

Optimal Impulse Control of Renewable Resources

Alain Jean-Marie¹ Mabel Tidball² Katrin Erdlenbruch³
Michel Moreaux⁴

¹INRIA/LIRMM, CNRS-Univ. Montpellier II, France.

²INRA, UMR LAMETA, Montpellier, France.

³Cemagref, UMR G-EAU, Montpellier, France.

⁴IDEI, UMR-LERNA, Toulouse, France.

Séminaire Économie-Optimisation
Université Paris Ouest – Nanterre – La Défense
Nanterre, 10 juillet 2009

Plan

- 1 Introduction
- 2 Continuous-time control models
- 3 The impulse control model
 - Dynamics, profits, policies
 - The auxiliary problems
 - Where is the solution to (AP)?
- 4 From discrete to continuous, and back
- 5 Conclusion

Context

Analysis of harvesting behavior in renewable resource economics (“Mathematical BioEconomics”).

Different types of mathematical models are used to describe the harvesting process, emphasizing different aspects of harvesting behavior and resulting in different harvesting policies.

Modeling : optimal control problems

- stochastic or deterministic
- finite or infinite horizon
- discrete or continuous time

discrete → cycles (!)

continuous → continuous extraction, or cycles (concavity, several dimensions, ...)

Related to the “discrete-continuous” interface.

A singular control model : Clark's model.

Continuous time, infinite horizon control problem

$$\max_{h(\cdot)} \int_0^{\infty} e^{-rt} [p - c(x(t))] h(t) dt$$

$$\dot{x}(t) = F(x(t)) - h(t) \quad x(0) = x_0, \quad 0 \leq h(t) \leq h_{\max}$$

- $x(t)$ is the level of the resource stock at time t , $F(x)$ is the natural growth function,
- p represents the resource price,
- $c(x)$ the unit harvest costs and r the discount rate.

A singular control model : Results

The solution is found as follows :

- The profit maximizing stock level leads to a steady state x^* solution of :

$$F'(x^*) - \frac{c'(x^*)F(x^*)}{p - c(x^*)} = r .$$

- If $x_0 < x^*$, the optimal control is $h(t) = 0$ as long as $x(t) < x^*$.
- If $x_0 > x^*$, the optimal control is $h(t) = h_{max}$ as long as $x(t) > x^*$.
- If $x(t) = x^*$, the optimal control keeps $x(t)$ constant.

⇒ A **turnpike** trajectory

The model dynamics

We consider a renewable resource, the dynamics of which, absent

any harvest is given by : $\dot{x}(t) = F(x(t))$, $t \geq 0$,

$x(t)$ is the size of the population at any time t

F , stationary through time, is the growth rate function.

- $\exists x_{sup}$ and x_{ns} , $0 < x_{ns} < x_{sup} < +\infty$.
- $F : (0, x_{sup}) \rightarrow \mathbb{R}$ is of class C^2
- positive over the interval $(0, x_{ns})$ and negative over the interval (x_{ns}, x_{sup}) ,
- $F(0) = F(x_{ns}) = 0$, where $\lim_{x \downarrow 0} F(x) = F(0)$, and $\lim_{x \uparrow x_{sup}} F(x) = -\infty$.

Impulse policies

An impulse policy $IP := \{(t_i, l_i), i = 1, 2, \dots\}$ as a sequence of harvesting dates t_i and instantaneous harvests l_i , one for each date.

- $0 \leq t_1$, and $t_i < t_{i+1}$ for each $i = 1, 2, \dots$
- if the sequence is finite with $n \geq 0$ values, then $t_i = +\infty$ for all $i > n$.
- $l_i \leq 0$ and $x_i - l_i \geq 0$, where x_i is the size of the population just before the harvesting date t_i .

The model : The profit function

We assume that the profit function is stationary through time so that whatever t_i , l_i and x_i , the current profits at time t_i amount to $\pi(x_i, l_i)$.

- the domain is $\mathcal{D} := \{(x, l), x \in (0, x_{sup}), l \in [0, x)\}$.
- It is of class C^2 and bounded above by $\bar{\pi} < +\infty$,
- $\pi(x, 0) = 0, \forall x \in (0, x_{sup})$.
- $(\partial\pi/\partial l)(x, l)$ admits a limit when $l \downarrow 0$ for all $x \in (0, x_{sup})$.

An impulse optimal control problem

(P) : Maximize over t_i, l_i for $i = 1 \dots \infty$

$$\mathcal{G}(\{t_i, l_i\}_{i=1}^{\infty}, x_0) = \sum_{i=1}^{\infty} e^{-rt_i} \pi(x_i, l_i)$$

s.t.

$$\dot{x}(t) = F(x(t)) \quad \text{if } t \geq 0, \quad t \notin t_i, \quad i = 1, 2, \dots \quad x(0) = x_0$$

$$\lim_{t \rightarrow t_i^+} x(t) = \lim_{t \rightarrow t_i^-} x(t) - l_i, \quad x_i = \lim_{t \rightarrow t_i^-} x(t),$$

$$l_i \leq x_i, \quad x(t) \in [0, x_{sup}].$$

Boundedness of the profit

Does the objective function have a finite supremum?

Denote $\pi^+(x, I) = \max(\pi(x, I), 0)$.

Property

Assume that for all x and some constant ℓ .

$$\pi^+(x, I) \leq \ell I .$$

Then for any $\Delta t \leq (F_{\max})^{-1}$,

$$\Pi \leq \ell \frac{x_{ns} + F_{\max} \Delta t}{1 - \exp(-r \Delta t)} .$$

Dynamic optimization from the book

Usual approach : Maximum Principle + Lagrangian/Hamiltonian.
 See Termansen, Léonard and van Long, Seierstad and Sydsaeter.
 The Hamiltonian :

$$H(x, \lambda) = \lambda(t)F(x(t)),$$

the discounted instantaneous cost $\pi(x, l, t) = e^{-rt}g(x, x - l)$.
 At the points without jumps ($t \neq t_j$) :

$$\dot{\lambda}(t) = -\lambda(t)\frac{\partial F}{\partial x}(x(t)), \quad \lambda(t) \geq \frac{\partial \pi(x(t), 0, t)}{\partial l}.$$

Dynamic optimization from the book (ctd.)

At the jump points :

$$\lambda_j^+ = \frac{\partial \pi(x_j^-, l_j^*, t_j)}{\partial l},$$

$$\lambda_j^+ - \lambda_j^- = -\frac{\partial \pi(x_j^-, l_j^*, t_j)}{\partial x},$$

$$H(x_j^+, \lambda_j^+) - H(x_j^-, \lambda_j^-) - \frac{\partial \pi(x_j^-, l_j^*, t_j)}{\partial t} = 0.$$

Notation : $x_j^- = \lim_{x \rightarrow t_i^-} x(t_i)$, and $x_j^+ = \lim_{x \rightarrow t_i^+} x(t_i)$. Likewise, for λ_j^- and λ_j^+ .

The dynamic programming principle

Value function approach (original in this context).

Theorem

The value function

$$v(x) = \sup_{IP \in \mathcal{F}_x} \Pi(IP)$$

is the unique solution of the following variational equation :

$$v(x) = \sup_{\substack{y \in [0, x_{sup}) \\ t \geq 0}} e^{-rt} [\pi(\phi(t, x), \phi(t, x) - y) + v(y)] ,$$

where $\phi(t, x)$ is the trajectory of the system at time t , solution of the dynamics with $x(0) = x$.

Cyclical policies

Consider $x_0 \leq \bar{x}$ and the family of policies

Cyclical Policy

A cyclical policy consists in :

- let the resource x_t grow until $\bar{x}$,
- harvest until $\underline{x}$

and repeat.

Define $\tau(x, y)$ as the time necessary for the dynamics to go from value x to y :

$$\tau(x, y) = \int_x^y \frac{1}{F(u)} du.$$

Special cases : $x = 0$ and $y = x_{ns}$.

Value of cyclical policies

Consider $x_0 \leq \bar{x}$. A cyclical policy has two parameters, $\underline{x}$ and $\bar{x}$, $l = \bar{x} - \underline{x}$. Define :

$$G(\underline{x}, \bar{x}, x_0) := \pi(\bar{x}, \bar{x} - \underline{x}) \frac{e^{-r\tau(x_0, \bar{x})}}{1 - e^{-r\tau(\underline{x}, \bar{x})}} .$$

G corresponds to $\mathcal{G}$ valued at : $t_1 = \tau(x_0, \bar{x})$,
 $t_i = t_1 + (i-1)\tau(\underline{x}, \bar{x})$, $i = 2, \dots$, $x_i = \bar{x}$, $x_i - l_i = \underline{x}$, $i = 1, \dots$

Limiting case : $\underline{x} = \bar{x}$. Then :

$$G_d(x) := G(x, x, x_0) = \pi_l(x, 0) \frac{F(x)}{r} e^{-r\tau(x_0, x)} .$$

Auxiliary problems

We define now :

Auxiliary problem (AP)

$$(AP) : \max_{\underline{x}, \bar{x}; \underline{x} \leq \bar{x}} G(\underline{x}, \bar{x}, x_0).$$

Under the assumption that (AP) has a unique solution $(\underline{x}^*, \bar{x}^*)$:

Auxiliary problem (TP)

$$(TP) : \max_{\substack{x, y; \\ 0 \leq y \leq x \leq x_{ns} \\ x_0 \leq x; y \leq \bar{x}^*}} e^{-r\tau(x_0, x)} [\pi(x, x - y) + G(\underline{x}^*, \bar{x}^*, y)] .$$

Relations between problems (P), (AP) and (TP)

Characterization of the solution to (P) :

Theorem

Under the assumptions on $F(\cdot)$ and $\pi(\cdot, \cdot)$, if :

- for every $a > b \geq c > d$,

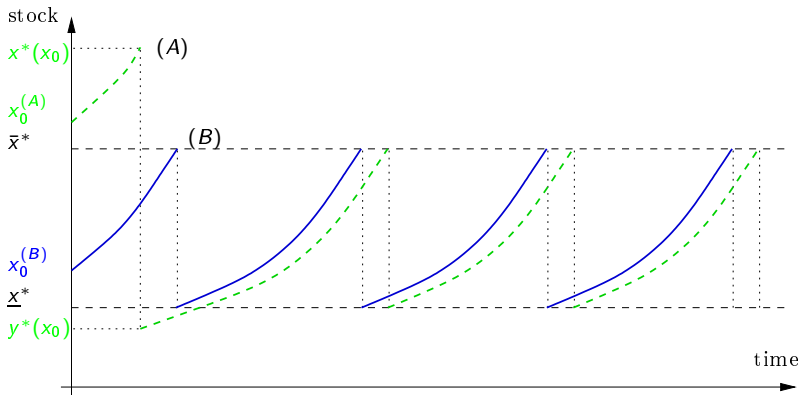
$$\pi(a, a - c) + \pi(b, b - d) \leq \pi(a, a - d) + \pi(b, b - c)$$

- Problem (AP) has a unique solution, $\underline{x}^*, \bar{x}^*$,

let $(x^*(x_0), y^*(x_0))$ solve the maximization problem (TP). Then the value function of (P) is : $v(x_0) =$

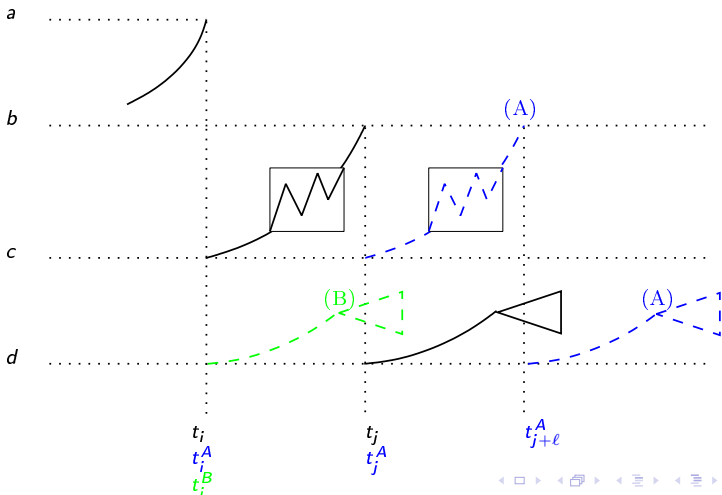
$$= \begin{cases} G(\underline{x}^*, \bar{x}^*, x_0) & \text{if } x_0 < \bar{x}^* \\ e^{-r\tau(x_0, x^*(x_0))} [\pi(x^*(x_0), x^*(x_0) - y^*(x_0)) + G(\underline{x}^*, \bar{x}^*, y^*(x_0))] & \text{if } x_{ns} \geq x_0 \geq \bar{x}^*. \end{cases}$$

Optimal trajectories



Submodularity (1)

For every solution to problem (P) which is not cyclical, there exists a cyclical solution with the same value.



Submodularity (2)

The assumption

$$\pi(a, a - c) + \pi(b, b - d) \leq \pi(a, a - d) + \pi(b, b - c)$$

is, with $g(x, y) = \pi(x, x - y)$,

Submodularity

For all $a \geq b \geq c \geq d$

$$g(a, c) + g(b, d) \leq g(a, d) + g(b, c)$$

A particular case, since $g(x, x) \equiv 0$:

Triangular constraint

$$g(a, c) + g(c, d) \leq g(a, d)$$

Existence of solutions to Problem (P)

Existence of solutions to (P) $\equiv$ location of solutions to (AP).

Theorem

Under the submodularity assumption, if the solution to (AP) is :

- in the interior $\underline{x} < \bar{x}$, there exists a solution to problem (P), and the solution can be chosen as cyclical.
- on the boundary $\underline{x} = \bar{x}$, there is no solution to (P), but sequences of ε -solutions corresponding to harvests $[\underline{x}, \underline{x} + \varepsilon]$.

On the diagonal or not ?

Back to the submodularity assumption.

$$\pi(a, a - c) + \pi(b, b - d) \leq \pi(a, a - d) + \pi(b, b - c)$$

If **equality** holds : there exists some integrable function $\gamma(\cdot)$:

$$\pi(\bar{x}, \bar{x} - \underline{x}) = \int_{\underline{x}}^{\bar{x}} \gamma(x) \, dx$$

Result

Assume that the function $G_d(\cdot)$ is of class C^1 , and is $\nearrow$, then $\searrow$, with an unique maximum at x_m .

If the function π satisfies the submodularity assumption

- in the strict sense, then all solutions to Problem (AP) are non-diagonal.
- with equality, then the solution of Problem (AP) is unique and given by $\underline{x} = \bar{x} = x_m$.

Exhausting the resource?

We note “ELB” (Exhaustion locally better). According to the form of the growth function $F(x)$ as $x \rightarrow 0$, we have :

$x = 0$ optimal ?

- i) If $F(x) \sim \alpha x^\beta$ with $\alpha > 0$ and $\beta > 1$, then the ELB property holds.
- ii) If $F(x) = \alpha x + O(x^2)$, and if $a = r/\alpha$, then :
 - ii.1) if $a > 1$, then ELB holds.
 - ii.2) if $a < 1$, then ELB does not hold.
 - ii.3) if $a = 1$... technical necessary condition for ELB involving $F()$ and $\pi()$.
- iii) If $F(x) \sim \alpha x^\beta$ with $\alpha > 0$ and $0 \leq \beta < 1$, then ELB does not hold.

Link with continuous control problem

If the solution of (AP) is on the boundary ($x^* = \underline{x} = \bar{x}$), **there does not exist a solution to (P)**.

However, there exists a sequence of cyclical impulse controls with $\bar{x} - \underline{x} = \epsilon$ (ϵ -optimal solutions of (P)) approaching the value $G_d(x^*) := G(x^*, x^*, x_0), \forall x_0$.

We have :

$$G_d(x) = \frac{\partial \pi}{\partial l}(x, 0) \frac{F(x)}{r} e^{-r\tau(x_0, x)}$$

Maximizing $G_d(x)$:

$$0 = \frac{\pi_{Ix}}{\pi_I}(x, 0) + \frac{F'(x)}{F(x)} - \frac{r}{F(x)} .$$

This value is the value of the solution to continuous control problem

$$\max_h \int_0^\infty e^{-rt} \frac{\partial \pi}{\partial I}(x, 0) h \, dt, \quad \dot{x} = F(x) - h.$$

The turnpike x^* solves $G'_d(x) = 0$.

Continuous to impulse

In the other direction, the singular control model “tends” to a mixed impulse/continuous control.

The turnpike :

- If $x_0 < x^*$, the optimal control is $h(t) = 0$ as long as $x(t) < x^*$.
- If $x_0 > x^*$, the optimal control is $h(t) = h_{max}$ as long as $x(t) > x^*$.
- If $x(t) = x^*$, the optimal control keeps $x(t)$ constant.

becomes, when $h_{max} \rightarrow \infty$:

- If $x_0 < x^*$, the optimal control is $h(t) = 0$ as long as $x(t) < x^*$.
- If $x_0 > x^*$, **harvest down to x^*** .
- If $x(t) = x^*$, the optimal control keeps $x(t)$ constant.

Discrete dynamics, continuous dynamics

Another issue in model comparison : how to map the discrete dynamics

$$x_{n+1} = f(x_n)$$

to the continuous one

$$\dot{x}(t) = F(x(t))$$

Answer : through the “reaching time function” τ :

$$\Delta t = \int_x^{f(x)} \frac{1}{F(u)} du$$

and the functional equation :

$$f'(x)F(x) = F \circ f(x) .$$

Conclusion

- Difficult to find conditions on F, π which ensure the existence of a solution to (P).
- Find instead conditions on F, π (cost) for the (AP) to have a solution with $\underline{x} < \bar{x} \dots$ or with $\underline{x} = \bar{x}$.
- Find a framework to handle at the same time continuous and impulse control.
- Go to higher dimension (discussions COMORE).

Bibliography

-  Clark, C.W. (1990). Mathematical Bioeconomics, The Optimal Management of Renewable Resources, John Wiley and Sons.
-  Dawid, H and Kopel, M. (1997). On the Economically Optimal Exploitation of a Renewable Resource : The Case of a Convex Environment and a Convex Return Function. Journal of Economic Theory 76 : 272-297.
-  Wirl, F. (1995). The Cyclical Exploitation of Renewable Resource Stocks May Be Optimal. Journal of Environmental Economics and Management 29 : 252-261.
-  Davis, M.H.A. (1993) *Markov Models and Optimization*.

Bibliography (end)

-  Léonard, D. and N. van Long (1998). Optimal Control Theory and Static Optimization in Economics. Cambridge University Press, 353 p.
-  Seierstad, A. and K. Sydsaeter (1987). Optimal Control Theory with Economic Applications. Amsterdam, Elsevier.
-  M. Termansen, Economies of scale and the optimality of rotational dynamics in forestry, Environ. Resource Econ. 37 (2007), 643-659.