

Tomographic Reconstruction - Exercises

Isabelle Bloch

LIP6, Sorbonne Université



SORBONNE
UNIVERSITÉ

isabelle.bloch@sorbonne-universite.fr

Definitions

- Radon transform:

$$\begin{aligned} R[f](u, \theta) &= p_{\theta}(u) \\ &= \int_{D_{\theta}} f(u \cos \theta - v \sin \theta, u \sin \theta + v \cos \theta) dv \end{aligned}$$

- Reconstruction:

$$\{p_{\theta}(u), \theta \in [0, \pi[, u \in \mathbb{R}\} \rightarrow \{f(x, y), (x, y) \in \mathbb{R}^2\}$$

- Backprojection:

- of a projection :

$$h_{\theta}(x, y) = p_{\theta}(x \cos \theta + y \sin \theta)$$

(value at (x, y) of the projection of angle θ
at point on which (x, y) projects)

- of all projections:

$$B[p](x, y) = \int_0^{\pi} p_{\theta}(x \cos \theta + y \sin \theta) d\theta$$

1 – Projection theorem

- Prove that

$$FT[p_\theta](U) = FT[f](U \cos \theta, U \sin \theta)$$

(FT = Fourier transform)

- How can this result be used to reconstruct f from the projections?

2 – Backprojection theorem

It can be proved that

$$B[p](x, y) = (f * h)(x, y)$$

with $h(x, y) = \frac{1}{\sqrt{x^2 + y^2}}$

- How can this result be used to reconstruct f from the projections?

3 – Filtered backprojection

- Prove that

$$f = B[\tilde{p}]$$

$$\text{with } \tilde{p}_\theta = IFT [FT[p_\theta](U) \cdot |U|]$$

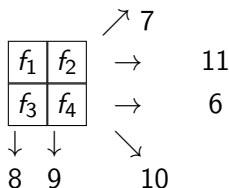
Note: this amounts to invert the order of backprojection and filtering operators.

- How can this result be used to reconstruct f from the projections?
- Summarize the main methods to reconstruct f from $\{p_\theta(u), \theta \in [0, \pi[, u \in \mathbb{R}\}$.

4 – Algebraic reconstruction methods

The problem is now expressed as $p = Rf$

Let us consider the following simple example: we want to reconstruct a 2×2 image, from the following projections:



- Write f, p, R .
- Why is it not sufficient to use the values 11, 6 (horizontal projections) and 8, 9 (vertical projections)?
- Find f using the ART method (using one p_j at each iteration):

$$f_i^{(k)} = f_i^{(k-1)} + R_{ji} \frac{p_j - R_j f^{(k-1)}}{\|R_j\|^2}$$

5 – Least square solution and regularization

- When the problem is ill-posed, instead of solving $Rf = p$, we may search for a least square solution, solving $\min_f ||Rf - p||^2$.
Calculate f .
- Since this solution may be unstable, we can add a regularization constraint, for instance minimizing the norm of f .
Calculate f under this additional constraint (use Lagrangian multiplier).