Conjectural Variations Equilibria

Learning and Dynamic Equilibria. Applications to natural resource management

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Dynamic conjectures, bounded rationality and learning

- The principle.
- A learning model.
- A natural resource management problem.

Consistent conjectures in a dynamic setting

- The principle.
- Consistent conjectures in differential games.
- A model of non-renewable natural resource management.

References

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«Incomplete information, learning and natural resource management»
To appear in European Journal of Operational Research.

N. Quérou, M. Tidball
« Consistent conjectures in a dynamic model of non-renewable resource management»
Manuscript.

Dynamic conjectures, bounded rationality and learning

The idea (Jean-Marie, Tidball, JEBO, 2006)

Ingredients

- Dynamic conjectures
- Limited rationality
- Updating of conjectures

Conjecture adjustment process

$$\dot{r}_{ij}(t) = \mu_i(r'_{ij}(t) - r_{ij}(t)), \quad r_{ij}(t+1) = (1-\mu_i)r_{ij}(t) + \mu_i r'_{ij}(t)$$

$\mu_i \longrightarrow$ speed of adjustment.
$r_{ij}(t) \longrightarrow$ conjecture of i about j.
$r'_{ij}(t) \longrightarrow$ conjecture to be used, based on observations.

The learning model

- n players, e_i strategy of i, e profile of strategies,
- e^b a given benchmark strategy,
- V^i instantaneous payoff of player i.

Player i makes a conjecture about j of the form

$$e_j = e_j^b + r_{ij}(e_i - e_i^b), \quad r_{ij} \in \mathbb{R}$$

and solves

$$\max_{e_i} V^i(e_i, (e_j^b + r_{ij}(e_i - e_i^b))_{i \neq j}) \; .$$

There exists a unique solution $e_i = \phi_i(e^b; r_i)$, ($r_i = (r_{ij})_{i \neq j}$).

Learning model (continued)

i observes that j has played e_j and concludes that her conjecture should have been r'_{ij} /

$$e_j = e_j^b + r'_{ij}\,(e_i - e_i^b), \qquad \Longrightarrow \qquad r'_{ij} = \frac{e_j - e_j^b}{e_i - e_i^b}$$

Adjustment process of conjectures

$$r_{ij}(t+1) = (1-\mu_i) r_{ij}(t) + \mu_i \frac{e_j(t) - e_j^b}{e_i(t) - e_i^b}$$

with $e_i(t) = \phi_i(e^b, r_i(t))$.

Properties of fixed points

Proposition 1: If $r_{ij}(t) \to r_{ij}$ as $t \to \infty$, then

$$r_{i_1 i_2} r_{i_2 i_3} ... r_{i_p i_1} = 1 \quad \forall i_1 ... i_p$$

in particular

$$r_{ji} = (r_{ij})^{-1}$$

The vector $(r_{i1} ... r_{ii-1}, 1, r_{ii+1} ... r_{in})$ is the direction of the line (passing through e^b) of the space of strategy profiles, on which player i chooses her own strategy.

$e_i = \phi_i(e^b, r_i)$ is the strategy played by i in the limit.

Properties of fixed points (continued)

Proposition 2: Pareto optimality
If e is a limit point obtained by the convergence of the adjustment recurrence then e is a candidate Pareto-optimal solution.
candidate i.e. it verifies necessary optimal conditions.

Proposition 3: In the case of identical players:
$\phi_i(e^b, r) = \phi(e^b, r)$, $e_i^b = e^b \ \forall i$; the recurrence converges to 1 for any $0 < \mu < 1$ and any (common) initial condition.

Example in a dynamic setting

x_t, the stock of natural resource at time t,
$e_{i,t}$, extraction at time t,
the evolution rule is:

$$x_{t+1} = [x_t - e_{1,t} - e_{2,t}]^\alpha, \quad x(0) = x_0, \quad \alpha \in (0,1).$$

The utility function for each player is:

$$V_i(e_i, e_j, x) = log(e_i) + \beta log[1/2(x - e_i - e_j)^\alpha],$$

$\beta \in (0,1)$ is the players' discount factor.

Learning process

The benchmark case, Lehvari and Mirman (1980)

Find the feedback Nash equilibrium of

$$\max_{e_{it}} \sum_{t=0}^{\infty} log(e_{it}) + \text{ dyn},$$

compare this solution to the cooperative outcome:

$$\max_{e_{1t},e_{2t}} \sum_{t=0}^{\infty} (log(e_{1t}) + log(e_{2t})) + \text{ dyn}.$$

Result:

$$x_{\infty}^{nashdyn} = \left(\frac{\alpha\beta}{2-\alpha\beta}\right)^{\alpha/(1-\alpha)} < (\alpha\beta)^{\alpha/(1-\alpha)} = x_{\infty}^{coopdyn}.$$

The one-shot game

Consider the static game where

$$V_i(e_i, e_j, x) = log(e_i) + \beta log[1/2(x - e_i - e_j)^\alpha],$$

Results:

$$e^{coop} = \frac{x}{2(1+\alpha\beta)} < e^N = \frac{x}{2+\alpha\beta},$$

$$x^N < x_\infty.$$

Return to the learning process. Results

Assuptions: $\bar{e}^1 = \bar{e}^2 = \bar{e}$, $\mu_1 = \mu_2 = \mu$.

The consumption plan $\{e^c_{i,t}\}_t$ when agent i learns according to the process defined previously, is, for all t:

$$e^c_{i,t} = \frac{x_t - (1 - r^i_t)\bar{e}}{(1 + r^i_t)(1 + \alpha\beta)}.$$

and

$$\lim_{t\to\infty} r^i_t = 1, \quad \lim_{t\to\infty} e^c_{i,t} = e^{coop}_\infty, \quad \lim_{t\to\infty} x^c_t = x_\infty$$

Return to the learning process. Results

$$x_\infty < x_\infty^{coopdyn}, \forall \alpha, \beta.$$

- If $\alpha\beta < 1/2$ then $x_\infty^{nashdyn} < x_\infty$
- If $\alpha\beta = 1/2$, then $x_\infty^{nashdyn} = x_\infty$
- If $\alpha\beta > 1/2$, then $x_\infty^{nashdyn} > x_\infty$

Return to the learning process. Results

- If $\frac{1-\alpha\beta}{1+\alpha\beta} < r_0 < 1$, then for all t, $e_t^c > 0$ and $x_t^c > 0$.

- If $\bar{e} > \frac{1}{2+\beta}\left[\frac{\alpha\beta}{1+\alpha\beta}\right]^{\frac{\alpha}{1-\alpha}}$, then the process is locally stable for all μ.

Conclusion

- We study a problem of resource management under incomplete information with conjectures, in a symmetric setting
- The steady state induced by the procedure leads to a (static) cooperative management of the resource once the stock has stabilized.
- For a large set of cases the steady state level of the resource lies in between the non cooperative and cooperative outcomes derived by Levahri and Mirman (1980).

Consistent conjectures in a dynamic setting

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Consistent conjectures in a dynamic setting

Ingredients

- Dynamic game.
- Conjectures on how the other players react
- Consistency: conjectures of each player $\equiv$ best response reactions of the others players

Principle. Jean-Marie, Tidball, Dyn. Games, 2005.

- n players, time horizon T
- $x(t) = (x_1(t), ...x_n(t)) \in \mathbb{R}^m$ state variable
- $e_i(t)$ control variable of i in $[t, t+1]$, $e(t)$

Dynamics

$$x(t+1) = f(x(t), e(t)), \quad x(0) = x_0$$

Payoff

$$V^i(x_0, e(0), ...e(T-1)) = \sum_{t=1}^{T} \theta^{t-1} \Pi^i(x(t), e(t))$$

Principle (continued)

Conjecture of i

$$e_j^c(t) = \phi_t^{ij}(x(t)) \quad \rightarrow \quad x(t+1) = \tilde{f}_i(x(t), e_i(t)) \; .$$

optimal control problem

optimal policy $e_i^{i*}(t)$ that we suppose unique. Player i can compute $e_j^{i*}(t)$ and $x^{i*}(t)$ via ϕ_t^{ij}.

Call $x^a(t)$ the actual trajectory (replacing e_i^{i*} in the dynamics).

Different definitions of consistency

Definition 1: $\phi_t^1, ...\phi_t^n$ is a state-consistent conjectural equilibrium $\iff$

$$x^{i*}(t) = x^a(t), \quad \forall i,\ t,\ x(0) = x_0$$

Definition 2: $\phi_t^1, ...\phi_t^n$ is a (weak) control-consistent conjectural equilibrium $\iff$

$$e^{i*}(t) = e^{j*}(t), \quad \forall i \neq j,\ t,\ x(0) = x_0 \ \ (x(0)\ given)$$

control-consistent c.e. $\implies$ state-consistent c.e.

Different definitions of consistency (continued)

Optimization problem: $\rightarrow e_i^{i*}(t) = \psi_t^i(x(t))$

Definition 3: $\phi_t^1, ...\phi_t^n$ is a feedback-consistent conjectural equilibrium $\iff$

$$\psi_t^i = \phi_t^{ji}, \quad \forall i \neq j,\ t,\ x(0) = x_0$$

as a consequence

$$\phi_t^{ji} = \phi_t^{ki}, \quad \forall i \neq j \neq k,\ t.$$

Consistency in differential games

Fershman and Kamien (1985) define consistent conjectures in differential games.

- Open-loop Nash equilibria are weak control-consistent conjectural equilibria
- Control-consistent conjectural equilibria and feedback Nash equilibria coincide

Different definitions of consistency (continued)

What happens when
Optimization problem: with $e_i^i(t) = \phi_t^i(x(t), e_j^j(t-1))$ + consistency

Non-renewable natural resource management model

The benchmark: The cooperative case

$$\max_{\{e_{1,t};e_{2,t}\}} \sum_{t=0}^{\infty} \beta^t \left[log(e_{1,t})+log(e_{2,t})\right], \quad x_{t+1} = x_t-e_{1,t}-e_{2,t}, x_0 \text{ given}$$

The Nash equilibrium

$$\max_{\{e_{i,t}\}} \sum_{t=0}^{\infty} \beta^t log(e_{i,t}), \quad x_{t+1} = x_t - e_{1,t} - e_{2,t}, \quad x_0 \text{ given.}$$

Result

$$x_t^{coop} = \beta^t x_0 > \left(\frac{\beta}{2-\beta}\right)^t x_0 = x_t^N, \quad \forall t.$$

State and strategy based beliefs

Consider that agent 1's beliefs regarding the behavior of agent 2 is:

$$e_{2,t} = a_2 x_t + b_2 e_{1,t-1}.$$

The problem is:

$$\max_{\{e_{i,t}\}} \sum_{t=0}^{\infty} \beta^t log(e_{i,t}),$$

subject to the following constraints (x_0, y_0 given):

$$x_{t+1} = x_t - a_j x_t - b_j y_t - e_{i,t} \quad y_{t+1} = e_{i,t}.$$

and to impose consistency!

State and strategy based beliefs. Results

If $e_0 = \frac{1-\beta}{2} x_0$, then: $e_t^{fc} = \beta e_{t-1}^{fc}, \quad x_t^{fc} = \beta^t x_0, \quad x_t^{fc} = x_t^{coop}$.

If $e_0 \neq \frac{1-\beta}{2} x_0$ then : $e_t = [1 - 2\frac{c_0}{x_0}]^t e_0, \quad x_t = [1 - 2\frac{c_0}{x_0}]^t x_0$.

- if $e_0 < \frac{1-\beta}{2} x_0$ then $a < 0$ and $b > 0$ and $x_t^{fc} > x_t^{coop} > x_t^N$;
- if $\frac{1-\beta}{2} x_0 < e_0 < \frac{1-\beta}{2-\beta} x_0$ then $a > 0$, $b > 0$ and $x_t^N < x_t^{fc} < x_t^{coop}$;
- if $\frac{1-\beta}{2-\beta} x_0 < e_0 < \frac{1}{2} x_0$ then $a > 0$ and $b < 0$ and $x_t^{fc} < x_t^N < x_t^{coop}$.

Conclusions

The state and strategy consistent solution gives

- better outcomes regarding the resource management in the long run compared to joint management if initial consumption is sufficiently low,
- or a more aggressive pattern than the non-cooperative benchmark if initial consumption is too high.
- For intermediate values of initial consumption, the optimal path lies in between the non-cooperative and cooperative benchmark cases.

Bibliography

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