Single approximation for Multiobjective Max TSP

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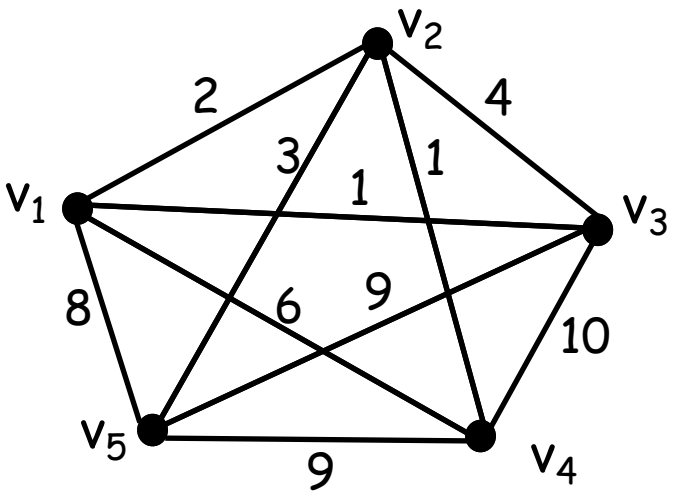
Outline

- Introduction :
  - the Multiobjective Max TSP
  - our approach
- Results for the Biobjective Max TSP
  - lower bounds
  - a generic algorithm
  - analysis of one case
- Dealing with many objectives
- Conclusion and further research

Max Traveling Salesman Problem

- Data: an undirected complete graph G=(V,E) with nonnegative weights on the edges. Let n=|V|.
- Return: an hamiltonian cycle of maximum weight (sum of its edges' weights)



NP-hard problem

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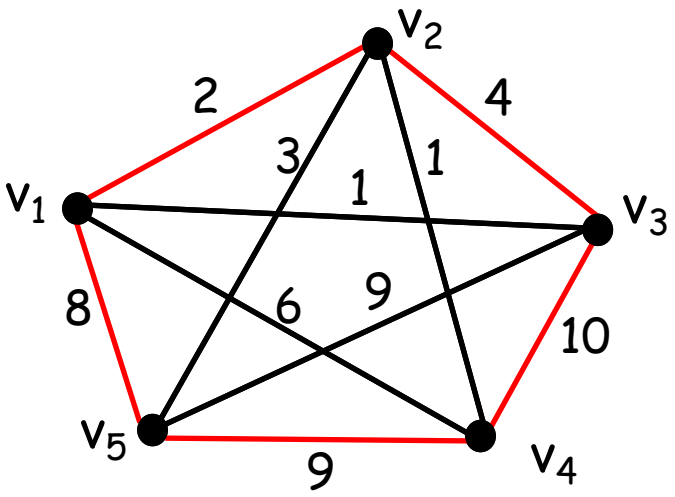
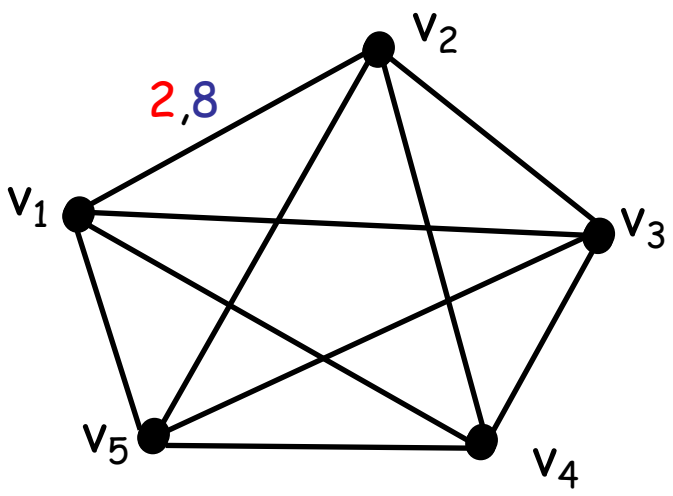


NP-hard problem

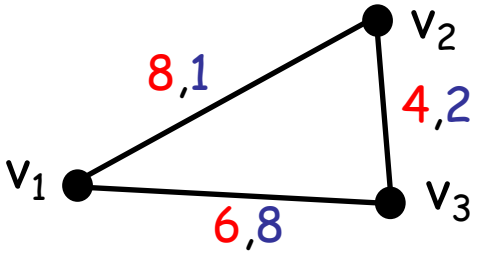
The Biobjective Max TSP

- Each edge has two weights : a and b.



$a(v_1,v_2)$ = weight of the objective a on edge $\{v_1,v_2\}$.
$a(T)$ = weight of objective a on the set of edges T.

Triangle inequality :
$\forall (v_1,v_2,v_3) \in V^3,\ a(v_1,v_2) + a(v_2,v_3) \geq a(v_1,v_3)$



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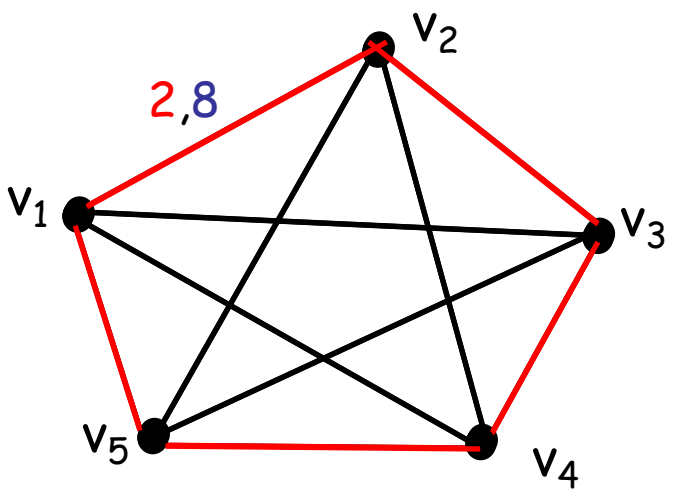

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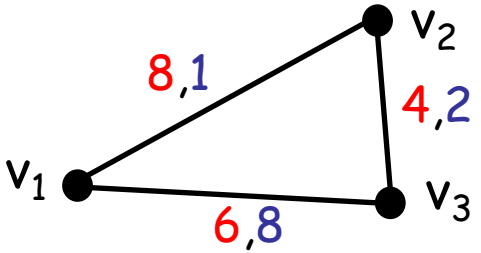

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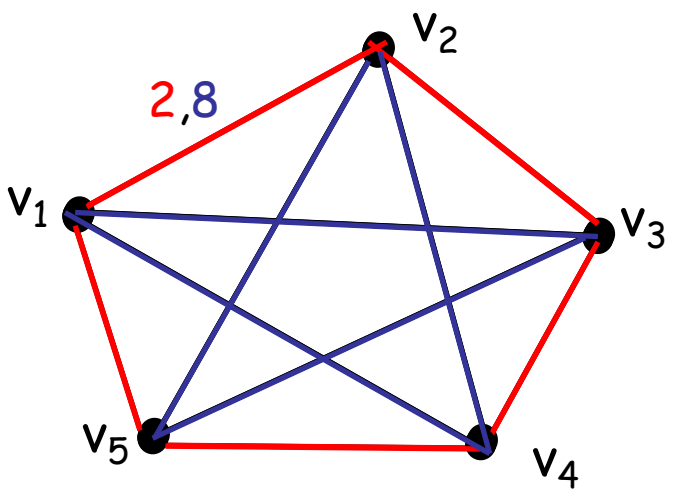

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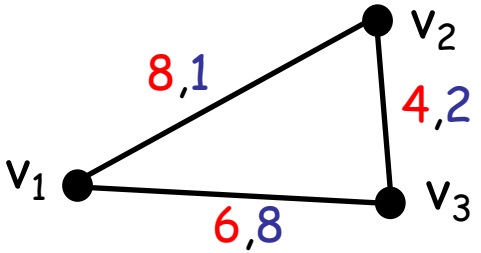

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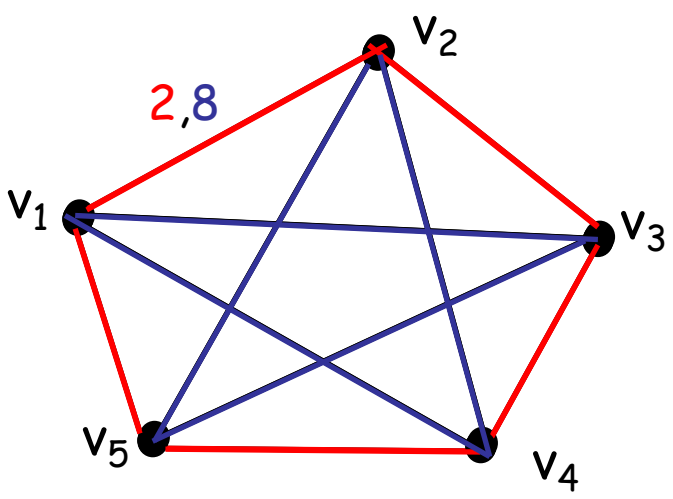

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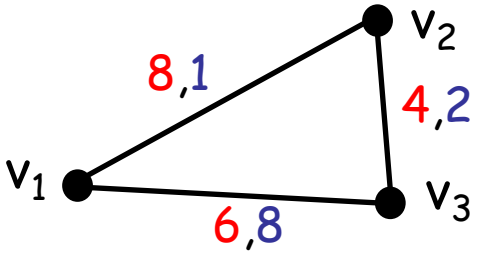


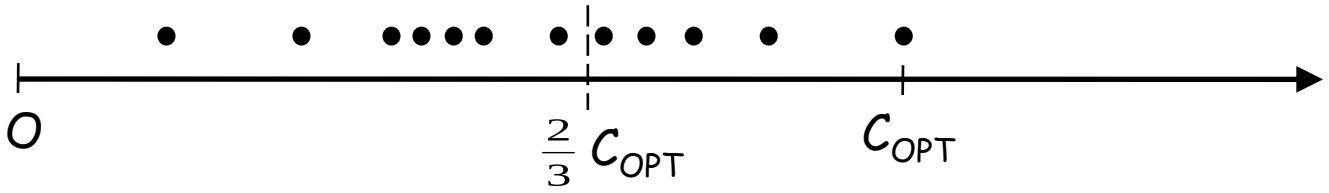
- Aim : output a good cycle for both objectives.
- 3 cases :

  0/1/2 objectives fulfill the triangle inequality

Approximation of the Max TSP

One objective: an algorithm is r-approximate ($r \leq 1$) iff

Cost (solution returned) $\geq$ r x Cost(optimal solution)



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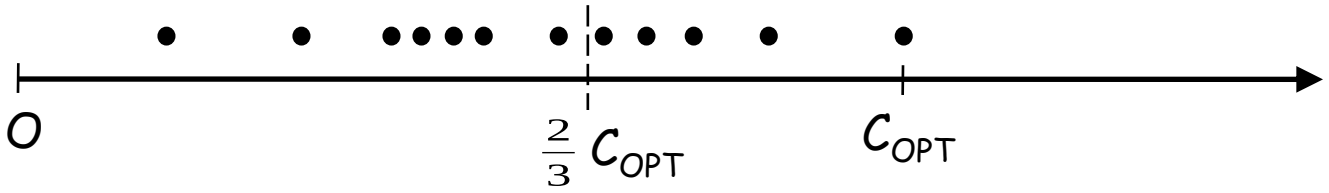

Best approximation ratio for the monobjective Max TSP:

- without the triangle inequality: $61/83 \cong 0.73$
  [Chen et al, Inf. Process. Letters, 1995]
- with the triangle inequality: $7/8 = 0.875$
  [Kowalik and Mucha, Theoretical Computer Science, 2009]

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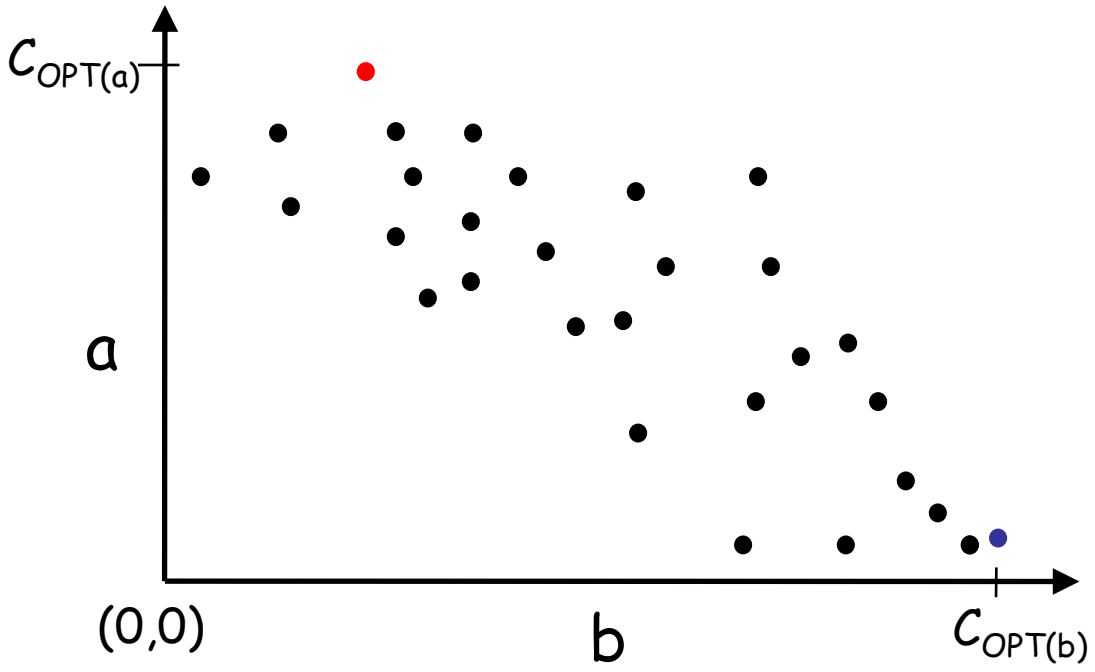

r-approximate algo = algo r-approximate on each objective.

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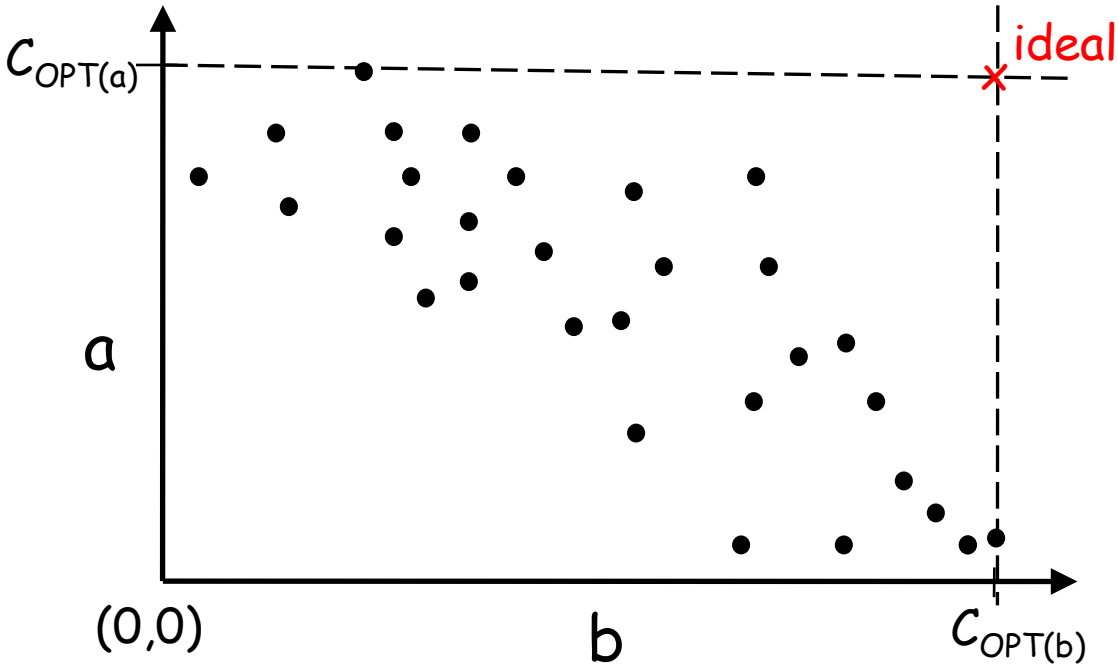

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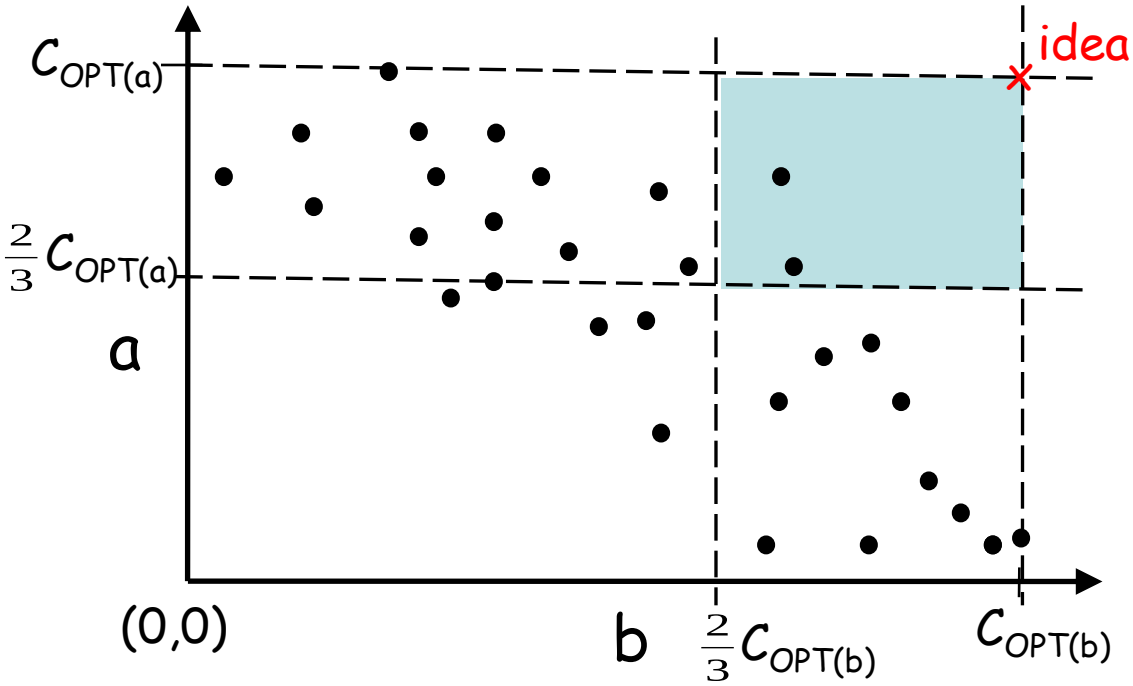

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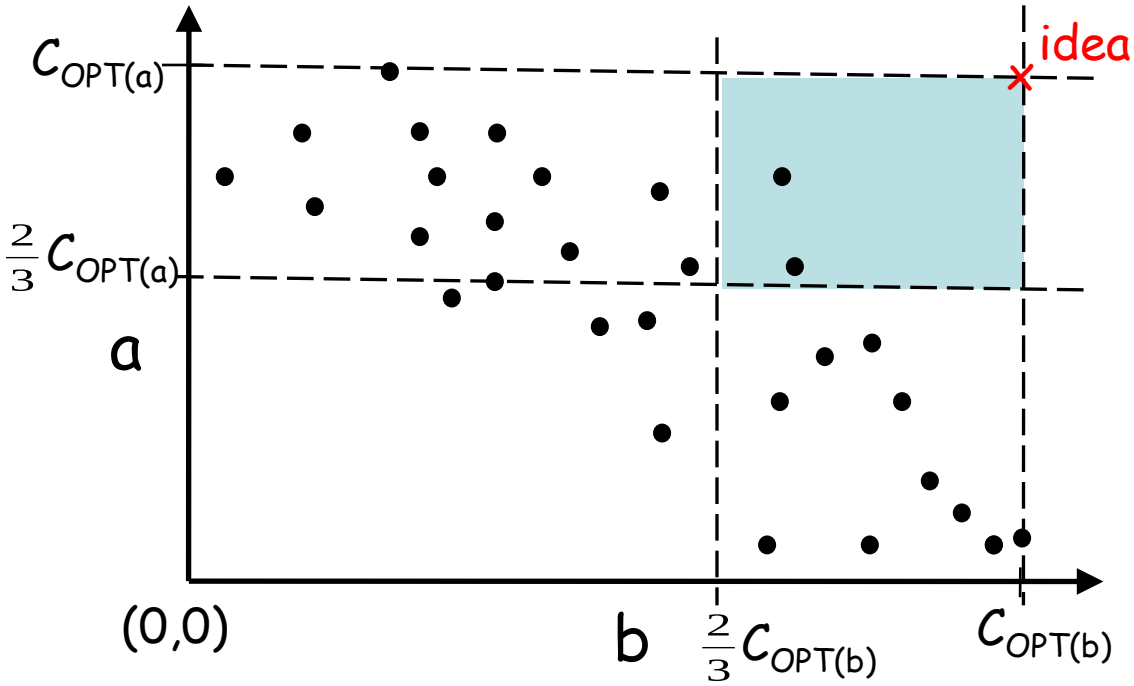

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What is maximum value of r such that there exists an r-approximate algorithm for the Max TSP?

Multiobj. Max TSP: state of the art

- Efficient set approach:
  - Randomized $(1/k - \varepsilon)$-approx. algo. for k objectives [Bläser et al. ESA'2008] ; improved to $(2/3 - \varepsilon)$ [Manthey, STACS'2009]
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- Ideal point approach (for k=2): [Manthey, STACS'2009]
  - There is no $(1/3 + \varepsilon)$-approximate algorithm
  - If there is a r-approx algo for the Max TSP $\Rightarrow$ a r/3-approximate algo for the biobjective Max TSP (i.e. 0.25-approx algo.; 0.29-approx algo. with 2 triangular inequalities).

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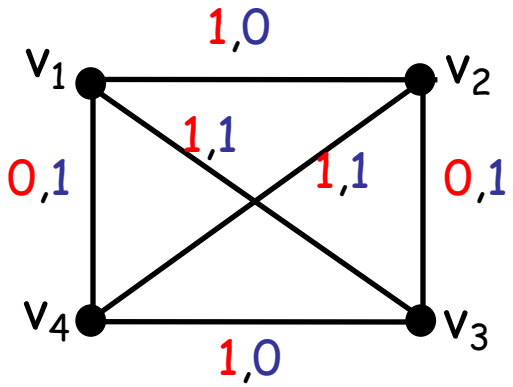

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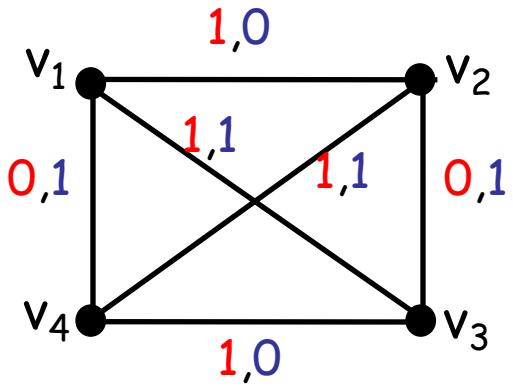
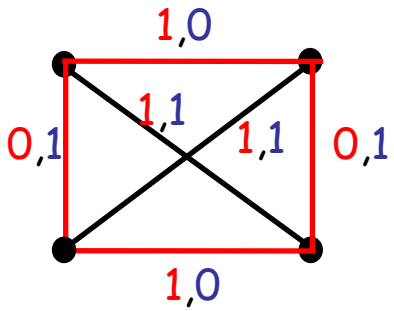
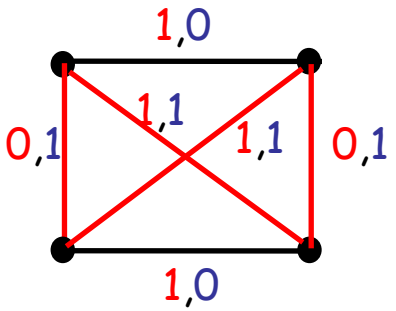


3 possible cycles :



Cost=(2,2)



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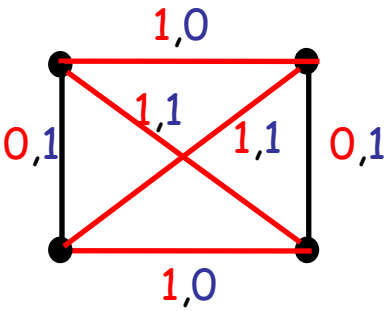

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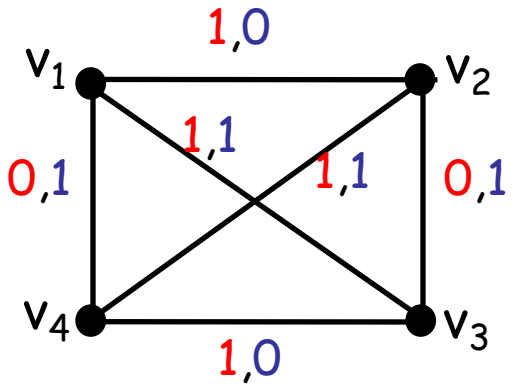

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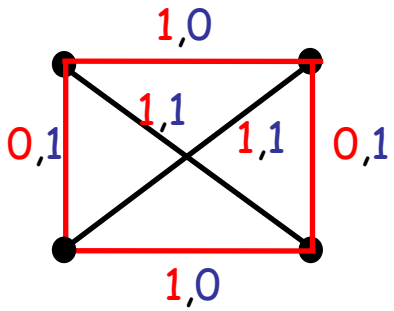

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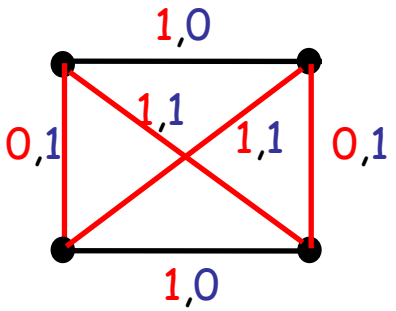

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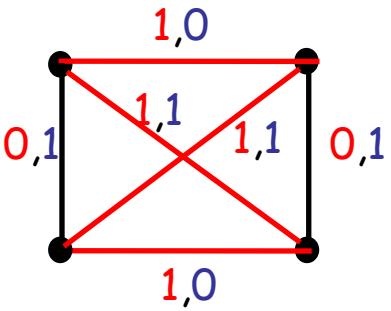

Cost=(4,2)

  - No $(\frac{3}{4}+\varepsilon)$-approximate algorithm (n large)

A generic algorithm

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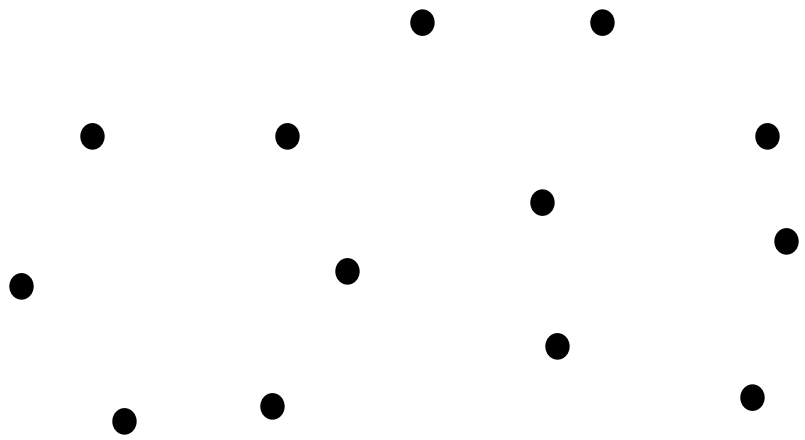

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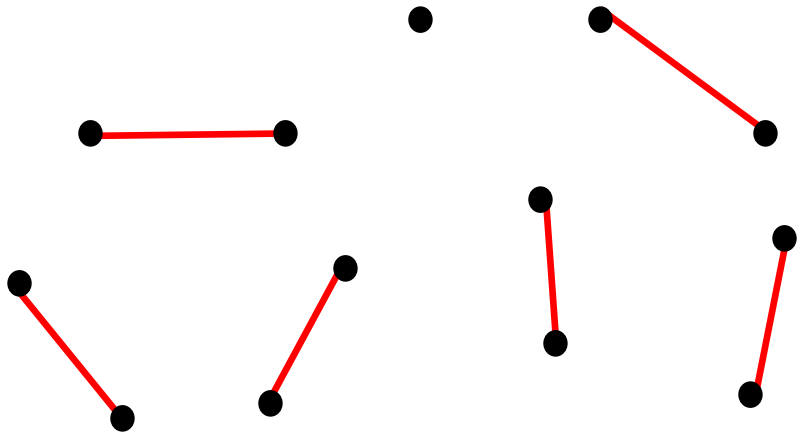

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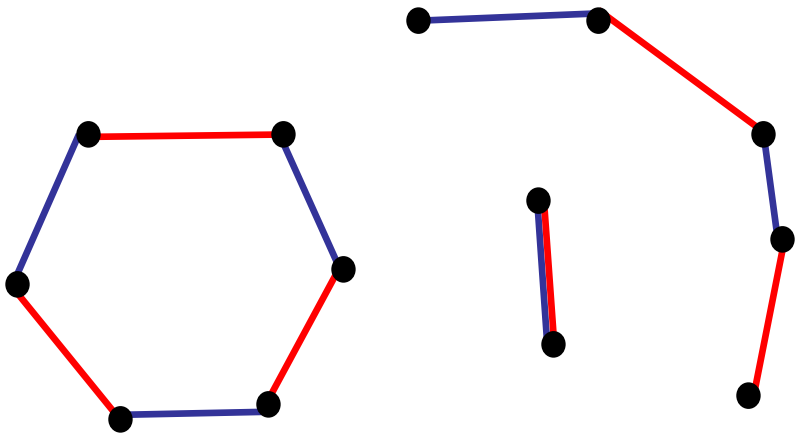

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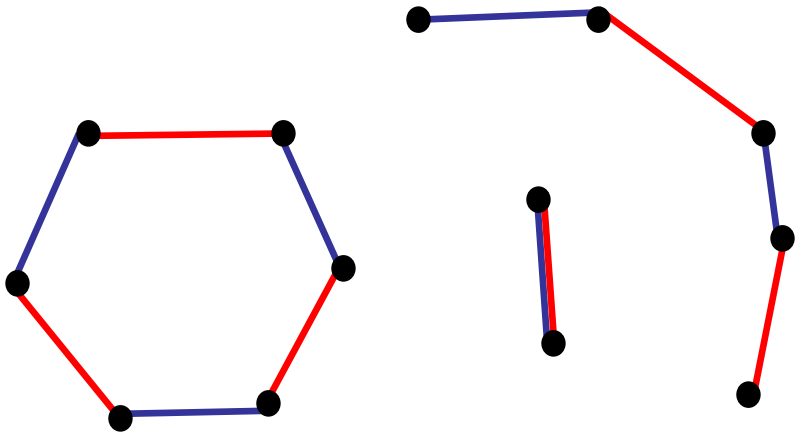

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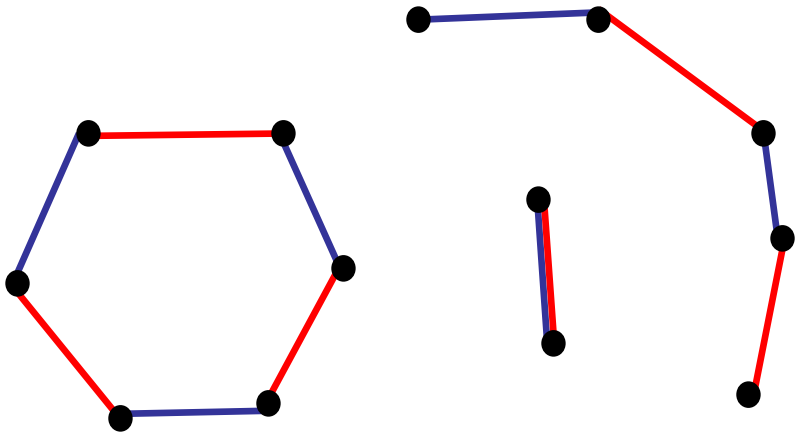

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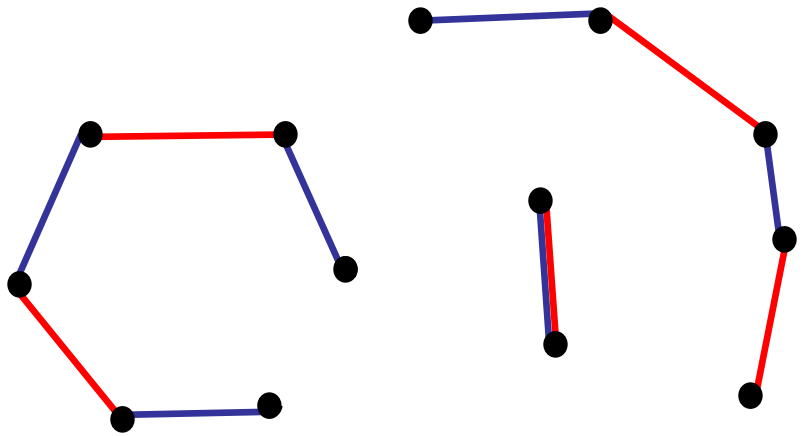

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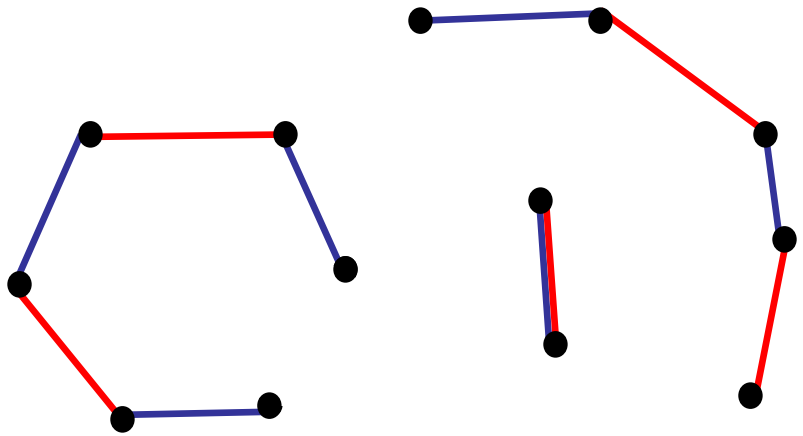

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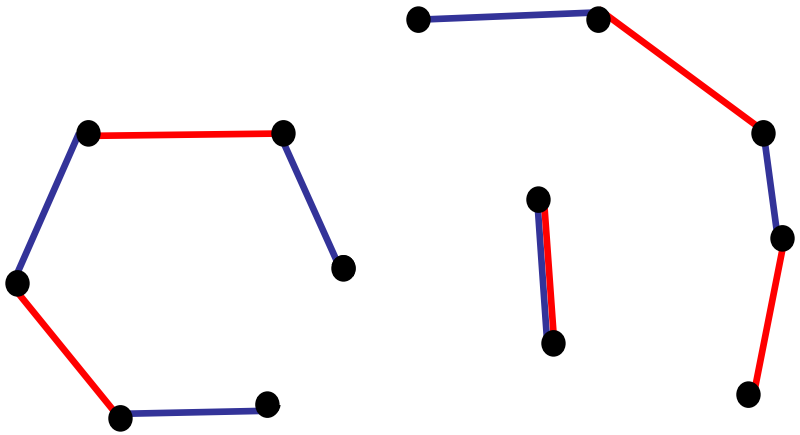

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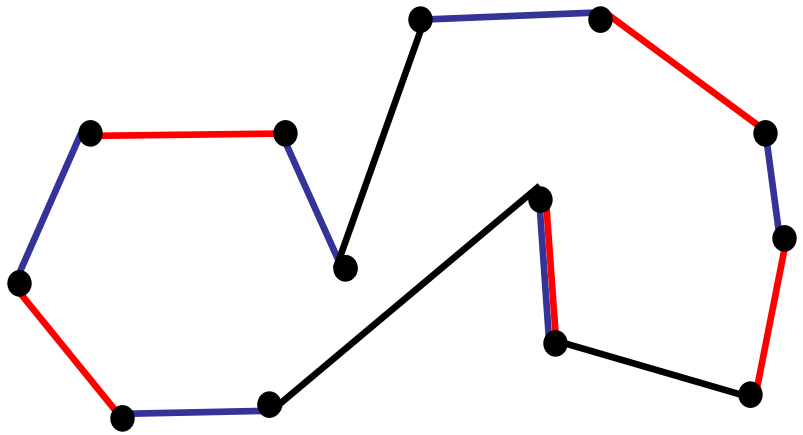

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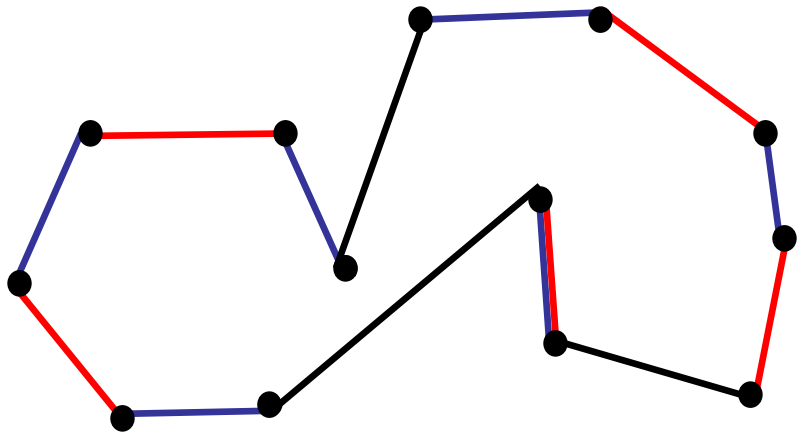

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2) We obtain a partial tour
3) We return this cycle.

Performance of this algorithm

We show that $\begin{cases} a(\text{cycle obtained}) \geq \alpha\ a(M_a) \\ b(\text{cycle obtained}) \geq \alpha\ b(M_b) \end{cases}$

Exemple: α=1/2 in the general case.

(Proof: we remove the edge with min value on a on each cycle
$\Rightarrow$ loss $\leq a(M_a)/2$)

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Property: If $\begin{cases} a(\text{cycle obtained}) \geq \alpha\ a(M_a) \\ b(\text{cycle obtained}) \geq \alpha\ b(M_b)\ , \end{cases}$

then the algorithm is $(\alpha/2)$-approximate.

Exemple: This algorithm is $\frac{1}{4}$-approximate.

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Exemple: This algorithm is $\frac{1}{4}$-approximate.

Property: This algorithm is $\frac{1+2\sqrt{2}}{14} \cong 0.27$ -approximate in the general case.

With the triangular inequality on a

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Proof: We show that $a(\text{output}) \geq 3/4\ a(M_a)$ (same for b).

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If at the end of Step (1):

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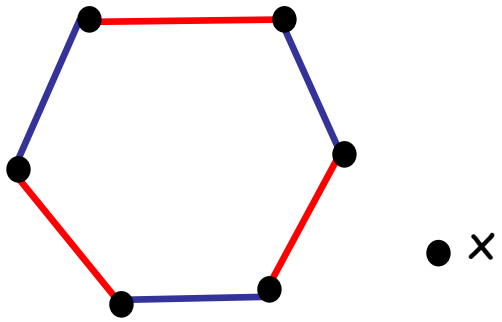
- There is one cycle and no path:
  - with n vertices: $a(\text{output}) \geq a(M_a)$
  - with (n-1) vertices:



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Property: This algorithm is 3/8-approximate.

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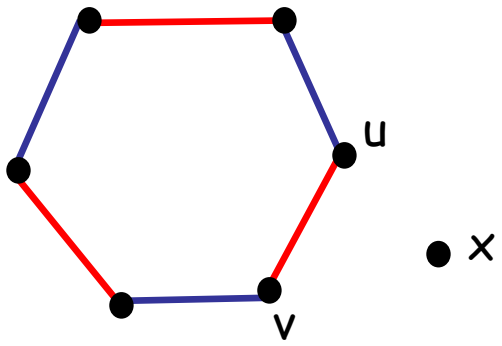

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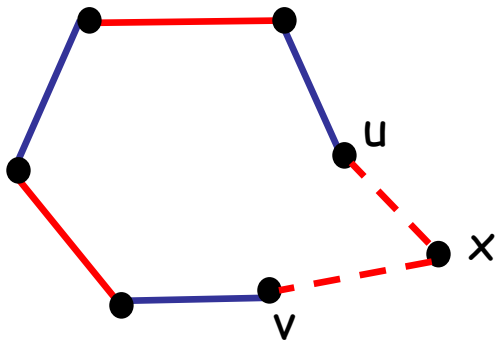

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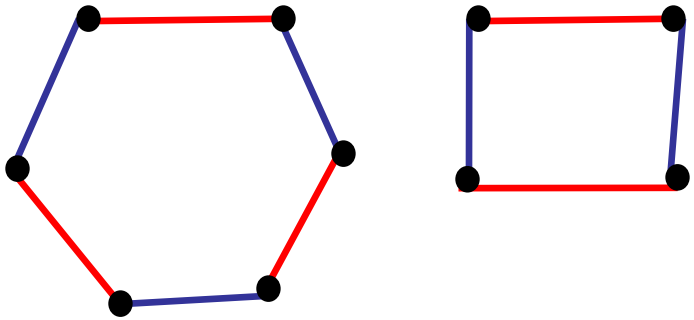

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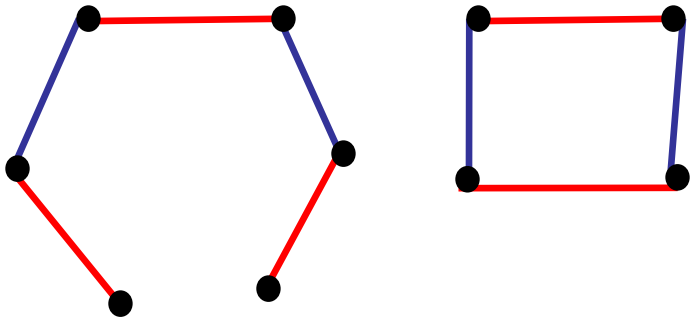

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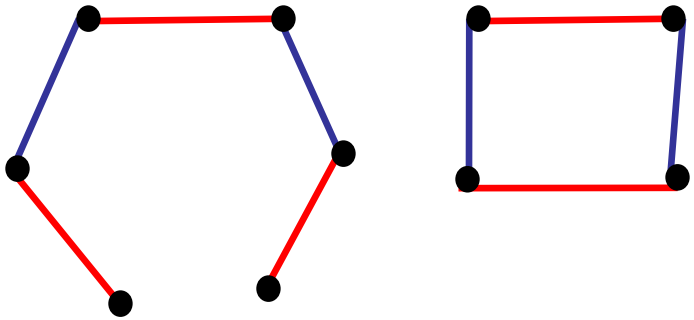

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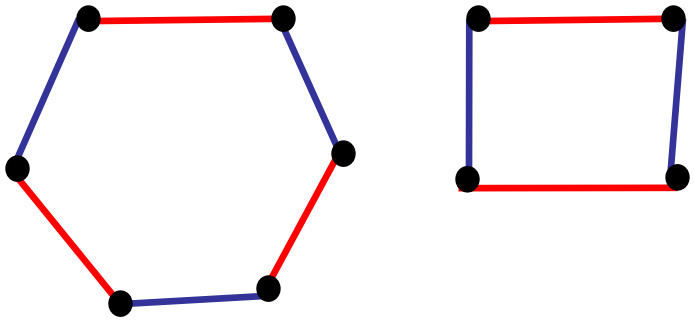

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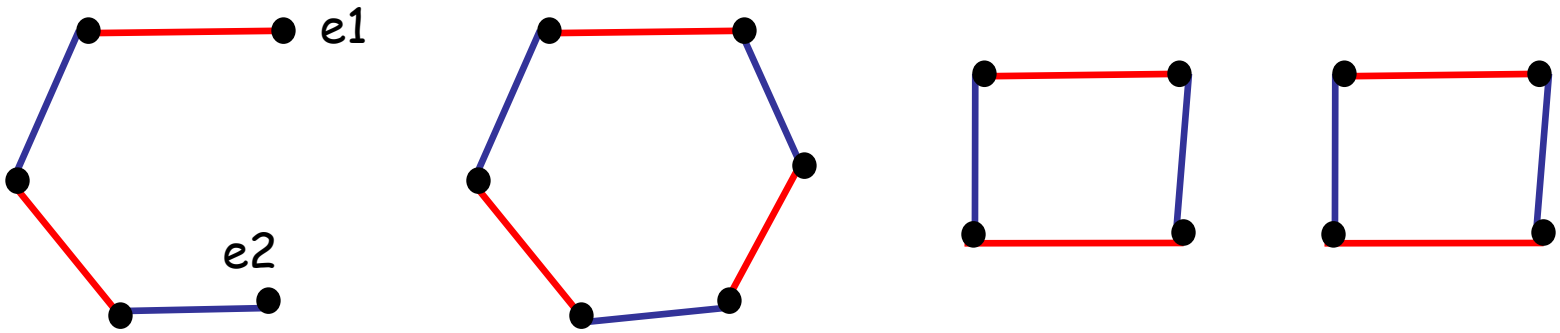

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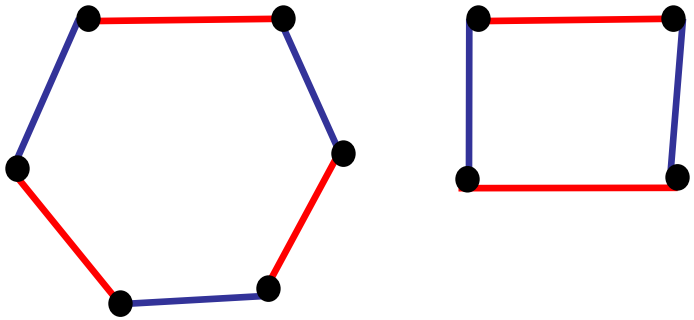

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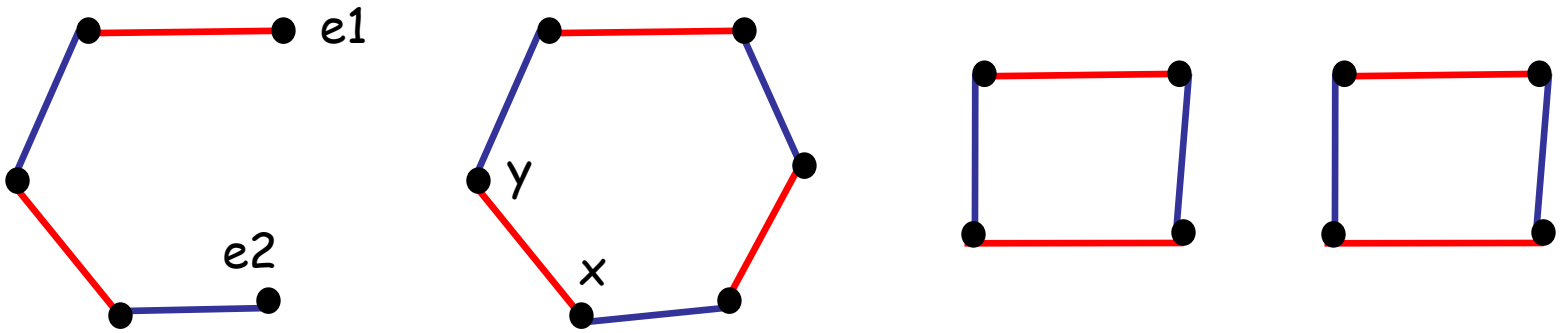

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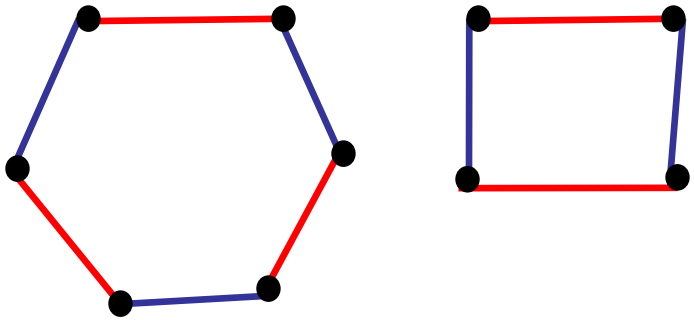

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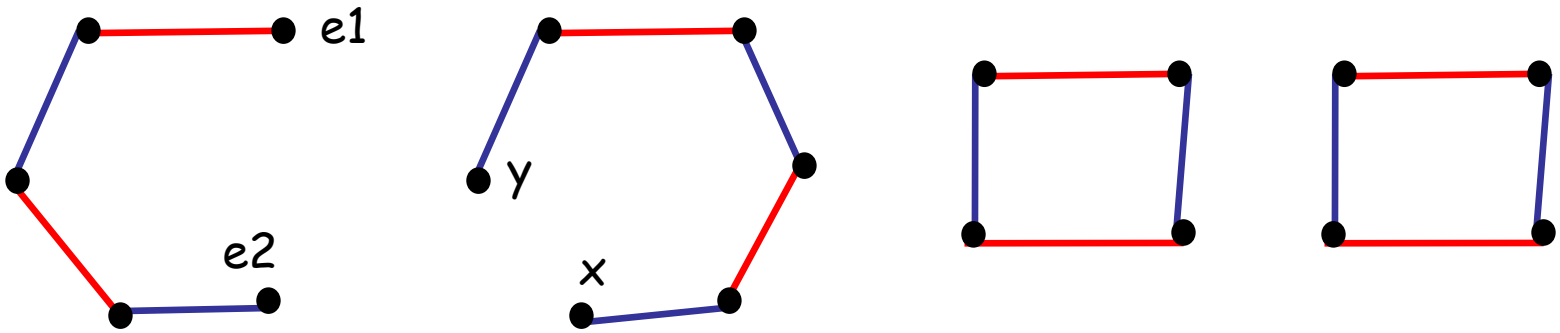

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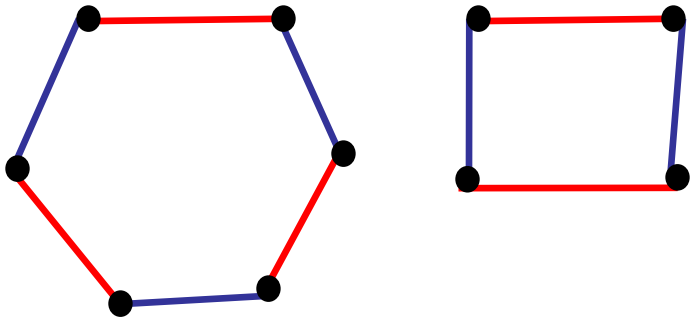

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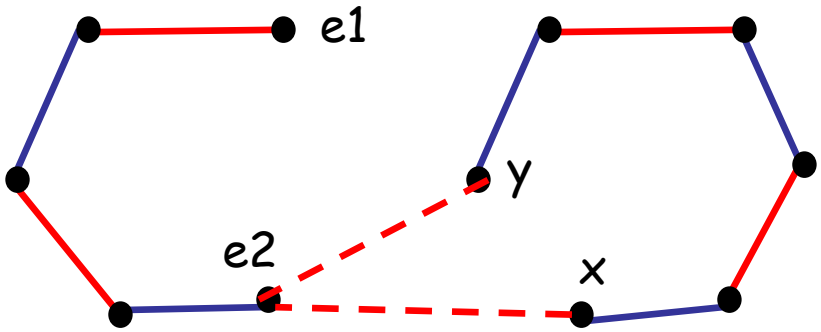

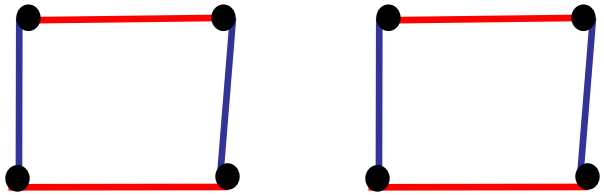

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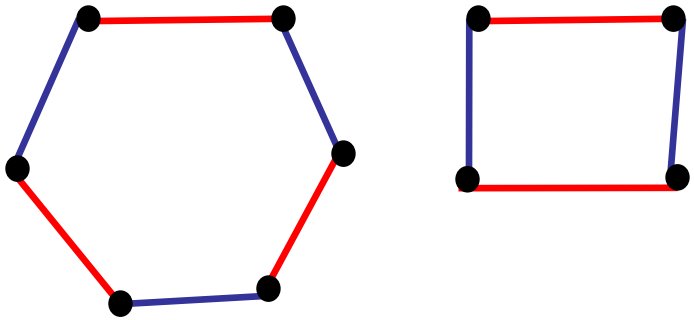

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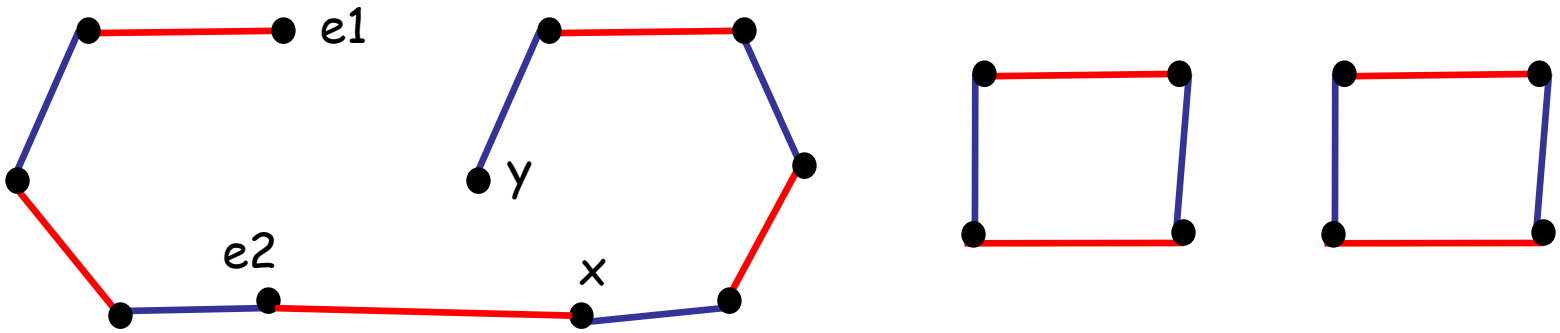

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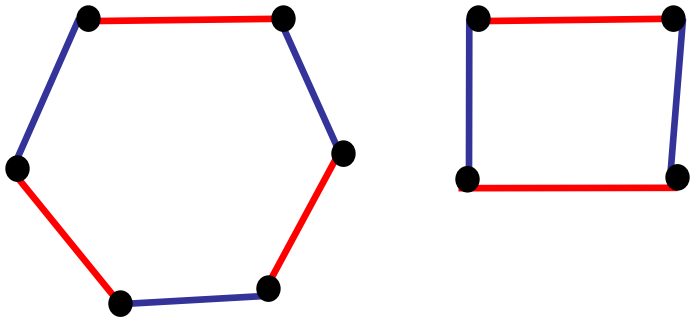

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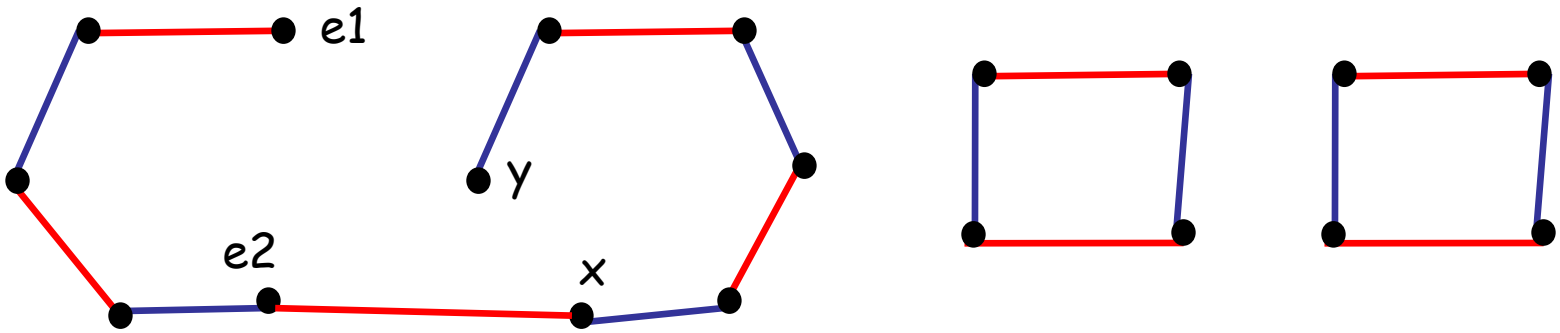

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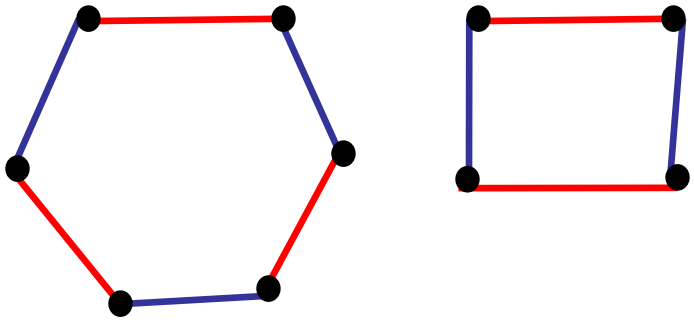

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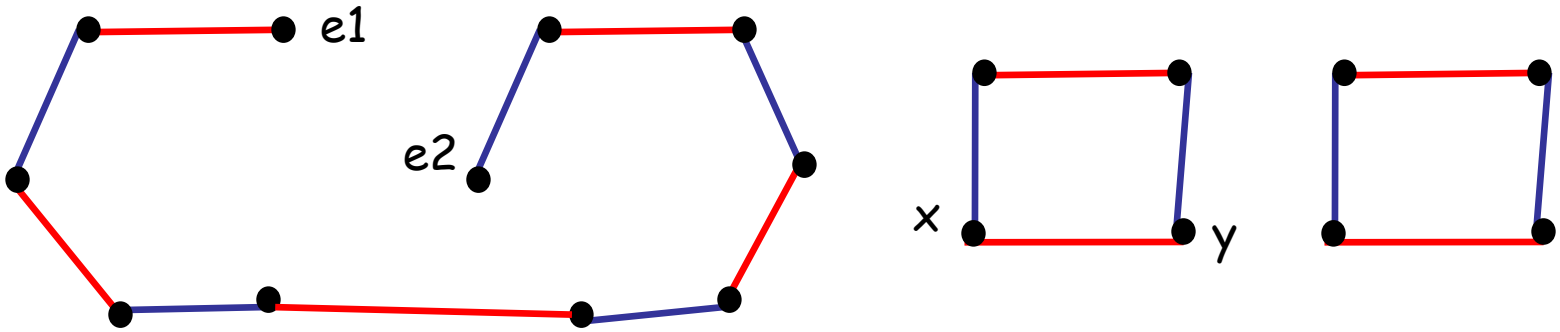

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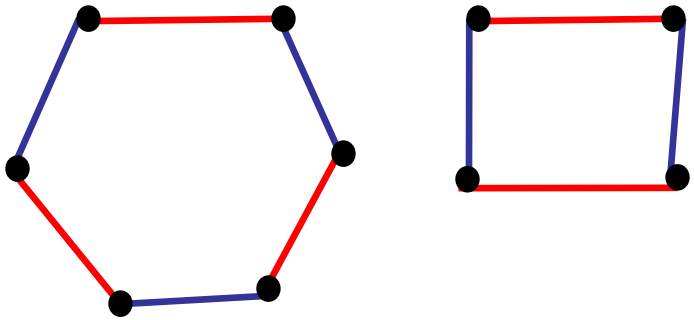

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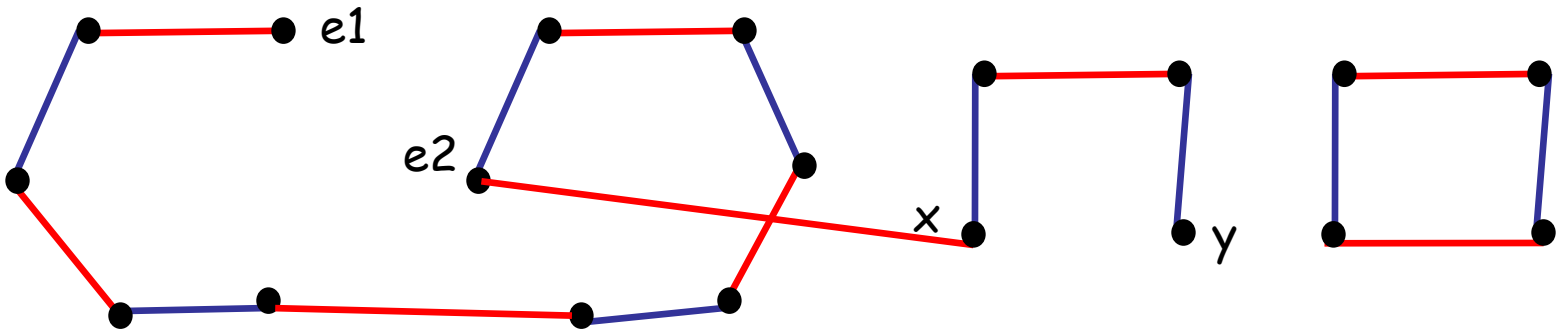

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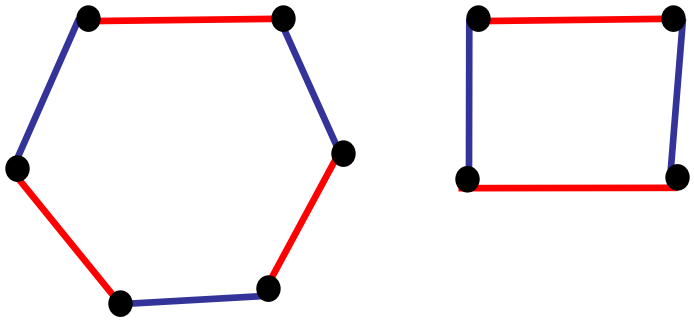

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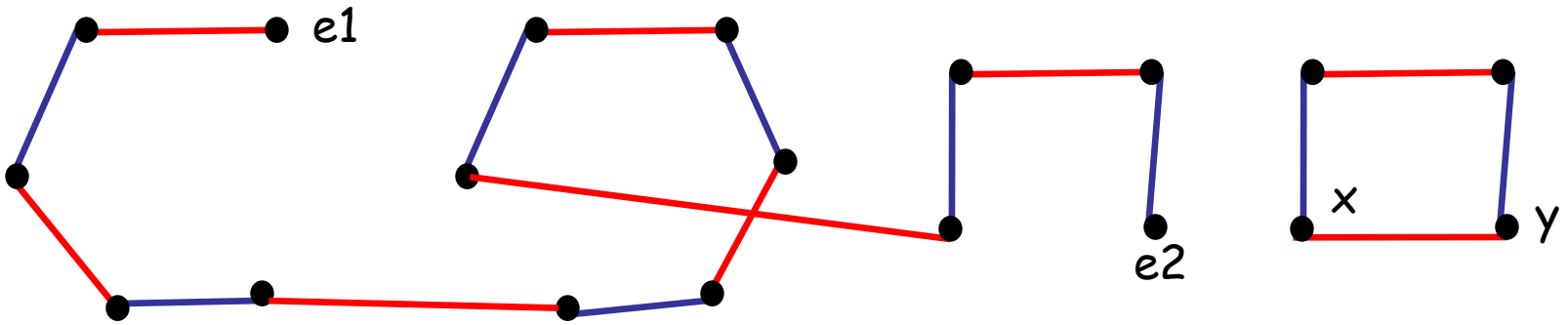

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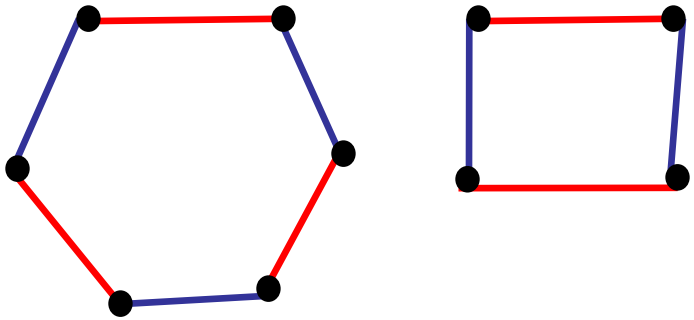

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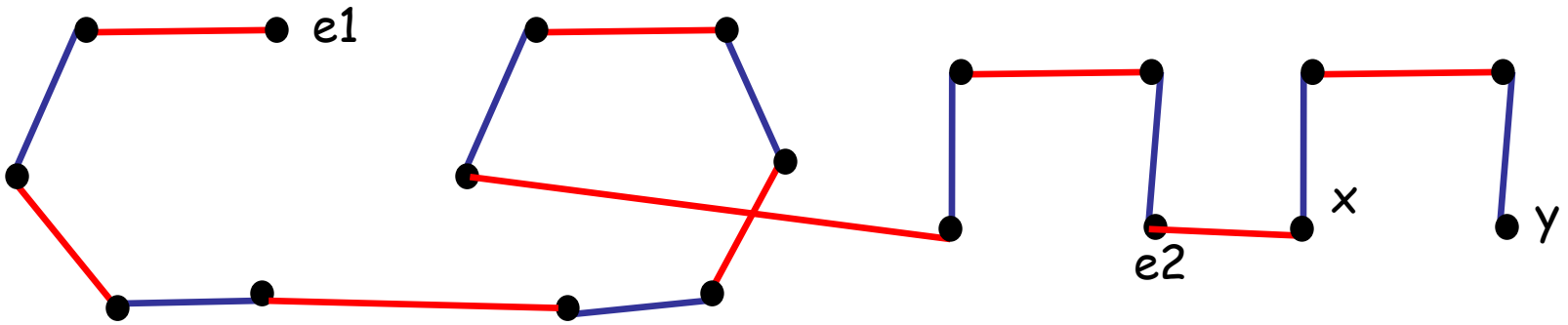

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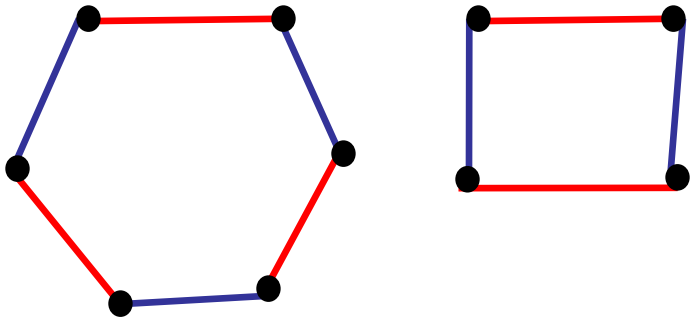

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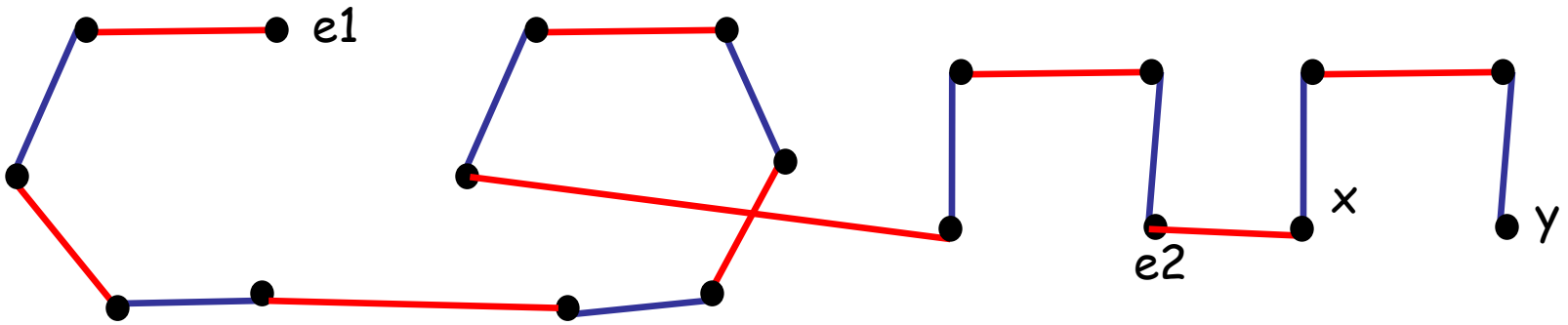

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Repeat this process until there is no more cycle.

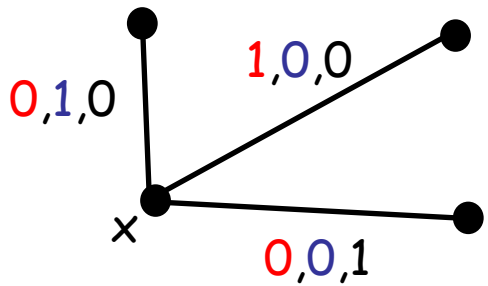
$\Rightarrow$ The cost of the output is $\geq 3/4\ a(M_a)$ and $\geq 3/4\ b(M_b)$

Outline

- Introduction :
  - the Multiobjective Max TSP
  - our approach
- Results for the Biobjective Max TSP
  - lower bounds
  - a generic algorithm
  - analysis of one case
- Dealing with many objectives
- Conclusion and further research

Dealing with many objectives

If no objective fulfills the triangle inequality, there is no approximate algorithm:



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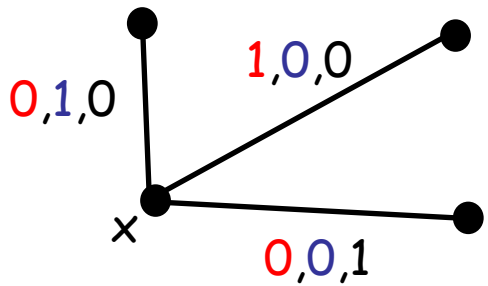
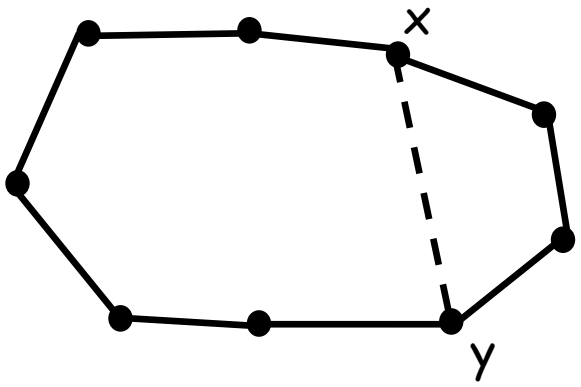


Property: If all the objectives fulfill the triangle inequality, then each cycle is n/2-approximate, and there is no (n/2+ε)-approximate algorithm.



Let T be any cycle.
Let T* be the optimal cycle for objective i,
and let {x,y} be its best edge : $i(x,y) \geq i(T^*)/n$.
Using the triangular inequality: $i(T) \geq 2\, i(T^*)/n$.

Conclusion and further research

- Approximation of the ideal point for the BiObjective Max TSP in 3 cases:
  - General case: $0.27 \leq r \leq 1/3$
  - One triangular inequality: $3/8 \leq r \leq 3/4$
  - Two triangular inequalities: $5/12 \leq r \leq 3/4$
- Many objectives : best approximation is n/2 (with triangular inequality)

- Intermediate cases: k-objective Max TSP?
- Multiobjective asymmetric Max TSP?
- Experimentations