Computing the Connected Components of Real Algebraic Curves

Elisabetta Rocchi
Joint work with Mohab Safey El Din

International Symposium on Symbolic and Algebraic Computation
Oldenburg, Germany
July 2026

Problem Statement

Real Algebraic Curves

$I = \langle f_1, \ldots, f_r \rangle \subset \mathbb{Q}[x_1, \ldots, x_n]$, $\dim(I) = 1$

$\mathcal{C}_{\mathbb{R}} = V_{\mathbb{R}}(I) = \{\boldsymbol{x} \in \mathbb{R}^n \mid f_1(\boldsymbol{x}) = \cdots = f_r(\boldsymbol{x}) = 0\}$

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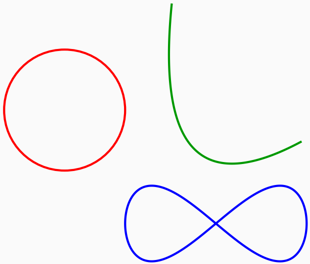
Theorem (folklore)

$\exists\ m \in \mathbb{N}$ such that $\mathcal{C}_{\mathbb{R}} = C_1 \sqcup \cdots \sqcup\ C_m$, where $C_1, \ldots, C_m$ are non-empty semi-algebraic connected sets.



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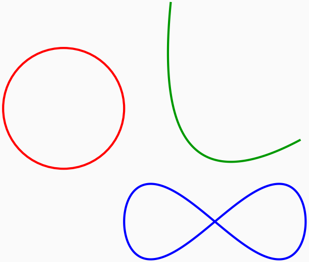

C_i **connected component**

- **Finite**: $m \leq d(2d-1)^{n-1},\ d = \max_i \deg f_i$
  [Milnor 1964]
- **Semi-algebraic** $\rightarrow$ **semi-algebraic polynomial constraints**

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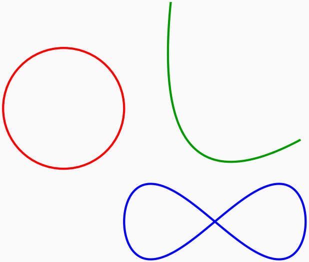

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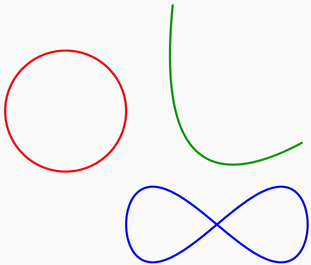

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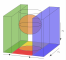
Motivation

This problem arises in optical systems; an application will be presented at the end.

State of the Art

Prior work (semi-algebraic sets)

[Brown-Davenport (2007)] [Collins (1974)] Cylindrical Algebraic Decomposition — doubly exp. in n



Maple Wolfram *Mathematica*

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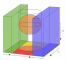

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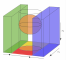

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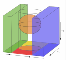

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Related work (algebraic curves)

[Cheng-Jin-Pouget-Wen-Zang (2024)] Topology of curves in $\mathbb{R}^3$ — $\widetilde{O}(d^{18} + d^{17}t)$

d: degree of polynomials, t: coeff. bitsize

[Islam-Poteaux-Prébet (2023)] Connectivity queries space curves — $\widetilde{O}(\delta^6 + \delta^5\tau)$

δ: degree of the curve ($\delta \sim d^{n-1}, \tau \sim d^{n-1}t$)

Contributions

Plane Curve: $\mathcal{C}_{\mathbb{R}} = V_{\mathbb{R}}(f)$ where $f \in \mathbb{Z}[x_1, x_2]$ is square-free of magnitude (δ, τ)

Main Theorem (Plane curve)

The connected components of $\mathcal{C}_{\mathbb{R}}$ can be computed in $\widetilde{O}(\delta^7 + \delta^6\tau)$ bit operations.

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$\mathbf{f} \subset \mathbb{Z}[x_1, \ldots, x_n]$ is radical of dimension 1 and magnitude (d, t)
$\mathbf{A}_1, \mathbf{A}_2 \in \mathrm{GL}_n(\mathbb{Z})$ **generic** with bit size of the entries α
$\varepsilon \rightarrow$ probability factor

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Given $\mathbf{f}, \mathbf{A}_1, \mathbf{A}_2$ and $0 < \varepsilon < 1$, the connected components of $\mathcal{C}_{\mathbb{R}}$ are computable in

$$\widetilde{O}(n^{12} d^{7n-7}(t + \alpha + 1) + (t + \log \tfrac{1}{\varepsilon})(n^{14} d^{6n-5} + \tbinom{n+d}{n} n^{16} d^{3n}))$$

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$$d^{O(n^4)} \rightarrow d^{7n-7}$$

$n = 3 \rightarrow \widetilde{O}(d^{14} + \ldots)$ $\quad \widetilde{O}(\underbrace{d^{18} + \ldots}_{\text{Topology}})$ $\quad \widetilde{O}(\underbrace{d^{12} + \ldots}_{\text{Connectivity}})$

d^2: degree of the curve

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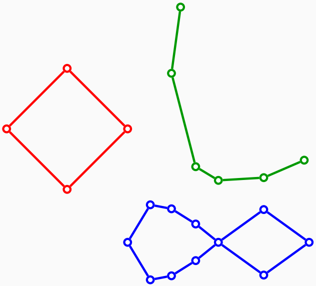

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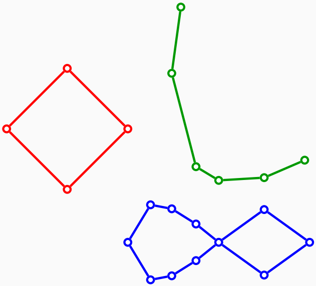

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Enrich the graph:
new vertices and edges

Semi-algebraic Real Isotopy Graph

$\mathcal{H} \subset \mathbb{R}^2$ encodes the topology of the curve $\mathcal{C}_{\mathbb{R}}$

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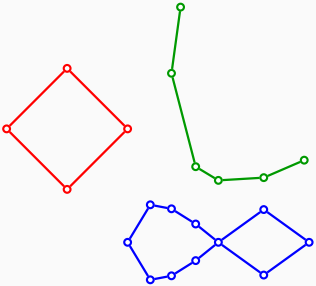

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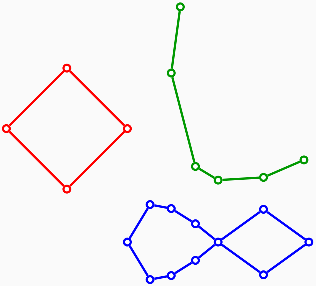

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Assumption: $\mathcal{S}(f) = \left\{p \in \mathcal{C}_{\mathbb{R}} \mid \exists k \in \{1, \ldots, \delta\} : \partial^k_{x_2}(f)(p) = 0\right\}$ is a finite set.

Plane Curve: Graph Construction

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Real Isotopy Graph $O(\delta^6 + \delta^5\tau)$ $\mathcal{G}$

$\pi_1 : (x_1, x_2) \mapsto x_1$

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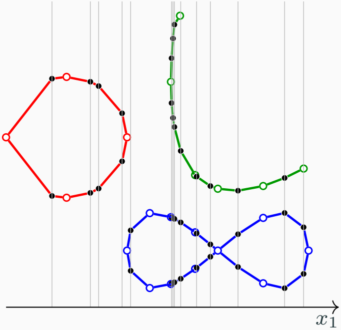

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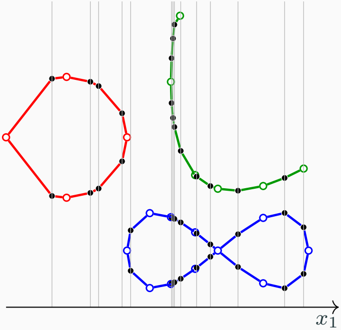

Semi-algebraic description of the edges of $\mathcal{H}$

Lemma

$\forall\ e \in \mathcal{E}(\mathcal{H})$, $\sigma_e = \left(\mathrm{sign}(\partial^k_{x_2} f)\right)_{k=1}^{\delta}$ is constant. Moreover, σ_e uniquely identifies e among all edges above I_e.

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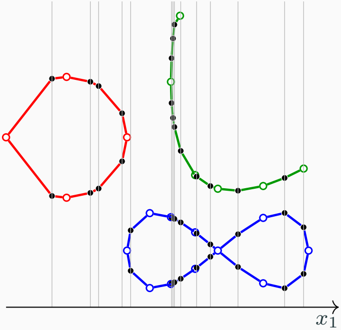

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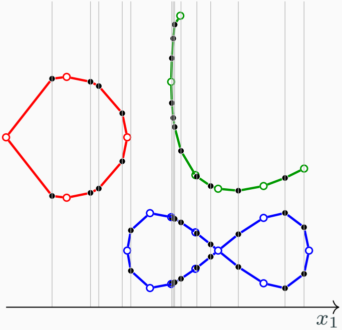

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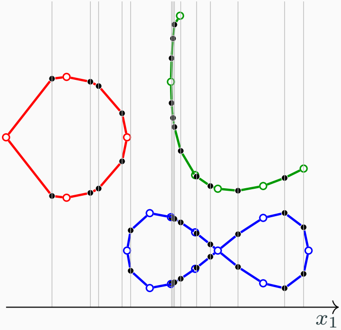

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semi-algebraic isotopy graph of $V_{\mathbb{R}}(f_i)$

Plane Curve: General case

$$\mathcal{S}(f) \;=\; \left\{p \in \mathcal{C}_{\mathbb{R}} \mid \exists k \in \{1,\ldots,\delta\} : \partial_{x_2}^k(f)(p) = 0\right\} \text{ is **not** finite}$$

$$\exists\, k \;:\; g = \gcd(f, \partial_{x_2}^k f) \neq 1 \;\longrightarrow\; f = g \cdot \frac{f}{g}$$

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$\mathcal{H}_j \cup \mathcal{H}_k$

- Compute additional vertices $V_{jk} = V(f_j, f_k)$
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$$\boxed{O(\delta^7 + \delta^6\tau)}$$

Space Curve: Preliminaries

$\mathbf{f}=(f_1,\ldots,f_{n-1}) \subset \mathbb{Z}[x_1,\ldots,x_n]$ radical, magnitude (d,h) $\longrightarrow$ $\mathcal{C}_{\mathbb{R}}=V_{\mathbb{R}}(\mathbf{f})$

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One-dimensional parametrization $\qquad (\lambda,\mu,(w,v_1,\dots,v_n)) \in \mathbb{Q}[\boldsymbol{x}]^2 \times \mathbb{Q}[u,v]^{n+1}$

[Giusti-Lecerf-Salvy (2001)] [Schost (2000)]

Birational Projection: $\quad \pi_{u,v} : \mathcal{C} \dashrightarrow \mathcal{C}_2$ such that $\pi_{u,v}(\boldsymbol{x}) = (\lambda(\boldsymbol{x}),\mu(\boldsymbol{x}))$

$$\mathcal{C}_2 = V\big(w(u,v)\big)$$

$$\mathcal{C} = \overline{\left\{ \left(\tfrac{v_1}{\partial w/\partial u}, \dots, \tfrac{v_n}{\partial w/\partial u} \right)(\theta,\eta) \mid w(\theta,\eta)=0,\ \tfrac{\partial w}{\partial u}(\theta,\eta) \neq 0 \right\}}$$

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Generic Projection (A_1)

[Islam-Poteaux-Prébet (2023)]

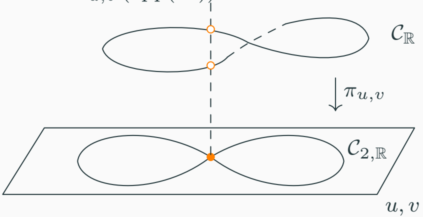
The connectivity of $\mathcal{C}_\mathbb{R}$ can be recover from the connectivity of $\mathcal{C}_{2,\mathbb{R}}$

$\mathrm{app}(\mathcal{C}_2) \subset \mathrm{sing}(\mathcal{C}_2)$ finite set

$\downarrow$

Nodes of $\mathcal{C}_2$ with 2 preimages

- $\mathrm{app}(\mathcal{C}_2)$
- $\pi_{u,v}^{-1}(\mathrm{app}(\mathcal{C}_2))$



Space Curve: From the Planar Case to Missing Points

Semi-algebraic real isotopy graph of $\mathcal{C}_2$

1. Connectivity of $\mathcal{C}_{2,\mathbb{R}}$
2. Semi-algebraic descriptions of $\mathcal{C}_{2,\mathbb{R}}$

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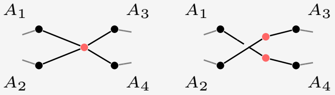
$\xrightarrow{1.}$

Recover Connectivity



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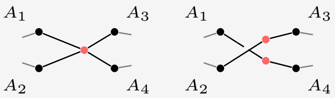

2.

Lift Description of edges and vertices

One-dimensional parametrization $+\ f_1, \dots, f_{n-1}$

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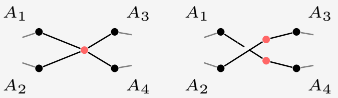

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Semi-algebraic description of the connected components of $\mathcal{C}_\mathbb{R}$ **outside** $\pi^{-1}_{u,v}(\mathrm{app}(\mathcal{C}_2))$

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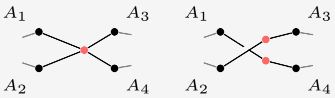

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Warning. Using $\pi_{u,v}^{-1}$ to compute the missing points is computationally expensive.

Space Curve: Recovery of Missing Points

Second Generic Projection (A_2)

$\pi_{u',v'} : \mathcal{C} \dashrightarrow \mathcal{C}'_2$

Semi-algebraic description of the connected components of $\mathcal{C}_{\mathbb{R}}$ outside $\pi^{-1}_{u',v'}(\mathrm{app}(\mathcal{C}'_2))$

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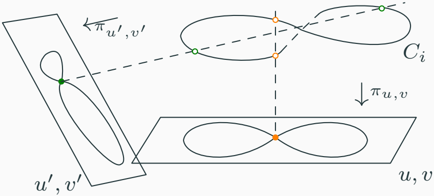

Theorem

For a generic choice of A_2, one has
$\pi^{-1}_{u',v'}(\mathrm{app}(\mathcal{C}'_2)) \cap \pi^{-1}_{u,v}(\mathrm{app}(\mathcal{C}_2)) = \varnothing$.

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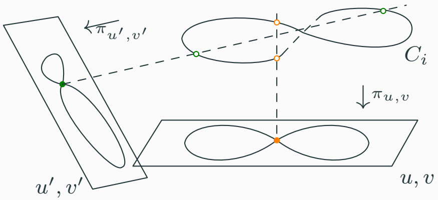

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C_i connected component of $\mathcal{C}_{\mathbb{R}}$

- $C_i \setminus \left(\pi^{-1}_{u,v}(\mathrm{app}(\mathcal{C}_2)) \cap C_i\right)$
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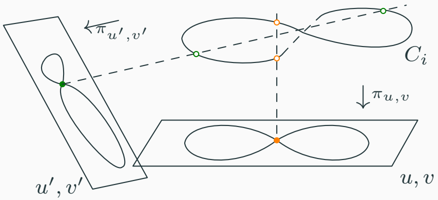

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By matching the two descriptions, we obtain a semi-algebraic description of each connected component of $\mathcal{C}_{\mathbb{R}}$.

Space Curve: Linking Procedure and Conclusion

First Projection

$\Theta_1, \ldots, \Theta_m$

Second Projection

$\Xi_1, \ldots, \Xi_m$

Space Curve: Linking Procedure and Conclusion

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$\mathcal{P} \subset \mathrm{Reg}(\mathcal{C})$ **finite** containing at least one point in each C_i

[Safey El Din-Schost (2003)]

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$p \in \Theta_i \xrightarrow{\text{same connected component}} p \in \Xi_j$

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$\downarrow$

Add the set $\mathcal{P}$ to the **graphs**

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$$\text{Space curve } (d,h) \longrightarrow \widetilde{O}\left(\underbrace{\textsc{OneDimParam} + \textsc{RegularPoints}}_{n^{15}d^{4n}(t+\log(1/\varepsilon)+\alpha)} + \underbrace{\text{PlaneCurve}}_{\delta^7+\delta^6\tau}\right)$$

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Conjecture
$\alpha \sim \log\left(\frac{1}{\varepsilon}\right) + O(n)\log d$

$\delta \sim d^{n-1}$
$\tau \sim ntd^{n-1}$

Application

Optical System Design $\longrightarrow$ **Semi-algebraic Sets**

needed to optimize performance — optical and geometrical constraints

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Optical systems in the same connected components are said Optically equivalent

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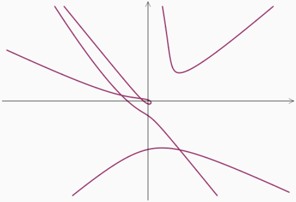
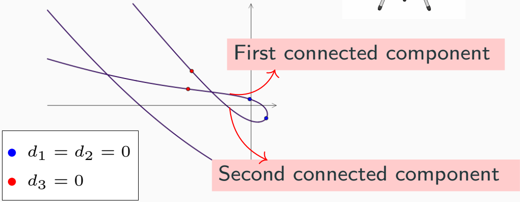
Three mirror double pass telescopes with telecentricity constraints

Semi-algebraic curve in 6 variables

Projected curve



the space curve is smooth



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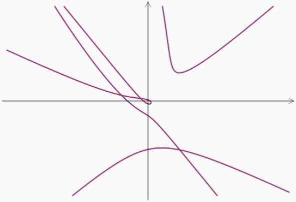

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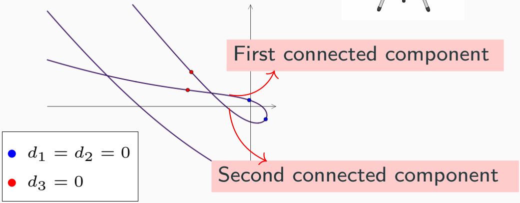

Running time: $8.9\,\mathrm{s}$

Implementation

Input: $f_1, \ldots, f_{n-1} \in \mathbb{Q}[x_1, \ldots, x_n]$ with random coefficients.

	$n = 3$	$n = 4$	$n = 5$	$n = 6$
CAD	3.66 s	> 6 d		
CurveComponents	0.28 s	0.73 s	15,71 s	34,84 m

- CAD: requires computing cell adjacencies (not implemented).
- CurveComponents: implemented in Julia using `AlgebraicSolving.jl` and `msolve`.

Laptop

- CPU: Intel® Core™ Ultra 7 265H
- Memory: 62 GB RAM

posso (lab server)

- CPU: 2 × Intel® Xeon® E7-4820 v4 @ 2.00 GHz
- Memory: 1.5 TiB RAM

Future Perspectives

- Control of the bit size α of generic changes of coordinates
- **Software Implementation**
  - Implementation in `AlgebraicSolving.jl`
  - Applications to problems arising in optical systems
- **Towards Semi-Algebraic Sets**
  - Semi-algebraic curves
  - Higher-dimensional semi-algebraic sets
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Thank you for your attention!