in E, \lambda \cdot (v + w) = \lambda \cdot v + \lambda \cdot w$
 - ▶ $+$ is distributive over $\cdot$: $\forall \lambda, \mu \in \mathbb{K}, \forall v \in E, (\lambda + \mu) \cdot v = \lambda \cdot v + \mu \cdot v$
 - ▶ 1 is the left neutral element of $\cdot$: $\forall v \in E, 1 \cdot v = v$
 - ▶ An element v of E is a vector, in the remaining E is a vector space

Vector space (2)

- ▶ **Vector subspace:** $F \subset E$ is a vector subspace of E if:
 - ▶ $F \neq \emptyset$
 - ▶ $\forall (\lambda, v, w) \in \mathbb{K} \times F \times F, \lambda \cdot (v + w) = \lambda \cdot v + \lambda \cdot w \in F,$
- ▶ In other words: F is stable for linear combination
- ▶ Example of vector spaces:
 - ▶ $(\mathbb{R}^n, +, \cdot), (\mathbb{R}^{\mathbb{N}}, +, \cdot)$
 - ▶ The set of continuous functions from $\mathbb{R}$ into $\mathbb{C}$ is an $\mathbb{C}$ – vector space (it is of infinite dimension)
- ▶ **Scalar product:** (or dot product, or inner product) the operation, denoted $\langle \cdot, \cdot \rangle$, such as:

$$\begin{aligned} E \times E &\rightarrow \mathbb{R} \\ (v, w) &\mapsto \langle v, w \rangle \end{aligned}$$

is a scalar product if

- ▶ bilinear (linear on left, linear on right)
 - ▶ symmetric: $\langle v, w \rangle = \langle w, v \rangle$
 - ▶ positive: $\langle v, v \rangle \geq 0$
 - ▶ definite: $\langle v, v \rangle = 0 \Rightarrow v = 0$
- ▶ **Norm:** the scalar product defines the norm $\|v\|^2 = \langle v, v \rangle$

Scalar product

- ▶ A fundamental operation: it allows two vectors to be compared, projecting one to another one
- ▶ Example of scalar product:
 - ▶ in $\mathbb{R}^n$: $v = (v_1, \dots, v_n)$, $w = (w_1, \dots, w_n)$ and

$$\langle v, w \rangle = \sum_{i=1}^n v_i \cdot w_i$$

- ▶ for the set of complex summable (or integrable) functions on $\mathbb{R}$:

$$\langle f, g \rangle = \int_{\mathbb{R}} f(t) \bar{g}(t) dt$$

- ▶ Euclidean space: a vector space with a scalar product
- ▶ Hilbert space: an Euclidean space of infinite dimension (space of functions)

Basis (1)

- ▶ A basis in E is a finite or countable (if E is of infinite dimension) set of vectors of E : $\mathcal{B} = \{b_1, \dots, b_n, \dots\}$ satisfying two conditions:
 1. linear independence property (free family): no element of $\mathcal{B}$ is a linear combination of others elements of $\mathcal{B}$:
$$\lambda_1 b_1 + \dots + \lambda_n b_n = 0 \Rightarrow \lambda_1 = \dots = \lambda_n = 0$$
 2. spanning property (spanning family): $\forall v \in E, \exists \lambda_1, \dots, \lambda_n, \dots$ such as $v = \sum_i \lambda_i b_i$
- ▶ **Orthogonal** basis: $\langle b_i, b_j \rangle = 0 \quad \forall i \neq j$
- ▶ **Orthonormal** basis: $\langle b_i, b_j \rangle = 0 \quad \forall i \neq j$ **and** $\langle b_i, b_i \rangle = 1 \quad \forall i$

Basis (2)

- ▶ Example in the Cartesian plane with the usual scalar product
 - ▶ the set reduced to the canonical vector $\vec{i} = (1 \ 0)$: linearly independent set
 - ▶ $\{\vec{i}, \vec{j}, \vec{i} + \vec{j}\}$: spanning set
 - ▶ $\{2\vec{i}, \vec{i} + \vec{j}\}$: basis
 - ▶ $\{2\vec{i}, \vec{j}\}$: orthogonal basis
 - ▶ $\{\vec{i}, \vec{j}\}$: orthonormal basis (canonical basis)
 - ▶ $(\frac{\vec{i} + \vec{j}}{\sqrt{2}}, \frac{\vec{i} - \vec{j}}{\sqrt{2}})$: orthonormal basis
- ▶ Consequences (without formal proof)
 - ▶ with a basis or a spanning set, one can represent any vector as $v = \sum_i \lambda_i b_i$
 - ▶ a linearly independent set can not represent all the vectors: for example, impossible to represent $\vec{j}$ as a linear combination of $\vec{i}$ (they are orthogonal)

Basis (3)

- ▶ Other consequences

- ▶ Redundancy: a spanning set which is not a basis is a redundant set: there are too many vectors (at least one)
- ▶ Redundancy: the representation of a vector is no more unique. For example with the spanning set $\{\vec{i}, \vec{j}, \vec{i} + \vec{j}\}$ and the vector $2 \cdot \vec{i} + \vec{j}$, one can exhibit two different linear combinations:

$$\begin{aligned}2 \cdot \vec{i} + \vec{j} &= 2 \cdot \vec{i} + 1 \cdot \vec{j} + 0 \cdot (\vec{i} + \vec{j}) \\ &= 1 \cdot \vec{i} + 0 \cdot \vec{j} + 1 \cdot (\vec{i} + \vec{j})\end{aligned}$$

- ▶ Non orthogonal basis: the representation is unique but the determination of coefficients λ_i is not easy. In general:

$$v = \sum_i \lambda_i b_i \neq \sum_i \langle v, b_i \rangle b_i$$

- ▶ Orthogonal basis: we have $\langle b_i, b_j \rangle = 0, i \neq j$ and

$$v = \sum_i \left\langle v, \frac{b_i}{\|b_i\|} \right\rangle \frac{b_i}{\|b_i\|}$$

determination of λ_i are direct with the scalar product.

- ▶ Use of an orthonormal basis simplifies calculus

Conclusion

- ▶ Goals of these recalls? Find suitable spaces of representation. Then find adapted basis.
- ▶ A well known example: Fourier Series! The T – periodic functions may write as:

$$x(t) = \sum_{n \in \mathbb{N}} a_n \cos\left(\frac{2\pi nt}{T}\right) + b_n \sin\left(\frac{2\pi nt}{T}\right)$$

$$a_n = \frac{2}{T} \int_0^T x(t) \cos\left(\frac{2\pi nt}{T}\right) dt \quad b_n = \frac{2}{T} \int_0^T x(t) \sin\left(\frac{2\pi nt}{T}\right) dt$$

- ▶ Alternative writing:

$$x(t) = \sum_{k \in \mathbb{Z}} c_k e^{\frac{2i\pi kt}{T}} \quad (1)$$

$$c_k = \frac{1}{T} \int_0^T x(t) e^{\frac{-2i\pi kt}{T}} dt \quad (2)$$

Here, we recognize the scalar product of a functional space:

$c_k = \left\langle x, e^{\frac{2i\pi kt}{T}} \right\rangle$ and an orthonormal basis: $\{\phi_k\}_{k \in \mathbb{Z}}$ with

$$\phi_k(t) = e^{\frac{2i\pi kt}{T}}$$

Content

Part 1: Fourier Transform, Short Time Fourier Transform

Recall: vector space spaces and important properties to know

Fourier transform

Short time Fourier Transform

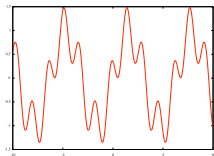
Part 2: Wavelets

Part 3: discrete wavelet transform for images, applications

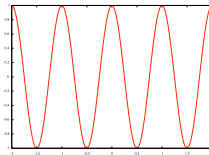
Fourier Series (1)

- ▶ Representation of the **periodic** functions
- ▶ Coefficient c_k are called Fourier coefficients
- ▶ The periodic function f is represented by the countable sequence $(c_k)_{k \in \mathbb{Z}}$
- ▶ Graphical interpretation:

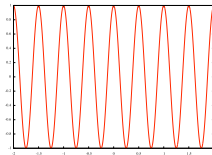
Given the following periodic signal:



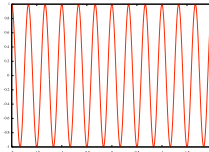
We have 8 non null Fourier coefficients:
 $c_{k_i} = c_{-k_i}$, $i = 1, \dots, 4$
describing the 4 modes (pure frequencies) of this signal



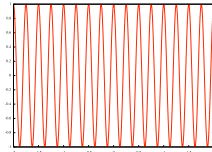
$$c_{k_1} = \left\langle x, e^{\frac{2i\pi k_1 t}{T}} \right\rangle$$



$$c_{k_2} = \left\langle x, e^{\frac{2i\pi k_2 t}{T}} \right\rangle$$



$$c_{k_3} = \left\langle x, e^{\frac{2i\pi k_3 t}{T}} \right\rangle$$



$$c_{k_4} = \left\langle x, e^{\frac{2i\pi k_4 t}{T}} \right\rangle$$

Fourier Series (2)

- ▶ Remark:

- ▶ x even function $\Rightarrow c_k = c_{-k}$
- ▶ x odd function $\Rightarrow c_k = -c_{-k}$

On the previous example: linear combination of 4 cosine functions with various frequencies $\Rightarrow$ even function.

- ▶ Exercises:

- ▶ show that the set $\{e^{\frac{2i\pi kt}{T}}\}_{k \in \mathbb{Z}}$ is an orthonormal basis
- ▶ determine the Fourier coefficients of the function $t \mapsto \cos(2\pi \frac{t}{T})$
- ▶ determine the Fourier coefficients of the Sawtooth wave (use a integration by parts to determine the integral of $t \mapsto te^{-2i\pi \frac{kt}{T}}$)

- ▶ See also: [BIMA lecture on Fourier Transform](#)

Fourier Transform (1)

Definition

- ▶ Applied on **non-periodic** function, the Fourier Series formulae does not work: $T = +\infty$ and $e^{2i\pi k \frac{t}{T}} = 1$, not a basis
- ▶ Extension to non-periodic functions: the Fourier Transform defined by

$$X(f) = \int_{\mathbb{R}} x(t) e^{-2i\pi ft} dt, f \in \mathbb{R}$$

- ▶ x must be an integrable function¹. X is a continuous function on $\mathbb{C}$ and is an element of a vector space:
 - ▶ with the scalar product $\langle f, g \rangle = \int_{\mathbb{R}} f(t) \bar{g}(t) dt$
 - ▶ with the orthonormal basis: $\{t \mapsto e^{2i\pi ft}\}_{f \in \mathbb{R}}$, an element of the basis is the function $t \mapsto e^{2i\pi ft}$ indexed by the real parameter f

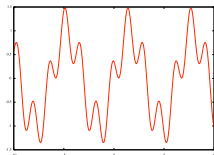
¹ f belongs to $L^2(\mathbb{R})$ space

Fourier Transform (2)

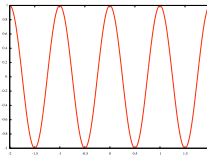
Graphical interpretation

- ▶ Same interpretation as the Fourier Series but on a continuous range of frequency

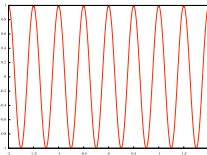
Given the following signal



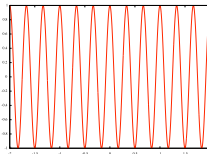
8 non null values for the Fourier transform:
 $X(f_i) = X(-f_i), i = 1, \dots, 4$ describing the 4 modes of this signal



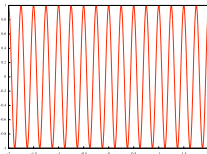
$$X(f_1) = \langle x, e^{2i\pi f_1 t} \rangle$$



$$X(f_2) = \langle x, e^{2i\pi f_2 t} \rangle$$



$$X(f_3) = \langle x, e^{2i\pi f_3 t} \rangle$$



$$X(f_4) = \langle x, e^{2i\pi f_4 t} \rangle$$

Fourier Transform (3)

Interpretation, reconstruction

► Interpretation:

- magnitude: $|X(f)| = \sqrt{X(f)\bar{X}(f)}$, or spectral amplitude, gives the quantity of “pure” frequency f available in the signal x
- phase: $\phi(f) = \arctan\left(\frac{\Re(X(f))}{\Im(X(f))}\right)$, gives the shift with the basis function $e^{2i\pi ft}$
- fundamental or null frequency, $f = 0$, is the integral of the signal:

$$X(0) = \int_{\mathbb{R}} x(t) dt$$

- As with Fourier Series, reconstruction is possible:

$$x(t) = \int_{\mathbb{R}} X(f) e^{2i\pi ft} df$$

FS versus FT

Fourier Series	Fourier Transform
$\times$ T -periodic functions	$\times$ integrable function
$c_k = \frac{1}{T} \int_0^T x(t) e^{-2i\pi \frac{k}{T} t} dt$	$X(f) = \int_{\mathbb{R}} x(t) e^{-2i\pi f t} dt$
$k \in \mathbb{Z}, c_k \in \mathbb{C}$	$X : \mathbb{R} \rightarrow \mathbb{C}$
$x(t) = \sum_{k \in \mathbb{Z}} c_k e^{2i\pi \frac{k}{T} t}$	$x(t) = \int_{\mathbb{R}} X(f) e^{2i\pi f t} df$

- ▶ To summary:
 - ▶ Fourier Series: periodic functions, countable orthonormal basis $(e^{2i\pi \frac{k}{T} t})_{k \in \mathbb{Z}}$
 - ▶ Fourier Transform: integrable functions, uncountable orthonormal basis $(e^{2i\pi f t})_{f \in \mathbb{R}}$

2-D Fourier Transform (1)

- ▶ An image is a non stationary function with a compact support, then is a non periodic function, Fourier Series are not suitable
- ▶ The 2-D Fourier Transform (for images) is built by separability:

$$X(f, g) = \int_{\mathbb{R}} \int_{\mathbb{R}} x(t, u) e^{-2i\pi(ft+gu)} dt du \quad (3)$$

$$= \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} x(t, u) e^{-2i\pi ft} dt \right\} e^{-2i\pi gu} du \quad (4)$$

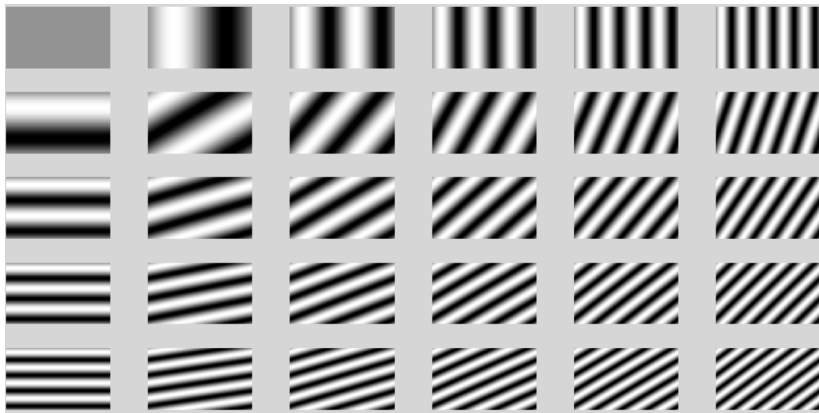
- ▶ $X : \mathbb{R}^2 \rightarrow \mathbb{C}$, (f, g) is a couple of vertical and horizontal frequencies
 - ▶ module of X (amplitude spectrum): $\sqrt{X\bar{X}}$, gives the amount of the element basis contained in signal x
 - ▶ basis: complex sinusoid $((f, g) \mapsto e^{2i\pi(ft+gu)})$
 - ▶ phase of X : gives the phase change between signal x and the element basis
- ▶ Signal x can be reconstructed from its spectrum X with the inverse Fourier transform:

$$x(t, u) = \iint_{\mathbb{R}^2} X(f, g) e^{2i\pi(ft+gu)} df dg$$

2-D Fourier Transform (2)

Inverse Fourier transform: any image is a linear combination of basis images

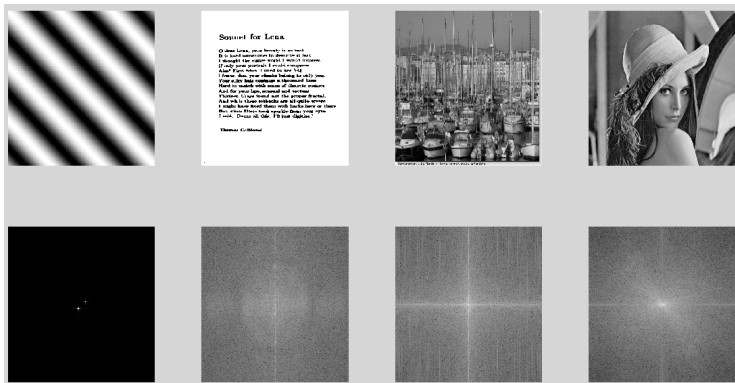
- ▶ an element of the basis, $(t, u) \mapsto \phi_{f,g}(t, u) = e^{2i\pi(ft+gu)}$, is an image!



2-D Fourier Transform (3)

Exemple sur des images

- ▶ module of spectrum: localize low and high frequencies, determine predominant orientations



Fourier transform: some mathematical tools (1)

Property (1-D or 2-D)

- ▶ linearity: $TF(\alpha x + \beta y) = \alpha X + \beta Y$
- ▶ scaling:

$$\begin{aligned}y(t) &= x(\alpha t) \\ Y(f) &= \frac{1}{\alpha} X\left(\frac{f}{\alpha}\right)\end{aligned}$$

- ▶ shift:

$$\begin{aligned}y(t) &= x(t - t_0) \\ Y(f) &= e^{-2i\pi f t_0} X(f) \\ |Y(f)| &= |X(f)|\end{aligned}$$

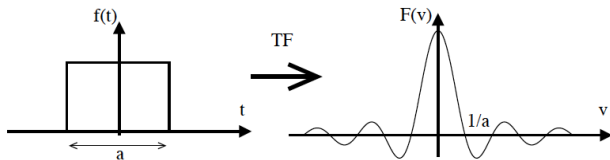
- ▶ rotation (for 2-D FT):

$$\begin{aligned}y(t, u) &= x(t \cos \theta + u \sin \theta, -t \sin \theta + u \cos \theta) \\ Y(f, g) &= X(f \cos \theta + g \sin \theta, -f \sin \theta + g \cos \theta)\end{aligned}$$

Fourier transform: some mathematical tools (2)

Fourier transform of some usual 1-D functions

- ▶ Rectangle function: $\text{Rect}(t) = \begin{cases} 1 & \text{si } |t| \leq \frac{1}{2} \\ 0 & \text{sinon} \end{cases}$
- ▶ $TF[t \mapsto \text{Rect}\left(\frac{t}{a}\right)](f) = \int_{-a/2}^{a/2} e^{-2i\pi ft} dt = a \frac{\sin(\pi af)}{\pi af} = a \text{sinc}(\pi af)$

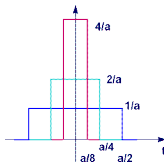


- ▶ Gaussian function:
 - ▶ $TF(t \mapsto e^{-b^2 t^2})(f) = \frac{\sqrt{\pi}}{|b|} e^{-\frac{\pi^2 f^2}{b^2}}$, also a Gaussian function!
 - ▶ standard deviation in the frequency domain is inversely proportional to standard deviation in the time domain

Fourier transform: some mathematical tools (3)

Fourier transform of some usual 1-D functions

- ▶ Dirac delta function: δ . A generalized function (or distribution), formally defined by:
 - ▶ $\delta(x) = 0 \quad \forall x \neq 0$
 - ▶ $\int_{\mathbb{R}} \delta(x) dx = 1$
- ▶ Can be seen as the limit of normal function: $\delta(t) = \lim_{a \rightarrow 0} \frac{1}{a} \text{Rect}\left(\frac{t}{a}\right)$



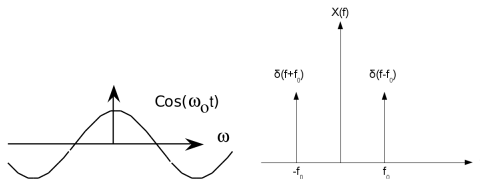
- ▶ Properties, for all function x
 - ▶ $x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0)$
 - ▶ $x \star \delta(t - t_0) = x(t - t_0)$, and then $x \star \delta(t) = x(t)$: δ neutral element of convolution
- ▶ Fourier transform:
 - ▶ $FT(t \mapsto \delta(t - t_0))(f) = e^{-2i\pi f t_0}$
 - ▶ $FT(t \mapsto e^{2i\pi f_0 t})(f) = \delta(f - f_0)$

Fourier transform: some mathematical tools (4)

Fourier transform of some usual 1-D functions

- Cosine function (Euler formulae):

$$FT[t \mapsto \cos(2\pi f_0 t)] = \frac{1}{2}(\delta(f - f_0) + \delta(f + f_0))$$



- Sine function: $FT[t \mapsto \sin(2\pi f_0 t)] = \frac{i}{2}(\delta(f - f_0) - \delta(f + f_0))$

Fourier transform: some mathematical tools (5)

Convolution theorem

- Recall, convolution:

$$z(t) = x \star y(t) = \int_{\mathbb{R}} x(t - t')y(t')dt'$$

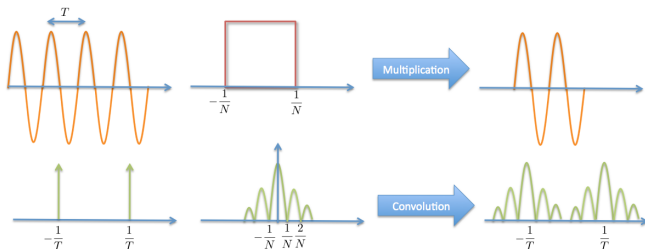
- Any linear filtering time invariant can be expressed by a convolution
- Convolution theorem:
 - if $z = x \star y$ then $Z = X \times Y$
 - if $z = x \times y$ then $Z = X \star Y$
- Important tool for calculation of Fourier transform! (see the next slide as an example)
- In 2-D (image), the convolution theorem still holds:

$$z(t, u) = x \star y(t, u) = \int_{\mathbb{R}} \int_{\mathbb{R}} x(t - t', u - u')y(t', u')dt' du'$$

- Consequence: filtering in the frequency domain is strictly equivalent to convolution in time (space) domain

Digitization and discrete Fourier transform (1)

- ▶ Practically: we analyze discrete signals and not real functions. A discrete tool is needed: the Discrete Fourier Transform (DFT)
- ▶ Formalization:
 1. the signal to analyze is windowed to obtain a bounded support function:
 - ▶ $x_L(t) = x(t) \text{Rect}(t/L)$
 - ▶ FT: $X_L(f) = L X \star \text{sinc}(\pi L f)$
- ▶ Example with a basic signal (cosine, pure frequency)



Digitization and discrete Fourier transform (2)

- ▶ Practically: we analyze discrete signals and not real functions. A discrete tool is needed: the Discrete Fourier Transform (DFT)
- ▶ Formalization:
 1. the signal to analyze is windowed: $x(t) \Rightarrow x_L(t) = x(t) \text{Rect}(t/L)$
 2. the windowed signal is sampled: a measure of this signal is done each T_s time step ($f_s = \frac{1}{T_s}$ is the sampling frequency):
 - ▶ $x_s(t) = x_L(t) \sum_{k \in \mathbb{Z}} \delta(t - kT_s)$ ($\sum_k \delta(t - kT_s)$: Dirac comb or train impulse)
 - ▶ Due to the windowing and the sampling frequency, we have $N = L/T_s$ measures
 - ▶ Fourier transform: $X_s(f) = X_L \star \sum_{k \in \mathbb{Z}} \delta(f - k/T_s)$ (the Fourier transform of Dirac comb is a Dirac comb). Hence:
$$X_s(f) = \sum_{k \in \mathbb{Z}} X_L(f - k/T_s)$$

$\Rightarrow$ Sampling implies a periodic spectrum (of period $f_s = 1/T_s$)!

Digitization and discrete Fourier transform (3)

Sampling: Shannon theorem

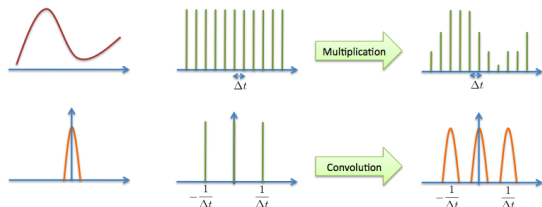


Figure: Sampling implies a periodic spectrum

- Let X be a bounded frequency support and let f_m be the maximal frequency of X :

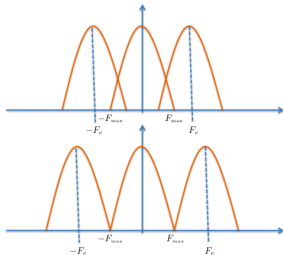
Theorem (Shannon)

If $f_s \geq 2f_m \Leftrightarrow T_s \leq \frac{1}{2}T_m$, then the signal can be reconstructed without loss

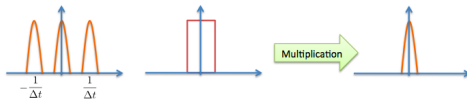
Digitization and discrete Fourier transform (4)

Sampling: Shannon theorem

- Spectrum overlapping if $f_m > f_s/2$ and limit case:



- Reconstruction: X_L is truncated with a Rectangle function, then an inverse Fourier Transform is applied: Shannon interpolation formula



Digitization and discrete Fourier transform (5)

- ▶ Practically: we analyze discrete signals and not real functions. A discrete tool is needed: the Discrete Fourier Transform (DFT)
- ▶ Formalization:
 1. the signal to analyze is windowed:
 - ▶ $x_L(t) = x(t) \text{Rect}(t/L)$
 - ▶ FT: $X_L(f) = L X \star \text{sinc}(\pi L f)$
 2. the windowed signal is sampled:
 - ▶ $x_s(t) = x_L(t) \sum_{k \in \mathbb{Z}} \delta(t - kT_s)$
 - ▶ FT: $X_s(f) = \sum_{k \in \mathbb{Z}} X_L(f - k/T_s)$
 3. X_s is sampled at frequencies $f = \frac{k}{Nf_s}$, $k = 0 \dots N-1$:
 - ▶ $\text{DFT}(x)(k) = X_s\left(\frac{k}{Nf_s}\right)$, $k = 0 \dots N-1$
 - ▶ $\text{DFT}(x)(k) = \sum_{n=0}^{N-1} x_s(n) e^{-2i\pi \frac{kn}{N}}$, $k = -\frac{N}{2} \dots \frac{N}{2} - 1$
- ▶ Practically: we denote $x(k) = x(kT_s)$ as the k -th sample of signal x , and the Discrete Fourier transform is defined as:

$$\text{DFT}(x)(k) = X(k) = \sum_{n=0}^{N-1} x(n) e^{-2i\pi \frac{kn}{N}}, k = -\frac{N}{2} \dots \frac{N}{2} - 1 \quad (5)$$

Discrete Fourier transform

Properties, and 2-D DFT

- ▶ DFT 2-D:

$$X(k, l) = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} x(n, m) e^{-2i\pi \left(\frac{kn}{N} + \frac{lm}{M} \right)}$$

- ▶ The DFT has the same properties than the continuous Fourier transform:
 - ▶ linearity, translation and rotation of the signal/image
- ▶ Practically, DFT is used for filtering discrete signal/image in the frequency domain
- ▶ Inverse 2-D DFT:

$$x(n, m) = \sum_{l=0}^{M-1} \sum_{k=0}^{N-1} X(k, l) e^{2i\pi \left(\frac{kn}{N} + \frac{lm}{M} \right)}$$

2-D discrete Fourier transform

Filtering in frequency domain vs time domain

- Filtering in the time domain:

$$y(n, m) = x \star h(n, m)$$

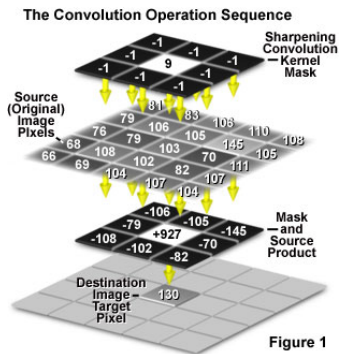
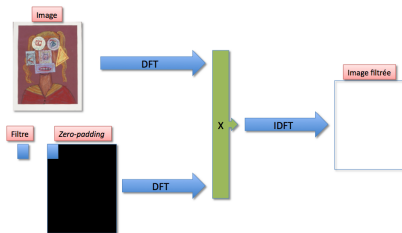


Figure 1

- Filtering in the frequency domain:

$$y(n, m) = TFD^{-1}[X(u, v) \times H(u, v)]$$



Filtering in the frequency domain

- ▶ Several types of filters:
 - ▶ low-pass: low frequencies are kept, high frequencies are attenuated
 - ▶ high-pass: low frequencies are attenuated, high frequencies are attenuated
 - ▶ band-pass: a range of frequencies is kept, others frequencies are attenuated: allow an multi-scale analysis (scale=size of structures)
- ▶ See BIMA course (<https://www-master.ufr-info-p6.jussieu.fr/parcours/ima/bima/>): lectures 3, 4, 5 and associated tutorial and practical works.

Content

Part 1: Fourier Transform, Short Time Fourier Transform

Recall: vector space spaces and important properties to know

Fourier transform

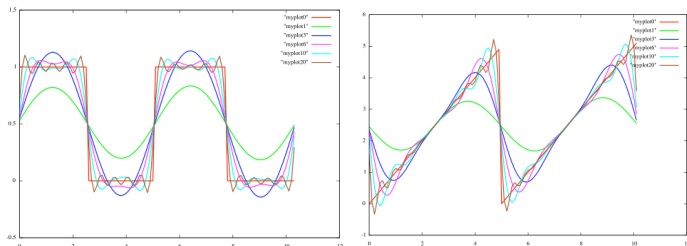
Short time Fourier Transform

Part 2: Wavelets

Part 3: discrete wavelet transform for images, applications

FT: limitations, issues (1)

- Compression, denoising: impossible to correctly represent edges (non derivable functions): Gibbs ringing artifacts appear after removing highest frequencies



- Visible in JPEG compression for example

FT: limitations, issues (2)

- ▶ In the Fourier space, structure size and orientation can be measured but it is **not possible** to localize (translation invariant): a wave has a period (size), an orientation (in 2-D), a phase, but not a localization.
- ▶ Two ways to represent a signal:
 - ▶ representation in time (or spatial if image) domain:

$$x(t) = \int_{\mathbb{R}} x(u) \delta(t - u) du$$

=> this basis localizes in time, but not in frequency (it can't see the size of structures)

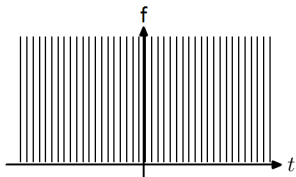
- ▶ representation in the frequency domain (inverse FT):

$$x(t) = \int_{\mathbb{R}} X(f) e^{2i\pi ft} df$$

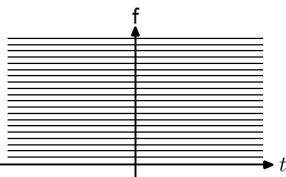
=> this basis localizes in frequency but not in time

FT: limitations, issues (3)

- ▶ Representation in time domain: null resolution in frequency, infinite resolution in time
- ▶ Representation frequency domain: infinite resolution in frequency, null resolution in time



(a) DIRAC



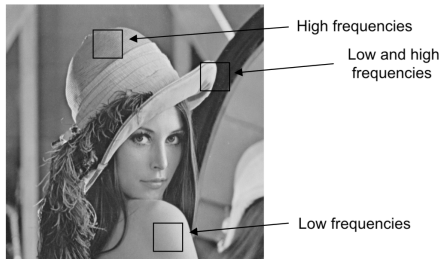
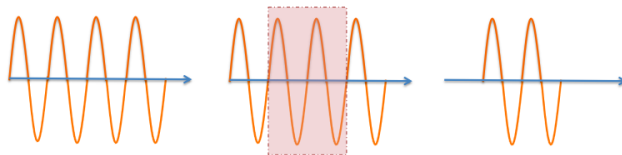
(b) FOURIER

- ▶ Consider two signals:
 - ▶ $y(t) = \sin(2\pi f_1 t) + \sin(2\pi f_2 t)$
 - ▶ $z(t) = \sin(2\pi f_1 t)u(t) + \sin(2\pi f_2 t)u(-t)$ with $u(t) = 1$ if $t > 0$ and 0 otherwise (Heavyside function)

y and z has the same spectrum!
- ▶ Need to analyze the signal both in time and in frequency domains!

Short Time Fourier Transform (1)

- ▶ Principle: perform a Fourier analysis on a window.
 - ▶ first the signal is windowed, the window being localized in the time domain, second a Fourier Transform is applied
 - ▶ the STFT has two parameters:
 - ▶ a parameter of time localization
 - ▶ a parameter of frequency localization



- ▶ Other name: Windowed Fourier Transform

Short Time Fourier Transform (2)

- ▶ Definition:

$$STFT(x)(f, b) = X(f, b) = \int_{\mathbb{R}} x(t) \bar{w}(t - b) e^{-2i\pi ft} dt$$

with w an *admissible* window, i.e. $\int_{\mathbb{R}} |w(t)|^2 dt = 1$

- ▶ Examples for w : Rectangle function, Triangle function, Gaussian function, ...
- ▶ The family of functions $\phi_{f,b}(t) = w(t - b) e^{2i\pi ft}$ is spanning but redundant set (two parameters f and b)
 - ▶ STFT: $\phi_{f,b}(t) = w(t - b) e^{2i\pi ft}$: localization in frequency f and in time b
 - ▶ FT: $\phi_f(t) = e^{2i\pi ft}$: localization only in frequency
- ▶ Reconstruction is available if w is an admissible window:

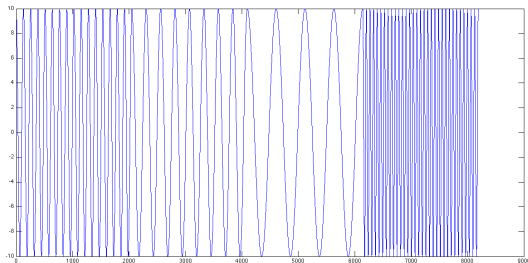
$$x(t) = \int_{\mathbb{R}} \int_{\mathbb{R}} X(f, b) w(t - b) e^{2i\pi ft} df db$$

- ▶ Exercise: prove the reconstruction formula

Short Time Fourier Transform (3)

Example

- Time-varying frequency signal:

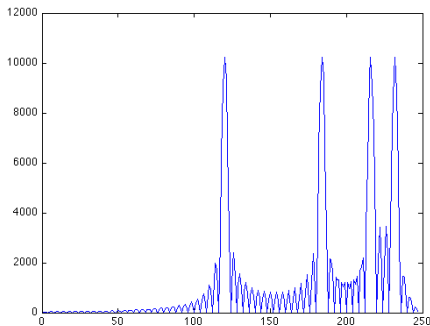


$$x(t) = \sum_{k=1}^4 \cos(2\pi f_k t) \text{Rect}\left(\frac{t - t_k}{w}\right)$$

Short Time Fourier Transform (4)

Exemple

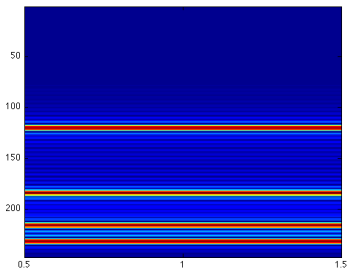
- Fourier transform of x : no localization in time!



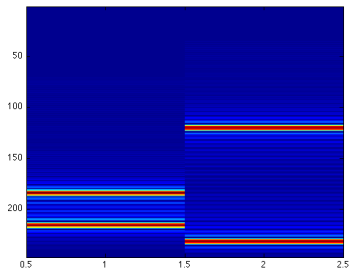
$$X(f) = \sum_{k=1}^4 \frac{\delta(f - f_k) + \delta(f + f_k)}{2} \star e^{-2i\pi f t_k} \text{sinc}(w\pi f)$$

Short Time Fourier Transform (5)

Example: representation time-frequency



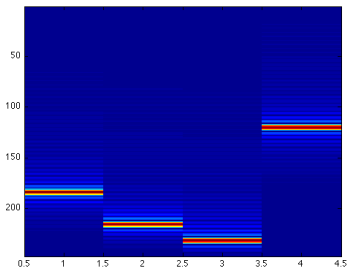
1 window: it is the standard Fourier Transform, so no localization in time



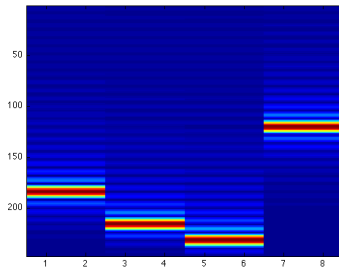
2 windows: gain in time resolution

Short Time Fourier Transform (6)

Example: representation time-frequency



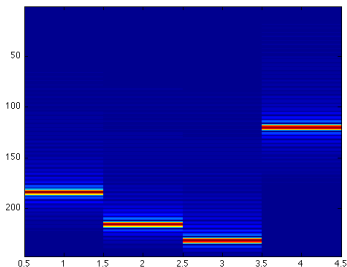
4 windows: gain in time resolution



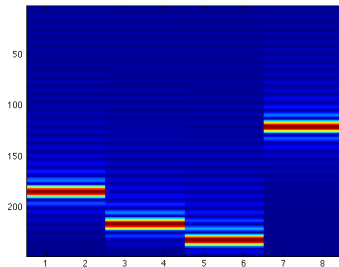
8 windows: loss of frequency localization and then frequency resolution! why?

Short Time Fourier Transform (6)

Example: representation time-frequency



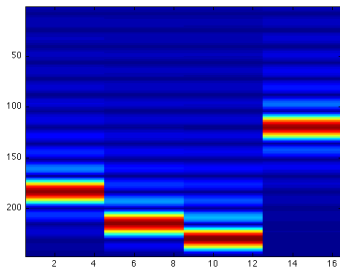
4 windows: gain in time resolution



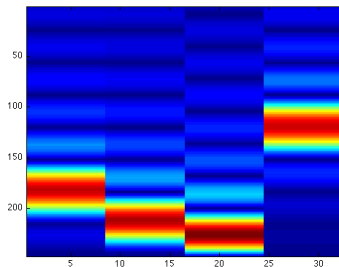
8 windows: loss of frequency localization and then frequency resolution! why? as the window becomes smaller, the FT (sinc) is lesser accurate

Short Time Fourier Transform (7)

Example: representation time-frequency



16 windows: loss of frequency resolution!



32 windows: loss of frequency resolution!

Short Time Fourier Transform (8)

Example: representation time-frequency

- ▶ Conclusion: there is an optimal configuration to analyze the x signal
 - ▶ with less than 4 windows: low time resolution but good frequency resolution
 - ▶ more than 4 windows: maximal time resolution, but low frequency resolution *résolution fréquentielle moins bonne*
 - ▶ 4 windows is the optimal in this case
- ▶ See Exercise 5 in tutorial works

Short Time Fourier Transform (9)

Limitations, issues

- ▶ Window length is a critical parameter:
 - ▶ must be the same order of value than the period of the signal to be analyzed
 - ▶ but not so large, because the time resolution will be degraded
- ▶ Let us formally define the time and frequency resolution of a x signal:

$$\text{▶ } \langle t \rangle = \frac{1}{E} \int_{\mathbb{R}} t |x(t)|^2 dt, \quad \langle f \rangle = \frac{1}{E} \int_{\mathbb{R}} f |X(f)|^2 df$$

$$\text{with } E = \int_{\mathbb{R}} |x(t)|^2 dt$$

- ▶ Time resolution (standard deviation, dispersion):

$$\sigma_t = \sqrt{\int_{\mathbb{R}} (t - \langle t \rangle)^2 |x(t)|^2 dt}$$

- ▶ Frequency resolution (standard deviation, dispersion):

$$\sigma_f = \sqrt{\int_{\mathbb{R}} (f - \langle f \rangle)^2 |X(f)|^2 df}$$

- ▶ small standard deviation $\Rightarrow$ high localization $\Rightarrow$ high resolution

Heisenberg uncertainty principle

- ▶ A general principle apply to any waves (and more):
 - ▶ impossible to localize both in time and in frequency with a infinite precision a signal
 - ▶ time and frequency resolution are bounded: $\sigma_t \sigma_f \geq \frac{1}{4\pi}$

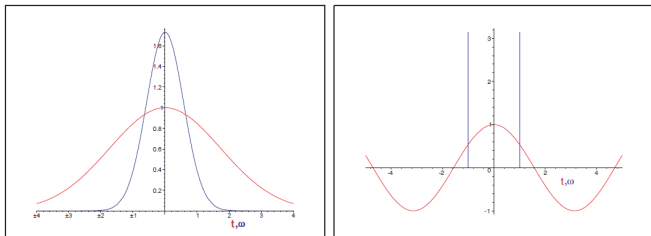
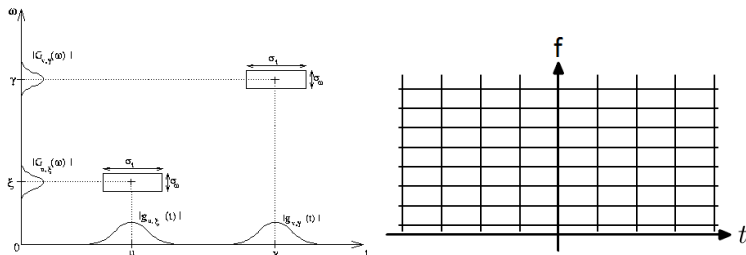


Figure: Left: Gaussian signal (red) and its spectrum, right: Cosine signal and its spectrum

- ▶ The bound is reached with the Gaussian function!

Heisenberg boxes

1. Time and frequency resolution can be represented using the Heisenberg boxes:



2. Here: σ_t and σ_f are constant.
3. Too large window: impossible to analyze non stationary signals (loss of localization in time)
4. Too small window: loss of localization in frequency
5. Idea of wavelets: analyze in time and frequency more suitable (i.e. Heisenberg boxes of various size), and design of an orthonormal basis (STFT is not a basis)

Content

Part 1: Fourier Transform, Short Time Fourier Transform

Part 2: Wavelets

Continuous wavelets

Dyadic wavelets

Multiresolution Analysis (MRA)

Haar wavelet

The discrete wavelet transform

Part 3: discrete wavelet transform for images, applications

Continuous wavelet transform (CWT): definition

- ▶ $E = L^2(\mathbb{R})$ set of real function squared integrable (a vector space)
- ▶ Let $x \in E$ be a signal, the continuous wavelet transform is a function $(a, b) \mapsto g(a, b)$ defined by:

$$g(a, b) = \frac{1}{\sqrt{a}} \int_{\mathbb{R}} x(t) \bar{\psi}_{a,b}(t) dt = \langle x, \psi_{a,b} \rangle$$

such as $a \neq 0$ and:

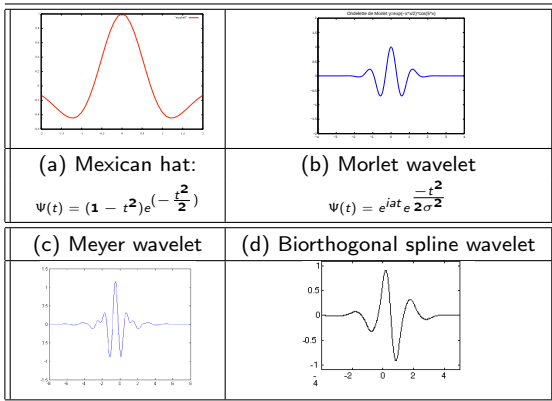
$$\psi_{a,b} = \frac{1}{\sqrt{a}} \psi \left(\frac{t - b}{a} \right)$$

where ψ is called **mother wavelet**

- ▶ Functions $\psi_{a,b}$ are translated/dilated version of ψ
- ▶ b : position (localization in time), a : scale (analog of the period of Fourier analysis)

Mother wavelet

- ▶ ψ must be *admissible*:
 - ▶ has a bounded support
 - ▶ is of mean null ($\int \psi = 0$)
 - ▶ be oscillating $|\psi| \neq \psi$
 - ▶ $\psi \in E$ (squared integrable)
 - ▶ $\psi(t) \in \mathbb{R}$ or $\mathbb{C}$
- ▶ Examples:



CWT versus STFT

- ▶ Similarity:
 - ▶ Both are redundant analysis (projection onto redundant spanning families)
 - ▶ Both localize in time and in frequency domains:
 - ▶ STFT: $\phi_{f,b} = w(t-b)e^{2i\pi ft}$
 - ▶ CWT: $\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right)$
 - ▶ Difference:
 - ▶ STFT: has a fixed resolution in time and in frequency (Heisenberg boxes have the same size)
 - ▶ CWT: has a variable resolution in time and in frequency
 - ▶ Interpretation for the CWT:
 - ▶ allow a multiscale analyze: the support in the time domain is more or less large (the mother wavelet is dilated at various size)
 - ▶ Let $\sigma_t^{a,b}$ et $\sigma_f^{a,b}$ be the respective time and frequency resolution of $\psi_{a,b}$:
 - ▶ $\sigma_t^{a,b} = a\sigma_t^{1,0}$
 - ▶ $\sigma_f^{a,b} = \frac{1}{a}\sigma_f^{1,0}$
- with $\sigma_t^{1,0}$ and $\sigma_f^{1,0}$ the time and frequency resolution of mother wavelet ψ

Heisenberg boxes

- Recall: Heisenberg uncertainty principle, $\sigma_t \sigma_f \geq \frac{1}{4\pi}$, boxes have a minimal area

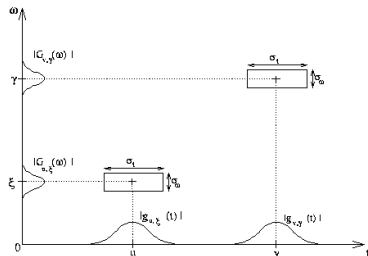


Figure: Heisenberg box of FT

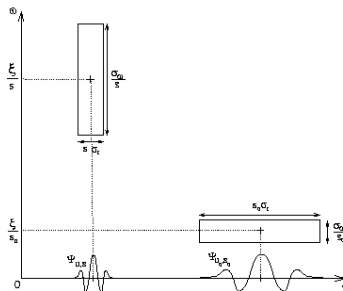


Figure: Heisenberg box of CWT

CWT: interpretation

Wavelet as a multi-scale analysis tool

► Findings:

1. low frequencies are less localized in time: a low frequency signal has a long period and is almost stationary
2. high frequencies are better localized in time (small period) and non stationary, their localization in time are important for analysis

► Wavelets: a frequency is analyzed at a suitable time resolution:

1. low frequency (scale a is large): low time resolution, high frequency resolution
2. high frequency (scale a is small): high time resolution, low frequency resolution

There is a compromise between time and frequency resolution
(Heisenberg)

Reconstruction

- Formally:

$$x(t) = \frac{1}{C_\psi} \int_{\mathbb{R}} \int_{\mathbb{R}} a^{-2} g(a, b) \psi_{a,b}(t) da db$$

with

$$C_\psi = \int_0^{+\infty} \frac{|\psi(f)|^2}{f} df$$

- If $C_\psi < \infty$ (admissibility condition), reconstruction is possible
- The family is redundant: practically, reconstruction is costly, but:
 - a countable set of values for $(a, b) \mapsto g(a, b)$ is sufficient to reconstruct x ,
 - practically, a continuous wavelet transform is not suitable for discrete signal: a discrete formulation of wavelet is requested

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Part 3: discrete wavelet transform for images, applications

Reducing redundancies: Dyadic wavelets

- ▶ The continuous wavelet transform is sampled using a *dyadic* position:
 - ▶ $a = 2^{-j}$
 - ▶ $b = k \times 2^{-j}, \quad k = 0, \dots, 2^j - 1$
- ▶ $j \in \mathbb{N}$ is the time resolution (or representation scale)
- ▶ $\psi_{a,b}(t) = \sqrt{2^j} \psi(2^j t - k) = \psi_k^j(t)$ has a support of length 2^{-j} and a position at k
- ▶ For j fixed, $\psi_k^j(t)$ functions have disjoint and contiguous supports. Let ψ be a mother wavelet with support on $[0, 1]$:
 - ▶ $j = 0$: $k = 0$. Only one function for this scale, $\psi_0^0(t) = \psi(t)$
 - ▶ $j = 1$: $k = 0$ or 1 . Two functions for this scale:
 - ▶ position 0: $\psi_0^1(t) = \sqrt{2}\psi(2t)$ with support on $[0, \frac{1}{2}]$
 - ▶ position 1: $\psi_1^1(t) = \sqrt{2}\psi(2t - 1)$ with support on $[\frac{1}{2}, 1]$
 - ▶ $j = 2$: $k = 0, 1, 2, 3$, 4 functions:
 - ▶ position 0: $\psi_0^2(t) = \sqrt{2}\psi(4t)$, support on $[0, \frac{1}{4}]$
 - ▶ position 1: $\psi_1^2(t) = \sqrt{2}\psi(4t - 1)$, support on $[\frac{1}{4}, \frac{1}{2}]$
 - ▶ position 2: $\psi_2^2(t) = \sqrt{2}\psi(4t - 2)$, support on $[\frac{1}{2}, \frac{3}{4}]$
 - ▶ position 3: $\psi_3^2(t) = \sqrt{2}\psi(4t - 3)$, support on $[\frac{3}{4}, 1]$
 - ▶ ...

Dyadic wavelets

- ▶ Redundancy is reduced: $(a, b) \in \mathbb{R}^2 \Rightarrow (j, k), j \in \mathbb{N}, 0 \leq k < 2^j$: countable family
- ▶ We obtain a discrete sequence of coefficients:

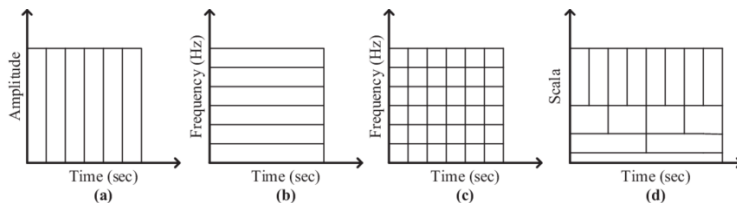
$$g_k^j = \langle x, \psi_k^j \rangle$$

- ▶ Reconstruction:

$$x(t) = \sum_{j \in \mathbb{N}} \sum_{k=0}^j g_k^j \psi_k^j(t)$$

- ▶ Remark: this transform applies on continuous signal (x is continuous as well the elements of the family, $t \mapsto \psi_k^j(t)$). We do not yet have a discrete transform.

Dyadic wavelets transform versus FT, STFT



- (a) Localization in time domain
- (b) Localization in frequency domain (FT)
- (c) Localization in time and frequency domains (STFT)
- (d) Localization in scale and time domains (dyadic wavelet)

Content

Part 1: Fourier Transform, Short Time Fourier Transform

Part 2: Wavelets

- Continuous wavelets

- Dyadic wavelets

- Multiresolution Analysis (MRA)**

- Haar wavelet

- The discrete wavelet transform

Part 3: discrete wavelet transform for images, applications

Multiresolution analysis (1)

Motivations

- ▶ Dyadic wavelets: the family is not redundant but the basis is not orthogonal (eg: $\langle \psi_k^j, \psi_{2k}^{j+1} \rangle \neq 0$)
- ▶ Multiresolution analysis: formalism to build wavelet orthonormal basis
- ▶ Principle: project the signal into nested vector subspaces



Multiresolution analysis (2)

Definition

- ▶ A multiresolution analysis of $E = L^2(\mathbb{R})$ is a sequence of subspaces $(V^j)_{j \in \mathbb{Z}}$ such as:
 1. information contained in resolution j is also contained in resolution $j + 1$: $\forall j \in \mathbb{Z} \quad V^j \subset V^{j+1}$
 2. intersection of all V^j is empty: $\bigcap_{j \in \mathbb{Z}} V^j = \lim_{j \rightarrow -\infty} V^j = \emptyset$
 3. union of all V^j is E : $\bigcup_{j \in \mathbb{Z}} V^j = \lim_{j \rightarrow +\infty} V^j = E$
 4. resolution j derives from resolution $j + 1$ by a dilation of factor 2:
 $\forall j \in \mathbb{Z} \quad f \in V^j \Leftrightarrow f(2 \cdot) \in V^{j+1}$
 5. it exists a function $\phi \in E$ such as the family $(\phi(\cdot - k))_{k \in \mathbb{Z}}$ is an orthonormal basis in V^0
- ▶ Consequences:
 - ▶ from 4. and 5. it comes: $\forall k \in \mathbb{Z} \quad f \in V^j \Leftrightarrow f(\cdot - k2^j) \in V^j$. In other words $(\phi(\cdot - k2^j))_{k \in \mathbb{Z}}$ is a basis in V^j
 - ▶ from 3.: one can reconstruct a signal $x \in E$ from its projections into V^j
- ▶ ϕ is known as **scaling function** (or wavelet father)
- ▶ V^j are known as the **approximation** subspaces

Multiresolution analysis (3)

scaling function: one example

1. Consider $\phi(t) = 1$ on $[0, 1[$, null otherwise
2. This is Haar scaling function
3. What does V^0 represent?, V^j ?

Multiresolution analysis (3)

scaling function: one example

1. Consider $\phi(t) = 1$ on $[0, 1[$, null otherwise
2. This is Haar scaling function
3. What does V^0 represent?, V^j ?

▶ $E = L^2(\mathbb{R})$, scalar product: $\langle f, g \rangle = \int_{\mathbb{R}} f(t) \bar{g}(t) dt$

▶ suppose $\phi(t - k)$ is a basis in V^0 then if $f \in V^0$,
$$f(t) = \sum_{k \in \mathbb{Z}} \langle f, \phi(\cdot - k) \rangle \phi(t - k) = \sum_k c_k \phi(t - k)$$

$\phi(t - k)$ is 1 on $[k, k + 1[$ and null otherwise

- ▶ then V^0 is the space of functions constant on intervals $[k, k + 1[$
- ▶ and then V^1 is the set of functions constant on intervals $[k/2, (k + 1)/2[$ if condition 4 holds.
- ▶ and then V^j is the set of functions constants on intervals $[2^{-j}k, 2^{-j}(k + 1)[$

Multiresolution analysis (3)

scaling function: one example

1. Consider $\phi(t) = 1$ on $[0, 1[$, null otherwise
2. This is Haar scaling function
3. What does V^0 represent?, V^j ?
4. Is Haar scaling function admissible to perform a multiresolution analysis of $E = L^2(\mathbb{R})$?

Multiresolution analysis (3)

scaling function: one example

1. Consider $\phi(t) = 1$ on $[0, 1[$, null otherwise
2. This is Haar scaling function
3. What does V^0 represent?, V^j ?
4. Is Haar scaling function admissible to perform a multiresolution analysis of $E = L^2(\mathbb{R})$?
 - ▶ condition 5. is true: $\phi(\cdot - k)$ is an orthonormal basis in V^0 , easy to verify
 - ▶ condition 1. ($V^j \subset V^{j+1}$): if $f \in V^j$ then f constant on intervals $[2^{-j}k, 2^{-j}(k+1)[$, and also constant on intervals $[2^{-(j+1)}k, 2^{-(j+1)}(k+1)[$ and we conclude $f \in V^{j+1}$
 - ▶ conditions 2. and 3. intuitively: integral of a function may be approximated by piecewise constant functions (integral definition in sense of Riemann)
 - ▶ condition 4. (transition j to $j+1$): similar proof than for condition 1, $f(2\cdot)$ is a dilatation of f by a factor 2, then $f(2\cdot) \in V^{j+1}$
5. Haar scaling function is an admissible solution for a multiresolution analysis of E (see Ex 6 tutorial works)

Multiresolution analysis (4)

Projection into V^j

- ▶ Let ϕ be an admissible scaling function in $E = L^2(\mathbb{R})$
- ▶ Let's define: $\phi_k^j(t) = \sqrt{2^j} \phi(2^j t - k)$, then:
 - ▶ $(\phi_k^j)_{k \in \mathbb{Z}}$ is an orthonormal basis in V^j
 - ▶ derives from conditions 4. and 5.
- ▶ Given $x \in E$, its projection into V^j is:

$$x^j(t) = (P_j x)(t) = \sum_k s_k^j \phi_k^j(t)$$

with:

$$s_k^j = \langle x, \phi_k^j \rangle_{V^j} = \int_{\mathbb{R}} \sqrt{2^j} x(t) \phi(2^j t - k) dt$$

we recognize a scalar product for V^j

- ▶ s_k^j are the **approximation coefficients** at resolution j
- ▶ Subspaces V^j are dyadic spaces

Multiresolution analysis (5)

Complementary subspaces (1)

- ▶ Last step: obtain an orthonormal basis
- ▶ Fundamental idea: as $V^j \subset V^{j+1}$ then

$$\exists W^j \text{ such as } V^{j+1} = V^j \oplus W^j$$

W^j is known as the **details** subspace for resolution j

- ▶ W^j is a complementary subspace orthogonal to V^j in V^{j+1}
- ▶ We call wavelets (or details functions) the set of functions $(\psi_k^j)_{k \in \mathbb{Z}}$ spanning W^j and pairwise orthogonal
- ▶ Having an orthonormal basis in V^j and in W^j , we have an orthonormal basis in V^{j+1} : $(\phi_k^j)_{k \in \mathbb{Z}} \cup (\psi_k^j)_{k \in \mathbb{Z}}$ and

$$x^{j+1}(t) = \underbrace{\sum_{k \in \mathbb{Z}} s_k^j \phi_k^j(t)}_{\text{projection into } V^j} + \underbrace{\sum_{k \in \mathbb{Z}} d_k^j \psi_k^j(t)}_{\text{projection into } W^j}$$

- ▶ $d_k^j = \langle x, \psi_k^j \rangle$ are known as the details coefficients

Multiresolution analysis (5)

Complementary subspaces (2)

- Recursively we have:

$$\begin{aligned}V^{j+1} &= V^j \oplus W^j = V^{j-1} \oplus W^{j-1} \oplus W^j \\&= V^0 \oplus W^0 \oplus W^1 \oplus \dots \oplus W^{j-1} \oplus W^j \\x^{j+1}(t) &= \sum_k s_k^0 \phi_k^0(t) + \sum_{i=0}^j \sum_k d_k^i \psi_k^i(t)\end{aligned}$$

- Basis in V^{j+1} contains:

- that of V^0
- that of W^0, W^1 , up to W^j

- $j \rightarrow +\infty$:

- $E = L^2(\mathbb{R}) = V^0 \bigoplus_{i=0}^{+\infty} W^i$

- $x(t) = \sum_k s_k^0 \phi_k^0(t) + \sum_{i=0}^{+\infty} \sum_k d_k^i \psi_k^i(t)$

Multiresolution analysis (5)

Complementary subspaces (3)

- ▶ Subspaces V^j are also nested when $j < 0$: $\dots \subset V^{-1} \subset V^0$
- ▶ Then:

$$\begin{aligned}E &= V^0 \bigoplus_{i=0}^{+\infty} W^i \\&= V^{-1} \oplus W^{-1} \bigoplus_{i=0}^{+\infty} W^i \\&= V^{-j} \oplus W^{-j} \oplus \dots \oplus W^{-1} \bigoplus_{i=0}^{+\infty} W^i \\&= \bigoplus_{j=-\infty}^{+\infty} W^j \\x(t) &= \sum_{j=-\infty}^{+\infty} \sum_k d_k^j \psi_k^j(t)\end{aligned}$$

Multiresolution analysis (6)

Conclusion

- ▶ The multiresolution analysis allows to build a basis of orthogonal wavelets (ψ_k^j)
- ▶ Subspaces V^j have a dyadic basis (ϕ_k^j) derived from the scaling function ϕ (also named father wavelet): $\phi_k^j(t) = \sqrt{2^j} \phi(2^j t - k)$
- ▶ Complementary subspaces W^j also have a dyadic basis derived from the mother wavelet ψ : $\psi_k^j(t) = \sqrt{2^j} \psi(2^j t - k)$
- ▶ Issue: choose ψ

Content

Part 1: Fourier Transform, Short Time Fourier Transform

Part 2: Wavelets

Continuous wavelets

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Multiresolution Analysis (MRA)

Haar wavelet

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Part 3: discrete wavelet transform for images, applications

Haar wavelet (1)

- ▶ $E = L^2([0, 1[)$, $x : E \rightarrow \mathbb{R}$
- ▶ Scaling function (Haar):

$$\phi(t) = \begin{cases} 1 & 0 \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$$

- ▶ Bases of subspaces V^j : $\phi_k^j(t) = \sqrt{2^j} \phi(2^j t - k)$:

$$\phi_k^j(t) = \begin{cases} \sqrt{2^j} & \frac{k}{2^j} \leq t < \frac{k+1}{2^j} \\ 0 & \text{otherwise} \end{cases}$$

- ▶ We conclude that:
 - ▶ V^0 is the set of constant functions on $[0, 1[$, spanned by ϕ_0^0
 - ▶ V^1 is the set of constant functions on $[0, \frac{1}{2}[$ and $[\frac{1}{2}, 1[$, spanned by ϕ_0^1 and ϕ_1^1
 - ▶ V^j is the set of constant functions on $[\frac{k}{2^j}, \frac{k+1}{2^j}[$, $k = 0, \dots, 2^j - 1$
 - ▶ V^{-1} do not make sense (functions constant on $[0, 2[$)

Haar wavelet (2)

- The mother wavelet can be chosen as:

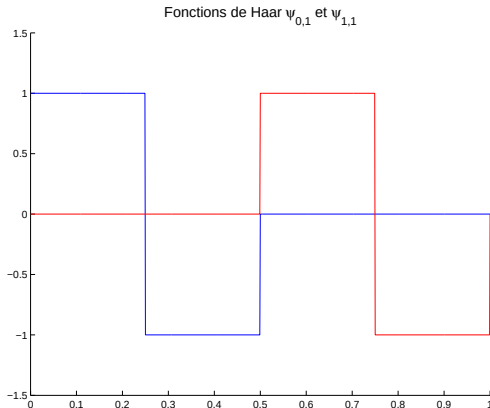
$$\psi(t) = \begin{cases} 1 & 0 \leq t < \frac{1}{2} \\ -1 & \frac{1}{2} \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$$

- And for other wavelets: $\psi_k^j(t) = \sqrt{2^j} \psi(2^j t - k)$:

$$\psi_k^j(t) = \begin{cases} \sqrt{2^j} & \frac{k}{2^j} \leq t < \frac{k}{2^j} + \frac{1}{2^{j+1}} \\ -\sqrt{2^j} & \frac{k}{2^j} + \frac{1}{2^{j+1}} \leq t < \frac{k+1}{2^j} \\ 0 & \text{otherwise} \end{cases}$$

Haar wavelet (3)

► $V^2 = \phi_0^2 \oplus \phi_1^2 \oplus \phi_2^2 \oplus \phi_3^2 = \phi_0^1 \oplus \phi_1^1 \oplus \psi_0^1 \oplus \psi_1^1$



► Easy to verify that (tutorial work):

► $\langle \psi_k^j, \psi_{k'}^{j'} \rangle = 0 \text{ } k \neq k'$

► $\langle \psi_k^j, \psi_k^{j'} \rangle = 0 \text{ } j \neq j'$

Haar wavelet (4)

Transition from resolution $j + 1$ to j (compression)

- ▶ ϕ_k^j scaling functions: approximation at resolution j
- ▶ ψ_k^j wavelet functions: details at resolution j
- ▶ By definition of ϕ_k^j and ψ_k^j , we have:

$$\phi_k^j = \frac{\phi_{2k}^{j+1} + \phi_{2k+1}^{j+1}}{\sqrt{2}} \quad \psi_k^j = \frac{\phi_{2k}^{j+1} - \phi_{2k+1}^{j+1}}{\sqrt{2}} \quad (6)$$

- ▶ And: $x^{j+1}(t) = \sum_{k=0}^{2^j-1} s_k^j \phi_k^j(t) + \sum_{k=0}^{2^j-1} d_k^j \psi_k^j(t) = \sum_{k=0}^{2^{j+1}-1} s_k^{j+1} \phi_k^{j+1}(t)$
- ▶ We derive:

$$s_k^j = \frac{s_{2k}^{j+1} + s_{2k+1}^{j+1}}{\sqrt{2}} \quad d_k^j = \frac{s_{2k}^{j+1} - s_{2k+1}^{j+1}}{\sqrt{2}}$$

Haar wavelet (5)

Transition from resolution j to $j + 1$ (decompression)

- Inversion of system (6)

$$\phi_{2k}^{j+1} = \frac{\phi_k^j + \psi_k^j}{\sqrt{2}} \quad \phi_{2k+1}^{j+1} = \frac{\phi_k^j - \psi_k^j}{\sqrt{2}}$$

- We have: $x^{j+1}(t) = \sum_{k=0}^{2^j-1} s_k^j \phi_k^j(t) + \sum_{k=0}^{2^j-1} d_k^j \psi_k^j(t) = \sum_{k=0}^{2^{j+1}-1} s_k^{j+1} \phi_k^{j+1}(t)$
- We derive:

$$s_{2k}^{j+1} = \frac{s_k^j + d_k^j}{\sqrt{2}} \quad s_{2k+1}^{j+1} = \frac{s_k^j - d_k^j}{\sqrt{2}}$$

Content

Part 1: Fourier Transform, Short Time Fourier Transform

Part 2: Wavelets

- Continuous wavelets

- Dyadic wavelets

- Multiresolution Analysis (MRA)

- Haar wavelet

- The discrete wavelet transform**

Part 3: discrete wavelet transform for images, applications

The discrete wavelet transform (1)

- ▶ Haar: scaling and details functions or coefficients at a given resolution derive from a linear combination of scaling and wavelet functions or coefficients at the superior resolution.

This can be generalized...

- ▶ $V^0 \subset V^1$:
 - ▶ then $\phi(t) \in V^0 \Rightarrow \phi(t) \in V^1$
 - ▶ then $\exists h(n)$ such as $\phi(t) = \sum_n h(n)\phi_n^1(t)$
 - ▶ then $\phi(t) = \sqrt{2} \sum_n h(n)\phi(2t - n)$
- ▶ This holds for any $V^{j-1} \subset V^j$ and generalizes as follow:
 - ▶ $\phi_k^{j-1}(t) \in V^{j-1} \Rightarrow \phi_k^{j-1}(t) \in V^j$
 - ▶ $\phi_k^{j-1}(t) = \sum_n h(n)\phi_{n+2k}^j(t) = \sqrt{2^j} \sum_n h(n)\phi(2^j t - n - 2k)$
- ▶ Consequence on approximation coefficients:
 - ▶ $s_k^{j-1} = \langle x, \phi_k^{j-1} \rangle$
 - ▶ $s_k^{j-1} = \sum_n h(n) \langle x, \phi_{n+2k}^j(t) \rangle$
 - ▶ $s_k^{j-1} = \sum_n h(n)s_{n+2k}^j = \sqrt{2} \sum_{n'} h(n' - 2k)s_{n'}^j$
 - ▶ $s_k^{j-1} = h^* \star s^j(2k)$ (with h^* the mirror filter of h)
- ▶ $\phi \leftrightarrow h$

The discrete wavelet transform (2)

- ▶ Same discussion on details subspaces W^j
- ▶ $W^0 \subset V^1$:
 - ▶ $\psi(t) \in W^0 \Rightarrow \psi(t) \in V^1$
 - ▶ $\exists g$ such as $\psi(t) = \sum_n g(n) \phi_n^1(t) = \sqrt{2} \sum_h g(n) \phi(2t - n)$
- ▶ Superior resolutions:
 - ▶ $\psi_k^{j-1}(t) = \sum_n g(k) \phi_{n+2k}^j(t) = \sqrt{2^j} \sum_n g(n) \phi(2^j t - n - 2k)$
- ▶ Consequence on details coefficients:
 - ▶ $d_k^{j-1} = \langle x, \psi_k^{j-1} \rangle$
 - ▶ $d_k^{j-1} = \sum_n g(n) \langle x, \phi_{n+2k}^j \rangle$
 - ▶ $d_k^{j-1} = \sum_n g(n) s_{n+2k}^j$
 - ▶ $d_k^{j-1} = g^* \star s^j(2k)$
- ▶ $\psi \leftrightarrow g$
- ▶ Reconstruction:

$$s_k^{j+1} = \sum_n s_n^j h(k - 2n) + \sum_m d_m^j g(k - 2m)$$

The discrete wavelet transform (3)

Link between ϕ and h

- ▶ Build an orthonormal basis, two ways: choose ϕ (see Haar scaling function), or choose h
- ▶ Indeed:
 - ▶ ϕ and h are linked ($V^0 \subset V^1$): $\phi(t) = \sqrt{2} \sum_n h(n) \phi(2t - n)$
 - ▶ Apply FT on previous equation, introduce $\omega = 2\pi f$, denote $\Phi = FT(\phi)$, and $H(\omega) = \sum_n h(n) e^{-in\omega}$
 - ▶ We have:

$$\Phi(\omega) = \frac{1}{\sqrt{2}} \Phi\left(\frac{\omega}{2}\right) H\left(\frac{\omega}{2}\right) = \prod_{j=1}^{+\infty} \frac{1}{\sqrt{2}} H\left(\frac{\omega}{2^j}\right)$$

- ▶ Then H can be derived from Φ and reciprocally
- ▶ H is a low-pass filter. Indeed:
 - ▶ $H(0) = \sqrt{2} \Phi(0) / \Phi(0/2) = \sqrt{2}$ ($\Phi(0) \neq 0$ because $\int \phi(t) dt$ can not be null)
 - ▶ from relation between Φ and H , it can be shown that $|H(\omega)|^2 + |H(\omega + \pi)|^2 = 2$, then $H(\pi) = 0$

The discrete wavelet transform (4)

Link between ψ and g , and h !

- ▶ Similarly, we have ($W^0 \subset V^1$): $\psi(t) = \sqrt{2} \sum_n g(n) \phi(2t - n)$ then:

$$\Psi(\omega) = \frac{1}{\sqrt{2}} \Phi\left(\frac{\omega}{2}\right) G\left(\frac{\omega}{2}\right) = \prod_{j=1}^{+\infty} \frac{1}{\sqrt{2}} G\left(\frac{\omega}{2^j}\right)$$

- ▶ G is a high-pass filter:
 - ▶ $G(0) = 0$ as $\Psi(0) = \int \psi(t) dt = 0$ by definition (oscillating)
 - ▶ Again: $|G(\omega)|^2 + |G(\omega + \pi)|^2 = 2$ and then $G(\pi) = \sqrt{2}$
- ▶ Moreover, one can prove that:
 - ▶ $G(\omega) = -\Lambda(\omega) \bar{H}(\omega + \pi)$ with Λ verifying this two conditions:
 $\Lambda(\omega + 2\pi) \pm \Lambda(\omega) = 0$
 - ▶ A solution is $\Lambda(\omega) = -e^{-i\omega}$
- ▶ Finally g can be derived from h :

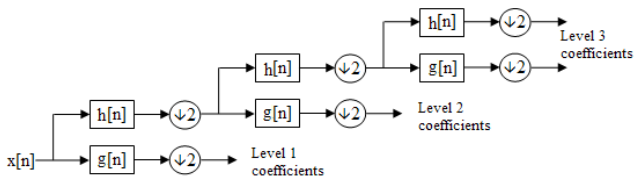
$$\begin{aligned} G(\omega) &= -e^{-i\omega} \bar{H}(\omega + \pi) \\ g(n) &= (-1)^n h(1 - n) \end{aligned} \tag{7}$$

- ▶ g is the conjugate and mirror filter of h

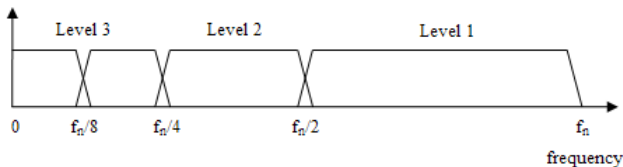
The discrete wavelet transform (5)

Cascade algorithm with mirror and conjugate filters

- ▶ The DWT is efficiently implemented using a series of low and high-pass filtering and sub-sampling (due to dyadic nature of MRA)



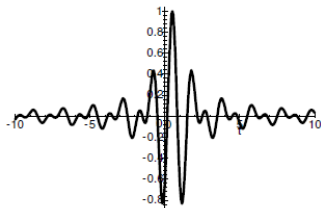
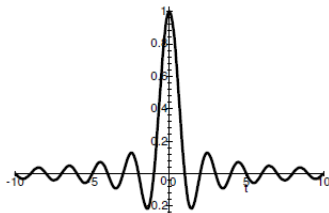
- ▶ low-pass filtering: low frequencies are captured with accurate frequency resolution, but poor time resolution
- ▶ high-pass filtering: high frequencies are captured with poor frequency resolution but an accurate time resolution



Other wavelet transforms (1)

Shannon wavelet

- ▶ We only know Haar wavelet: $h(n) = \begin{pmatrix} 1 & 1 \end{pmatrix}$, and $g(n) = \begin{pmatrix} 1 & -1 \end{pmatrix}$
(Important: do not forget to divide by $\sqrt{2}$ in practice!)
- ▶ Shannon wavelet (dual of Haar):
 - ▶ Haar: $\phi(t) = \text{Rect}(t) \Rightarrow \Phi(f) = \text{sinc}(\pi f)$
 - ▶ Shannon: $\phi(t) = \text{sinc}(\pi t) \Rightarrow \Phi(\omega) = \text{Rect}(\omega)$
 - ▶ We derive $H(\omega)$ then h : $h(n) = \text{sinc}\left(\frac{n\pi}{2}\right)$
 - ▶ then $G(\omega)$ from $g(n) = (-1)^n h(1-n) = (-1)^n \text{sinc}\left(\frac{(1-n)\pi}{2}\right)$
 - ▶ then $\Psi(\omega)$ and finally $\psi(t) = \frac{\cos(\pi t) - \sin(2\pi t)}{\pi t}$



Other wavelet transforms (2)

Daubechies wavelet (1)

- ▶ Motivation: build a basis with n null moments and compact support
- ▶ ψ has n null moments if:

$$\int_{\mathbb{R}} t^k \psi(t) dt = 0 \quad \forall k = 1, \dots, n$$

- ▶ In other words: $\langle \psi(t), t^k \rangle = 0$, the mother wavelet is orthogonal to polynomials of degree $\leq n$
- ▶ Interest: the more a wavelet function has null moments, the more the signal representation is sparse. Essential property for compression.
- ▶ Properties of wavelet basis having many null moments:
 - ▶ the scaling function better approximates smooth signals
 - ▶ the wavelet function is dual: it better captures signal discontinuities

Other wavelet transforms (3)

Daubechies wavelet (2)

- ▶ Daubechies with 4 null moments (denoted D_4 or db2 with Matlab)
- ▶ Filters h et g are of length 4
- ▶ If $h = (h_0, h_1, h_2, h_3)$ then $g = (h_3, -h_2, h_1, -h_0)$ (eq.(7))
- ▶ Constraints to determine the coefficients:
 - ▶ ψ of null mean $\Rightarrow h_3 - h_2 + h_1 - h_0 = 0$
 - ▶ ψ with 4 null moments $\Rightarrow h_3 - 2h_2 + 3h_1 - 4h_0 = 0$
 - ▶ $\langle \psi(t), \psi(t-1) \rangle = 0 \Rightarrow h_1 h_3 + h_2 h_0 = 0$
 - ▶ $\|\phi\| = 1 \Rightarrow h_0 + h_1 + h_2 + h_3 = 2$
- ▶ We find: $h_0 = \frac{1+\sqrt{3}}{4}$ $h_1 = \frac{3+\sqrt{3}}{4}$ $h_2 = \frac{3-\sqrt{3}}{4}$ $h_3 = \frac{1-\sqrt{3}}{4}$

Other wavelet transforms (3)

Daubechies wavelet (3)

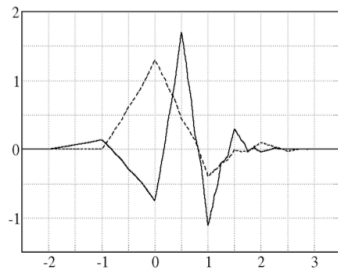
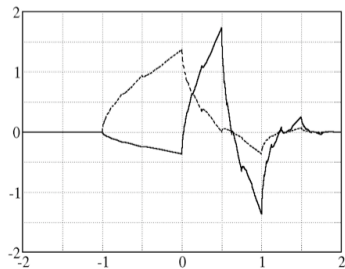


Figure: Daubechie scaling and wavelet functions with 4 null moments (db2) and 6 null moments (db3)

Content

Part 1: Fourier Transform, Short Time Fourier Transform

Part 2: Wavelets

Part 3: discrete wavelet transform for images, applications

Discrete wavelet transform for images

Applications

2-D DWT for images

- ▶ 2-D Haar decomposition for a 2-D signal
- ▶ Two approaches:
 - ▶ the standard decomposition: 1-D DWT on one direction (lines), then 1-D DWT on the other direction (columns)
 - ▶ non standard decomposition: the 1-D DWT is alternated on lines and columns
 - ▶ both approaches lead to two specific 2-D Haar bases
- ▶ Advantages:
 - ▶ standard: only 1-D transforms
 - ▶ non standard, faster: $\frac{8}{3}(n^2 - 1)$ operations against $4(n^2 - n)$ for standard one

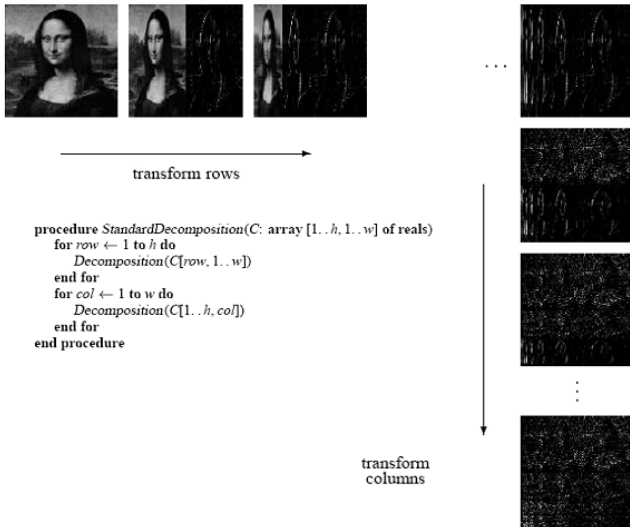
2-D DWT: standard decomposition (1)

- ▶ Basis of the Haar standard decomposition is a tensor product between the 1-D bases:

$$\Psi_{k,k'}^{j,j'}(x,y) = \psi_k^j(x)\psi_{k'}^{j'}(y)$$

- ▶ Algorithm:
 1. apply a DWT on each line to obtain an intermediary image, repeat up to the finest resolution $j = 0$.
 2. then, apply a DWT on each column of this image, repeat up to the finest resolution
- ▶ we obtain an unique approximation coefficient and a set of details coefficients for all resolutions

2-D DWT: standard decomposition (2)



2-D DWT: standard decomposition (3)

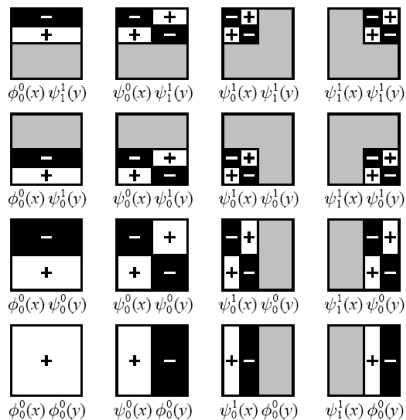


Figure: Haar standard basis

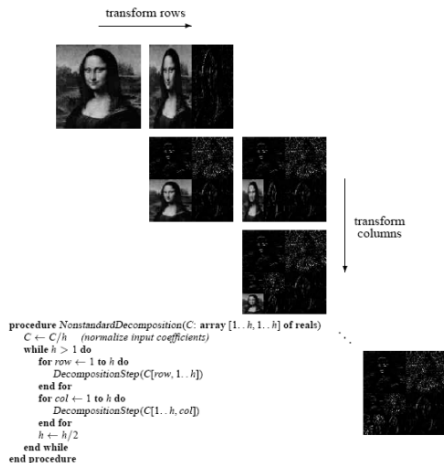
2-D DWT: non standard decomposition (1)

- ▶ Principle: perform an MRA of $L^2(\mathbb{R}^2)$
- ▶ Let's define $\mathcal{V}^j = V^j \otimes V^j$
- ▶ The details spaces are $\mathcal{W}^j$ such as $\mathcal{V}^{j+1} = \mathcal{V}^j \oplus \mathcal{W}^j$
- ▶ Then, we have:

$$\begin{aligned}\mathcal{V}^{j+1} &= V^{j+1} \otimes V^{j+1} \\ &= (V^j \oplus W^j) \otimes (V^j \oplus W^j) \\ &= (V^j \otimes V^j) \oplus (W^j \otimes V^j) \oplus (V^j \otimes W^j) \oplus (W^j \otimes W^j) \\ &= \mathcal{V}^j \oplus \mathcal{W}^j\end{aligned}$$

- ▶ Basis of $\mathcal{W}^j$: $\psi_k^j(x)\phi_{k'}^j(y), \phi_k^j(x)\psi_{k'}^j(y), \psi_k^j(x)\psi_{k'}^j(y), \quad k, k' \in \mathbb{Z}$

2-D DWT: non standard decomposition (2)



The DWT is alternated on lines and columns:

1. one iteration of 1-D DWT on each lines
2. one iteration of 1-D DWT on each column
3. repeat stages 1. and 2. on approximation image up to resolution $j = 0$

2-D DWT: non standard decomposition (3)

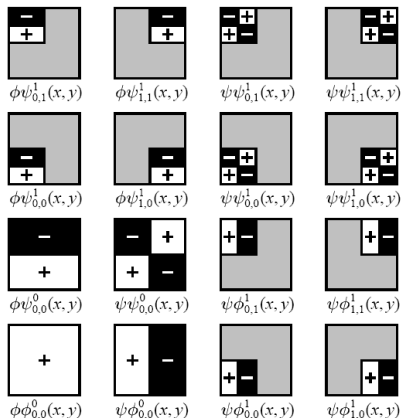
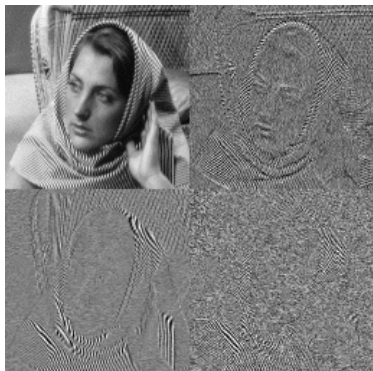
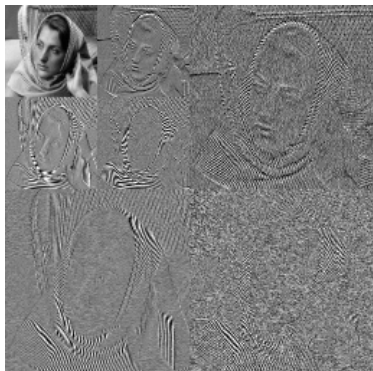


Figure: Base non standard de Haar 2-D

2-D DWT: Examples with Matlab²



```
[S1,H1,V1,D1] = dwt2(X,'haar');  
imagesc([S1,H1;V1,D1])
```



```
[S2,H2,V2,D2] = dwt2(S1,'haar');  
imagesc([S2,H2;V2,D2],H1;V1,D1)
```

²Python: use PyWavelets package

Content

Part 1: Fourier Transform, Short Time Fourier Transform

Part 2: Wavelets

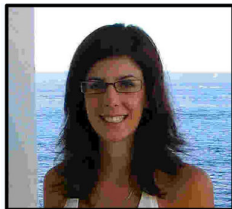
Part 3: discrete wavelet transform for images, applications

Discrete wavelet transform for images

Applications

Application: compression (1)

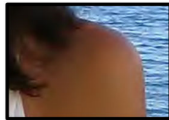
- ▶ Famous application (JPEG2000)
- ▶ JPEG compression (Fourier based): suppression of high frequencies
⇒ edges are degraded (Gibbs phenomena)
- ▶ Suitable wavelet basis for edges representation: Haar (the Haar scaling function is basically an edge)



Jpeg



Jpeg 2000



Application: compression (2)

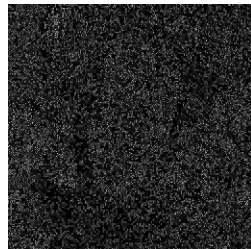
- ▶ Principle: keep only the biggest details coefficients
- ▶ We apply an threshold:



Image



Reconstruction with
a threshold value of 10



error: 1%

- ▶ 47% of details coefficients are zero (hence lesser than 10)
- ▶ without compression: 10% are zero

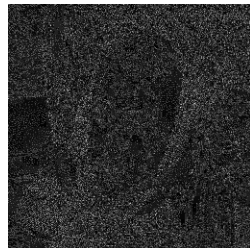
Application: compression (3)



Image



Reconstruction with
a threshold value of 40



error: 4.3 %

- ▶ 89% of the details coefficients are zero.
- ▶ Drawback (Haar): high compression rate makes appear blocs in the image

Application: denoising (1)

- ▶ Y image acquisition having an additive noise B
- ▶ Retrieve X such as

$$Y = X + B$$

- ▶ Practically, we look for an operator D minimizing the reconstruction error:

$$E(\|X - D(Y)\|) = \sum_{i=1}^N E(X(i) - D(Y)(i))^2 \quad (8)$$

- ▶ Many methods! Depending on the noise characteristics
- ▶ If B centered Gaussian, a wavelet filtering gives good results
- ▶ Method:
 - ▶ projection on a wavelet basis (encoding)
 - ▶ hard threshold: details coefficients lesser than threshold S are nullified
 - ▶ soft threshold: details coefficients lesser than threshold S are nullified, other are attenuated
 - ▶ How to choose S ?

Application: denoising (2)

- ▶ An optimal value minimizing (8) with respect to B be Gaussian of standard deviation σ :

$$S = \sigma \sqrt{2 \ln N}$$

- ▶ Estimation of σ ?

$$\hat{\sigma} = \frac{M_s}{0,6745}$$

with M_s median value of details coefficients at the finest resolution

- ▶ Wavelet basis?
 - ▶ Haar
 - ▶ Daubechies
 - ▶ others: curvelets, ridgelets, ...

Application: denoising (3)



Image



Gaussian noise



Haar



Daubechies (db3)

Other applications

- ▶ 3-D mesh: approximation of a volume by decomposition on Haar wavelets
- ▶ Pattern recognition: for example, faces characterization, by projection on a wavelets basis
- ▶ Texture characterization and modeling
- ▶ Image watermarking: the trademark is projected on a wavelets basis, highest coefficients are retained and added to image details coefficients
- ▶ Sparse representation: wavelets allow sparse representations i.e. having a minimal number of coefficients
 - ▶ with Fourier: an edge is encoding in all modes, so many coefficients are necessary. Remove one of them has a global impact
 - ▶ with wavelets: an edge is encoded locally, by few coefficients. Remove one of them has a local impact.