

TADI - Wavelets - Tutorial works

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Fourier Analysis

Reminder:

$$\begin{aligned}\cos(p) + \cos(q) &= 2 \cos\left(\frac{p+q}{2}\right) \cos\left(\frac{p-q}{2}\right) & \cos(a+b) &= \cos(a)\cos(b) - \sin(a)\sin(b) \\ \cos(p) - \cos(q) &= -2 \sin\left(\frac{p+q}{2}\right) \sin\left(\frac{p-q}{2}\right) & \cos(a-b) &= \cos(a)\cos(b) + \sin(a)\sin(b) \\ \sin(p) + \sin(q) &= 2 \sin\left(\frac{p+q}{2}\right) \cos\left(\frac{p-q}{2}\right) & \sin(a+b) &= \sin(a)\cos(b) + \cos(a)\sin(b) \\ \sin(p) - \sin(q) &= 2 \sin\left(\frac{p-q}{2}\right) \cos\left(\frac{p+q}{2}\right) & \sin(a-b) &= \sin(a)\cos(b) - \cos(a)\sin(b) \\ \cos(x) &= \frac{e^{ix} + e^{-ix}}{2} & 2 \cos(a) \cos(b) &= \cos(a+b) + \cos(a-b) \\ \sin(x) &= \frac{e^{ix} - e^{-ix}}{2i} & 2 \sin(a) \sin(b) &= \cos(a-b) - \cos(a+b) \\ e^{inx} &= \cos(nx) + i \sin(nx) & 2 \sin(a) \cos(b) &= \sin(a+b) + \sin(a-b) \\ & & X : f \mapsto X(f) &= \int_{-\infty}^{+\infty} x(t) e^{-i2\pi f t} dt \\ & & \text{int. by part : } \int u'v &= [uv] - \int uv'\end{aligned}$$

Exercise 1: Fourier Series

1. Let E be the vector space of continuous functions defined on $[-\pi, \pi[$ valued in $\mathbb{C}$, with associated scalar product:

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(t) \bar{g}(t) dt$$

- Prove that the family $\mathcal{F}$ defined by $\{\cos(nx) \mid n \in \mathbb{N}\} \cup \{\sin(nx) \mid n \in \mathbb{N}^*\}$ is orthogonal.
- Is $\mathcal{F}$ a basis for F , the vector space of 2π periodic functions?
- Propose an orthonormal basis for F .

2. Fourier Series: all 2π periodic function writes:

$$\begin{aligned}f(x) &= \frac{1}{2}a_0 + \sum_{n=1}^{+\infty} a_n \cos(nx) + b_n \sin(nx) \\ &= \sum_{n \in \mathbb{Z}} c_n e^{inx}\end{aligned}$$

Express coefficients c_n as function of a_n and b_n .

3. Prove that the family $\{e^{inx}\}_{n \in \mathbb{Z}}$ is an orthogonal basis for $L^2([0, 2\pi])$.

Exercise 2: Fourier transform

1. Let $z(t) = x(t - \tau)$, prove that $Z(f) = e^{-2i\pi f\tau} X(f)$.
2. Prove that if x is even (respectively odd), its Fourier transform is even (respectively odd).
3. Let x be such as $\lim_{t \rightarrow \pm\infty} x(t) = 0$, let $y = x'$, prove that $Y(f) = 2i\pi f X(f)$.
4. Let $y(t) = t x(t)$, prove that $X'(f) = -2i\pi Y(f)$.

Exercise 3: Fourier transform of usual functions

1. $\text{Rect}\left(\frac{t}{T}\right)$, with $\text{Rect}(t) = \begin{cases} 1 & \text{si } |t| \leq \frac{1}{2} \\ 0 & \text{sinon} \end{cases}$ (Gate or Rectangular function)
2. $x(t) = e^{-\alpha|t|}$, $\alpha > 0$
3. $g(t) = e^{-b^2 t^2}$,
 - prove that $g'(t) + 2b^2 t g(t) = 0$
 - deduce that $G'(f) + \frac{2\pi^2}{b^2} G(f) = 0$
 - and that $G(f) = \frac{\sqrt{\pi}}{|b|} e^{-\frac{\pi^2 f^2}{b^2}}$
4. $k(t) = e^{-\alpha t} \mathbb{1}_{t \geq 0}$, $\alpha > 0$
5. $z(t) = t \mathbb{1}_{t \in]-a, a[}$

Exercise 4: frequency resolution and windowing

Let us consider the Sine function $x(t) = \cos(2\pi f_0 t)$ and the Rectangular function $r(t) = \text{Rect}\left(\frac{t}{L}\right)$. We recall that $X(f) = \frac{1}{2}(\delta(f - f_0) + \delta(f + f_0))$.

1. Determine the Fourier transform of $z(t) = x(t)r(t)$.
2. What can we conclude about the frequency resolution?

Exercise 5: Short-time Fourier Transform

Reminder:

$$\text{TFF}(x)(f, b) = \int_{\mathbb{R}} x(t) \bar{w}(t - b) e^{-2i\pi f t} dt$$

Let's consider the following 1-D signal:

$$x(t) = \cos(2\pi f_1 t) \text{Rect}\left(\frac{t - T_1}{2T_1}\right) + \cos(4\pi f_1 t) \text{Rect}\left(\frac{t - 3T_1}{2T_1}\right)$$

with $f_1 = \frac{1}{T_1}$

1. Draw the graph of signal $x(t)$.
2. Determine and represent the spectrum of x for various disjoint temporal windows of length respectively $\frac{4}{f_1}$ (1 window), $\frac{2}{f_1}$ (2 windows), $\frac{1}{f_1}$ (4 windows), $\frac{1}{2f_1}$ (8 windows).
3. What previous windows can separate in time and in frequency the components of x ? What is the best compromise?

Haar Wavelets

Let's consider the vector space of square integrable functions defined on $\mathbb{R}$ and denoted $E = L^2(\mathbb{R})$. Let ϕ be the scale function defined on $\mathbb{R}$ by:

$$\phi(t) = \begin{cases} 1 & t \in [0, 1[\\ 0 & \text{otherwise} \end{cases}$$

and $\phi_k^j(t) = \sqrt{2^j} \phi(2^j t - k)$:

$$\phi_k^j(t) = \begin{cases} \sqrt{2^j} & t \in [\frac{k}{2^j}, \frac{k+1}{2^j}[\\ 0 & \text{otherwise} \end{cases}$$

Exercise 6: multiresolution analysis of E

1. Prove that ϕ is an admissible scale function to build a multiresolution analysis of E .
2. Describe V^0, V^j .

We recall the multiresolution analysis of a vector space E :

1. $\forall j \in \mathbb{Z} \quad V^j \subset V^{j+1}$
2. $\lim_{j \rightarrow -\infty} V^j = \bigcap_{j \in \mathbb{Z}} V^j = \emptyset$
3. $\lim_{j \rightarrow +\infty} V^j = \bigcup_{j \in \mathbb{Z}} V^j = E$
4. $\exists \phi$ such as $\{\phi(\cdot - k)\}_{k \in \mathbb{Z}}$ is an orthonormal basis of V^0
5. $\forall j \in \mathbb{Z}, f \in V^j \Leftrightarrow f(2\cdot) \in V^{j+1}$
6. (consequence) $\forall j, k \in \mathbb{Z}, f \in V^j \Leftrightarrow f(\cdot - 2^j k) \in V^j$

Exercise 7: orthonormal basis of Haar scale functions

Let's consider the space of square integrable functions defined on $[0, 1[$: $E = L^2([0, 1[)$. We apply a multiresolution analysis using ϕ , we have:

- V^0 the space of constant functions on $[0, 1[$ (dimension 1)
 - V^1 the space of constant functions on $[0, \frac{1}{2}[$ and $[\frac{1}{2}, 1[$ (dimension 2)
 - V^j the space of constant functions on the 2^j intervals $[\frac{k}{2^j}, \frac{k+1}{2^j}[$, $k = 0, \dots, 2^j - 1$ (dimension 2^j)
1. Verify that the family of functions $(\phi_k^j)_{k \in \{0, \dots, 2^j - 1\}}$ is an orthonormal basis of V^j .
 2. Draw the graph of functions ϕ_0^1 et ϕ_1^1 (basis of V^1), and of functions $\phi_0^2, \phi_1^2, \phi_2^2, \phi_3^2$ (basis of V^2).

Exercise 8: orthonormal basis of Haar details functions

1. Determine the two elements ψ_0^1 and ψ_1^1 of the basis of the vector space W^1 such as $V^2 = V^1 \oplus W^1$.
 - Compression: express $\phi_0^1, \phi_1^1, \psi_0^1$ and ψ_1^1 as functions of $(\phi_k^2)_{k \in \{0,1,2,3\}}$
 - Uncompression: express $(\phi_k^2)_{k \in \{0,1,2,3\}}$ as functions of $\phi_0^1, \phi_1^1, \psi_0^1$ and ψ_1^1
 - Determine the function ψ such as $\psi_k^1(t) = \sqrt{2}\psi(2t - k)$
2. Generalization: deduce the definition of the 2^j Haar details functions $(\psi_k^j)_{k \in \{0, \dots, 2^j - 1\}}$ as an orthonormal basis of spaces W^j such as $V^{j+1} = V^j \oplus W^j$.
 - a) Compression: express (ϕ_k^j) and (ψ_k^j) as function of (ϕ_k^{j+1})
 - b) Uncompression: express (ϕ_k^{j+1}) as function of (ϕ_k^j) and (ψ_k^j)
 - c) Compression: express coefficients (s_k^j) et (d_k^j) as fonction of coefficients (s_k^{j+1})
 - d) Uncompression: express coefficients (s_k^{j+1}) as function of coefficients (s_k^j) and (d_k^j)
 - e) Why the relation between coefficients s and d is the same than between functions ϕ and ψ ?

Exercise 9: representation of a signal in the Haar wavelet basis

Let S be the following discrete signal $[2, 4, 8, 12, 14, 0, 2, 1]$.

1. Project S in the space $V^0 \oplus W^0 \oplus W^1 \oplus W^2$.
2. Draw the signal obtained for each resolution level (i.e. projection of S in V^2, V^1 and V^0), and details coefficients (i.e. projection of S in W^2, W^1 and W^0).
3. The module of the Discrete Fourier Transform of S is: $[9, 8, 11, 24, 44, 24, 11, 8]$ (rounded to the closest integer).

Discuss the interpretation in terms of scale space of the signal in the Haar wavelet space and in the Fourier space.