Computing points per connected components of semi-algebraic sets

Edern Gillot
Joint work with Jérémy Berthomieu and Mohab Safey El Din

10 March 2025

Problem Exposition

Real polynomial systems
with **constraints**

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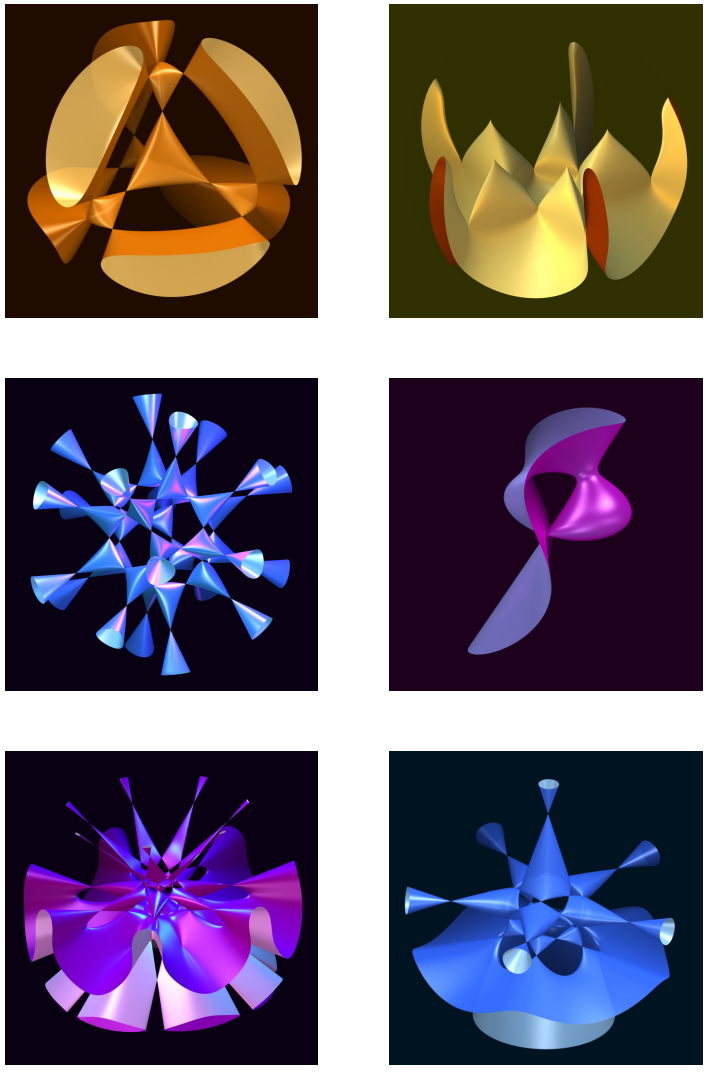

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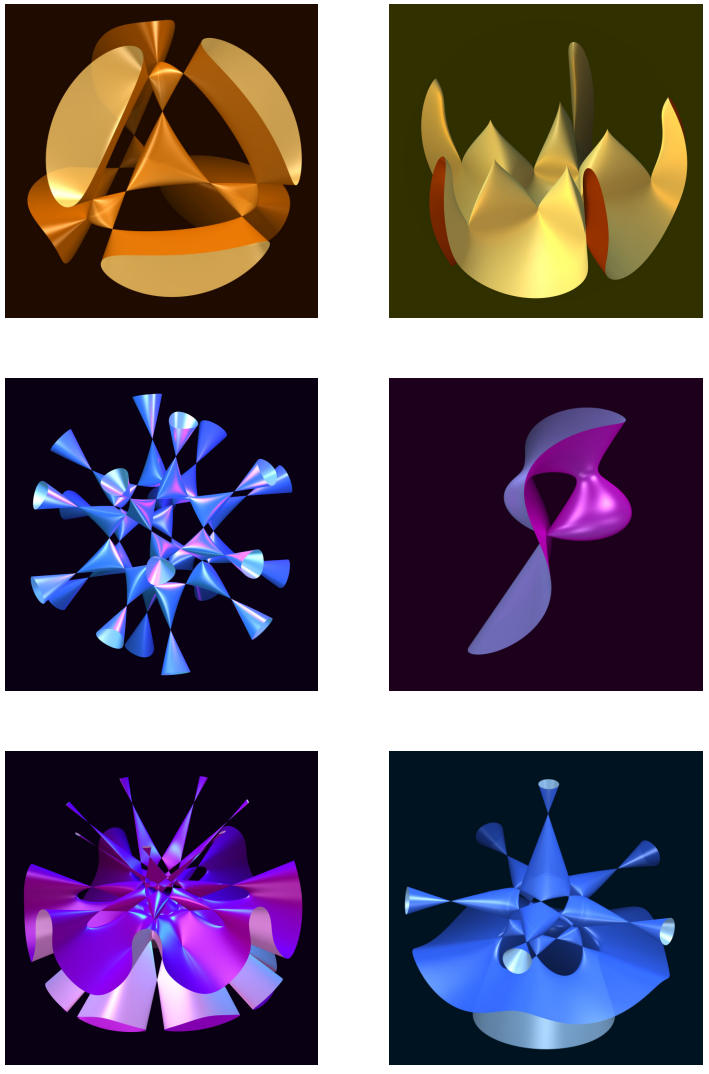

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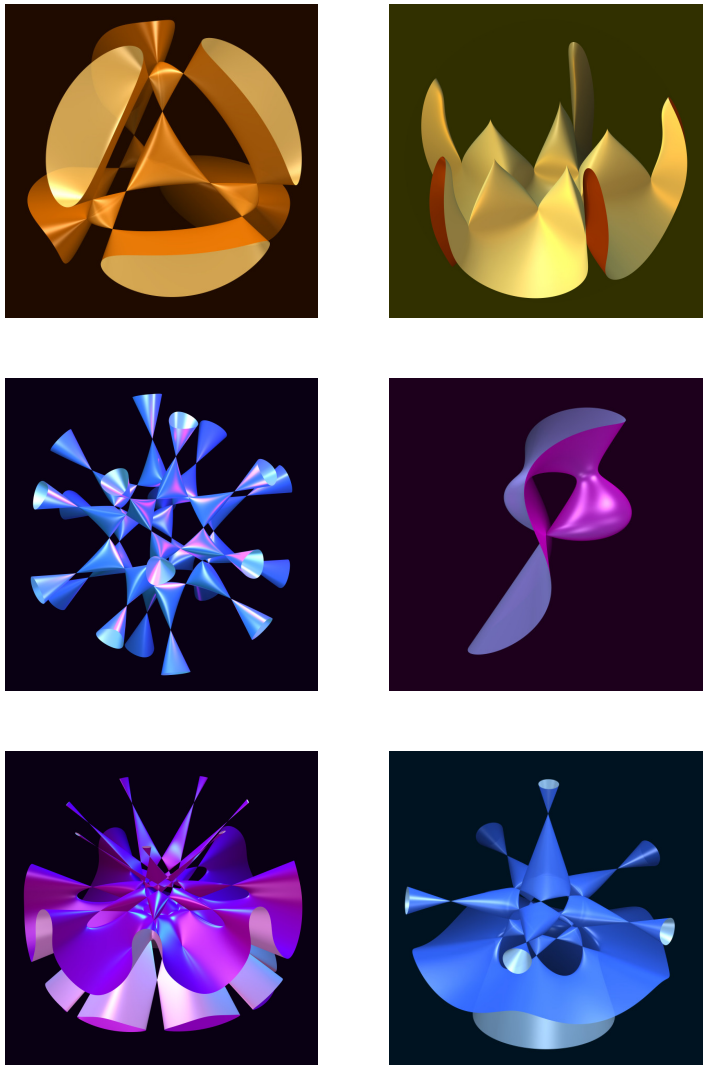

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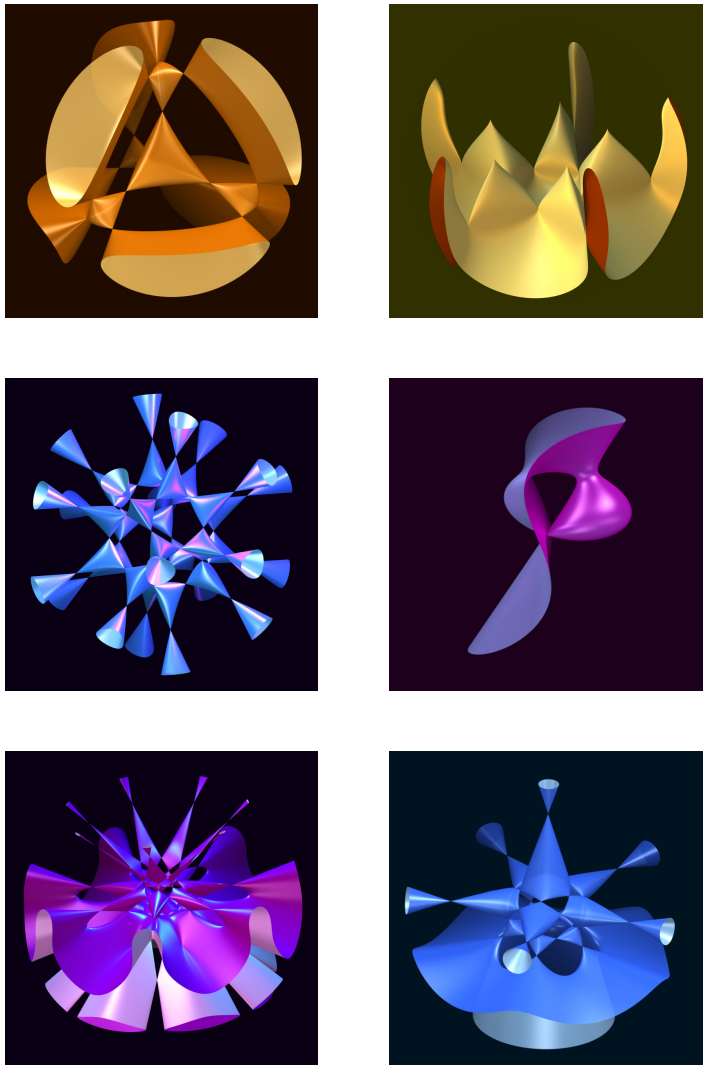

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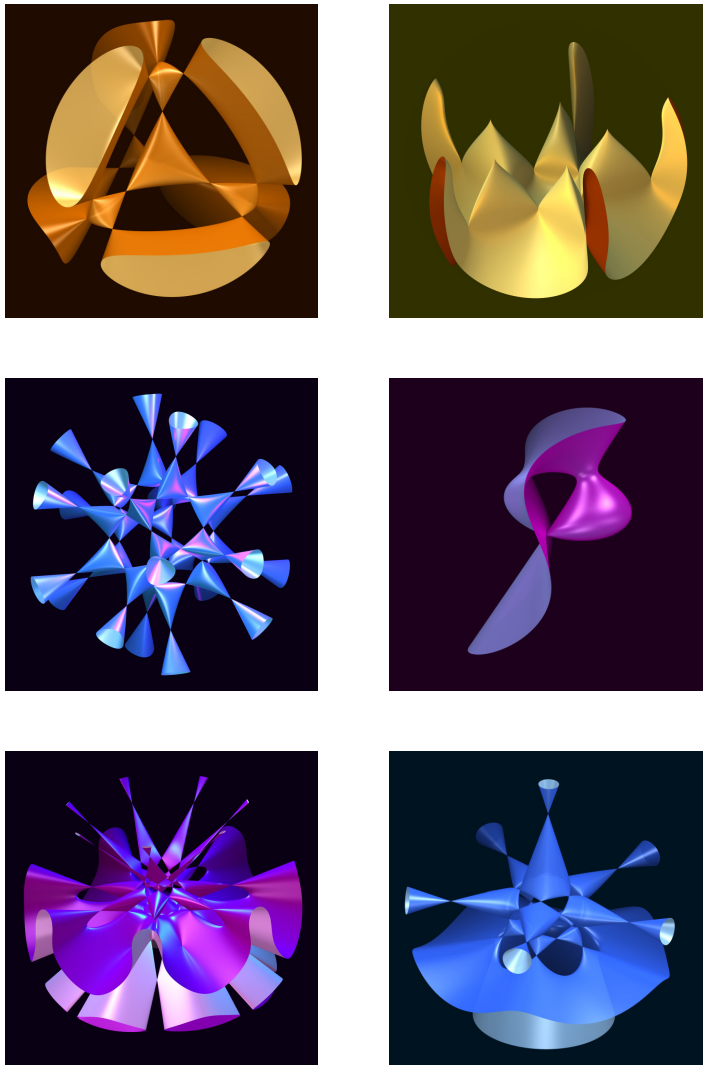

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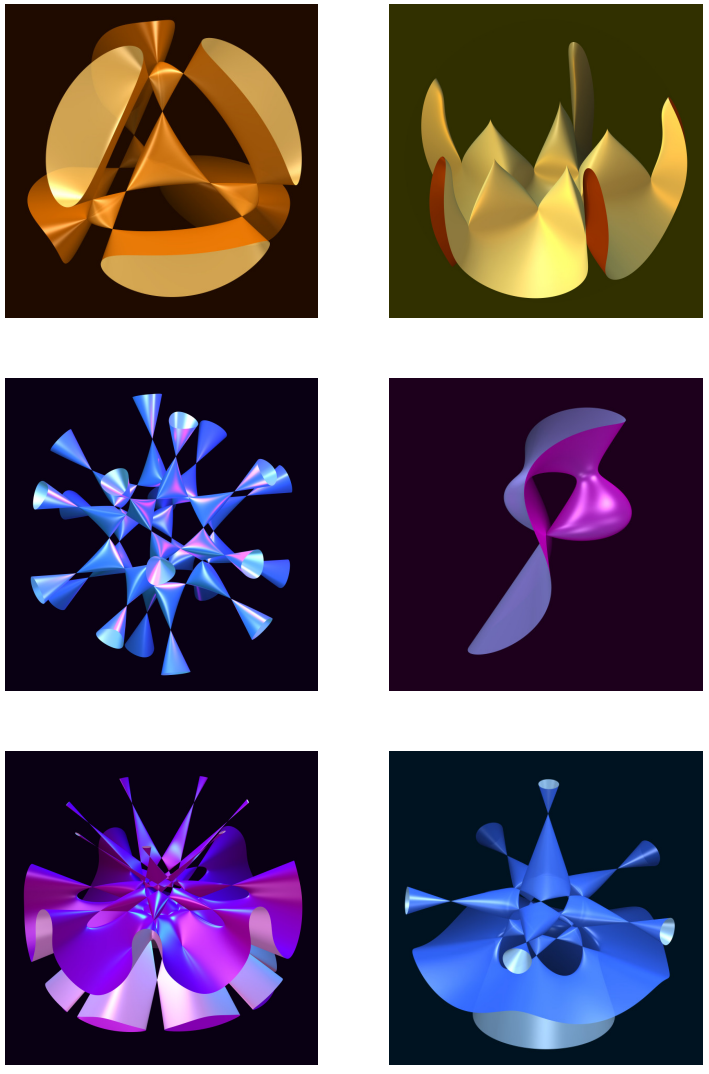

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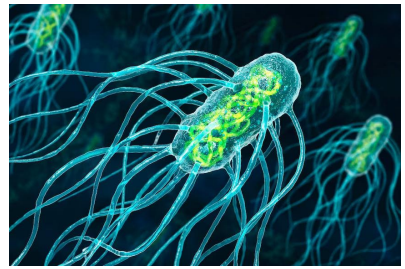

Applications:

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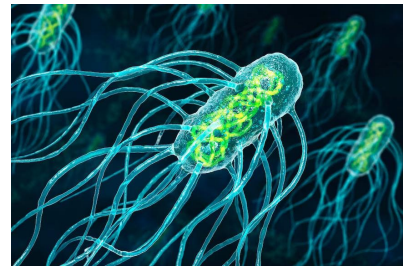

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Make use of **inherent structure**

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Related problems have **structural** results
[Faugère–Labahn–Safey El Din–Schost–Vu 2023] [Safey El Din–Schost 2018]

Contributions & Perspectives

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Long-term aims: generalise to **several** polynomials; **apply** to problems from other scientific fields; **study** other structures