Brick wall excursions

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Short random walks

- d is the dimension,
- $\nu = \dfrac{d}{2} - 1$,
- $m = \#$ steps,
- A_k is a random step, $|A_k| = 1$.

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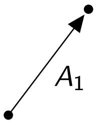

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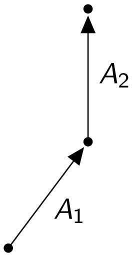

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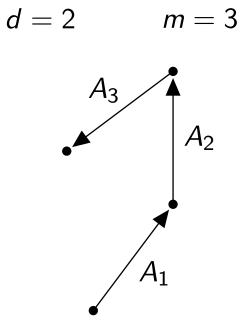

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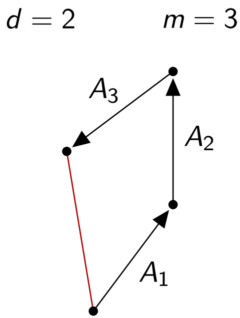

Key object: moments $W_m(\nu, n) = \mathbb{E}\left(|A_1 + \ldots + A_m|^n\right)$

- Fact: for any $m, n \in \mathbb{Z}_{\geqslant 0}$,
  $W_m(0, 2n)$ and $W_m(1, 2n)$ are integers. Interpretation?

Matrix form for $d = 2$ ($\nu = 0$). Example: $m = 2$

Fact:

$$W_m(0, 2n) = \sum_{\ell=0}^{n} [M^{m-1}]_{n\ell}\,, \qquad \text{where} \quad M = \left(\binom{i}{j}^2 \right)_{i,j \geqslant 0}$$

Example: $m = 2$,

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 1 & 4 & 1 & 0 & \cdots \\ 1 & 9 & 9 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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Example: $m = 2$, $W_2(0, 2n) = \binom{2n}{n}$

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 1 & 4 & 1 & 0 & \cdots \\ 1 & 9 & 9 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \begin{matrix} \to & 1 \\ \to & 2 \\ \to & 6 \\ \to & 20 \\ \vdots & \vdots \end{matrix}$$

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$W_2(0, 4) = 6$:

UUDD
UDUD
UDDU
DUUD
DUDU
DDUU

Matrix form for $d = 2$ $(\nu = 0)$. Example: $m = 3$

Example: $m = 3$

$$M^2 = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 0 & 0 & \cdots \\ 6 & 8 & 1 & 0 & \cdots \\ 20 & 46 & 10 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Matrix form for $d = 2$ ($\nu = 0$). Example: $m = 3$

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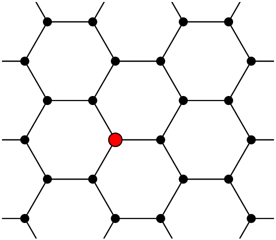
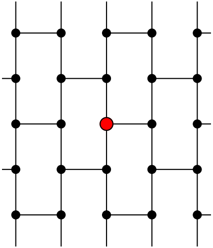


$\rightsquigarrow$



$W_3(0, 4) = 15$:

RLRL
RLUD
RLDU
RUDL
RDUL
ULRD
UDRL
DLRU
DURL
UUDD
UDUD
UDDU
DUUD
DUDU
DDUU

Interpretation for $d = 2$ ($\nu = 0$)

Let $A_k \in \mathbb{C}, \quad |A_k| = 1 \qquad (k = 1, \ldots, m).$

$$W_m(0, 2n) \quad = \quad \mathbb{E} \left| A_1 + \ldots + A_m \right|^{2n}$$

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$$W_m(0, 2n) \quad = \quad \mathbb{E}\left|A_1 + \ldots + A_m\right|^{2n}$$

$$\overset{(1)}{=} \quad \mathbb{E}\left(\left(A_1 + \ldots + A_m\right)\left(A_1^{-1} + \ldots + A_m^{-1}\right)\right)^n$$

1 $1 = |A_k|^2 = A_k \bar{A}_k \qquad \Rightarrow \qquad A_k^{-1} = \bar{A}_k$

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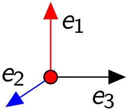
$$\stackrel{(1)}{=} \mathbb{E}\left(\left(A_1 + \ldots + A_m\right)\left(A_1^{-1} + \ldots + A_m^{-1}\right)\right)^n$$

$$\stackrel{(2)}{=} \left[A_1^0 \ldots A_m^0\right]\left(\left(A_1 + \ldots + A_m\right)\left(A_1^{-1} + \ldots + A_m^{-1}\right)\right)^n$$

1. $1 = |A_k|^2 = A_k \bar{A}_k \qquad \Rightarrow \qquad A_k^{-1} = \bar{A}_k$
2. $\mathbb{E}\left(A_1 A_2^{-1} A_3 A_1^{-1} A_3 A_2^{-1}\right) = \mathbb{E}\left(A_2^{-2} A_3^2\right) = \mathbb{E}\left(A_2^{-2}\right) \cdot \mathbb{E}\left(A_3^2\right) = 0$

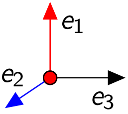
Interpretation for $d = 2$ ($\nu = 0$) and $m = 3$

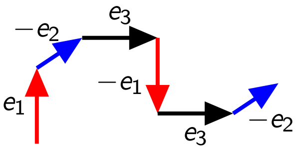
$$\mathbb{E}\Big(\big(\underset{\substack{\downarrow\\ e_1}}{A_1} + \underset{\substack{\downarrow\\ e_2}}{A_2} + \underset{\substack{\downarrow\\ e_3}}{A_3}\big)\big(\underset{\substack{\downarrow\\ -e_1}}{A_1^{-1}} + \underset{\substack{\downarrow\\ -e_2}}{A_2^{-1}} + \underset{\substack{\downarrow\\ -e_3}}{A_3^{-1}}\big)\Big)^n$$



Interpretation for $d = 2$ ($\nu = 0$) and $m = 3$

$$\mathbb{E}\Big(\big(\underset{\displaystyle e_1}{\overset{\displaystyle A_1}{\downarrow}} + \underset{\displaystyle e_2}{\overset{\displaystyle A_2}{\downarrow}} + \underset{\displaystyle e_3}{\overset{\displaystyle A_3}{\downarrow}}\big)\big(\underset{\displaystyle -e_1}{\overset{\displaystyle A_1^{-1}}{\downarrow}} + \underset{\displaystyle -e_2}{\overset{\displaystyle A_2^{-1}}{\downarrow}} + \underset{\displaystyle -e_3}{\overset{\displaystyle A_3^{-1}}{\downarrow}}\big)\Big)^n$$



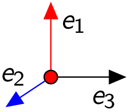


$$A_1 A_2^{-1} A_3 A_1^{-1} A_3 A_2^{-1}$$

Interpretation for $d = 2$ ($\nu = 0$) and $m = 3$

$$\mathbb{E}\Big(\big(A_1 + A_2 + A_3\big)\big(A_1^{-1} + A_2^{-1} + A_3^{-1}\big)\Big)^n$$

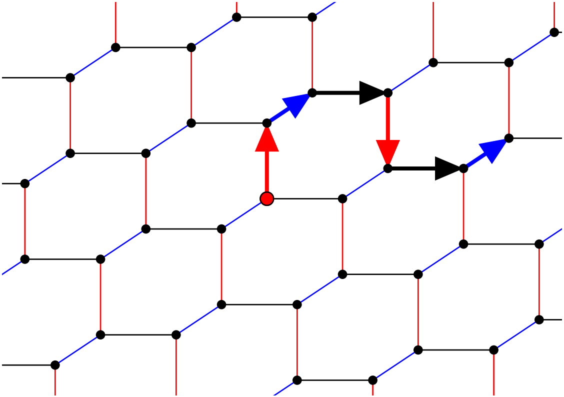
A_1	A_2	A_3	A_1^{-1}	A_2^{-1}	A_3^{-1}
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
e_1	e_2	e_3	$-e_1$	$-e_2$	$-e_3$



terms

$\downarrow$

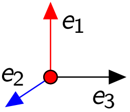
paths



Interpretation for $d = 2$ ($\nu = 0$) and $m = 3$

$$\mathbb{E}\Big(\big(\;A_1 \;+\; A_2 \;+\; A_3\;\big)\big(\;A_1^{-1} \;+\; A_2^{-1} \;+\; A_3^{-1}\;\big)\Big)^n$$

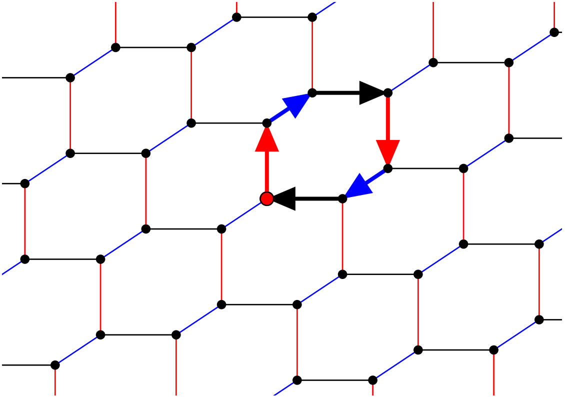
$$\begin{array}{cccccc} A_1 & A_2 & A_3 & A_1^{-1} & A_2^{-1} & A_3^{-1} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ e_1 & e_2 & e_3 & -e_1 & -e_2 & -e_3 \end{array}$$



constant term

$\downarrow$

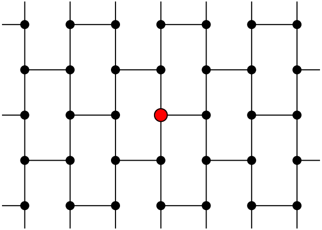
closed paths



Matrix form revisited for $d = 2$ ($\nu = 0$) and $m = 3$

Paths $\quad \leftrightarrow \quad$ words on $\{U, D, R, L\}$:

- R on odd positions,
- L on even positions.



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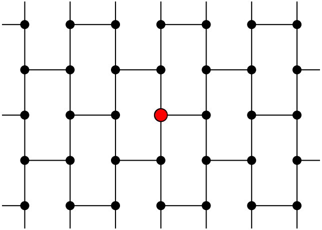

Let

- $2n = \#$ steps,
- $k = \#U = \#D$,
- $n - k = \#R = \#L$.

Then $$W_3(0, 2n) = \sum_{k=0}^{n} \binom{n}{n-k} \binom{n}{n-k} \binom{2k}{k}$$

Matrix form revisited for $d=2$ ($\nu=0$) and $m=3$

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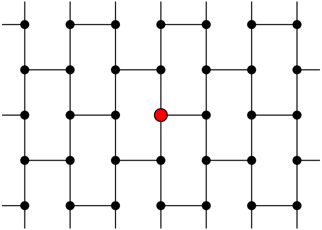

Let

- $2n = \#$ steps,
- $k = \#U = \#D$,
- $n-k = \#R = \#L$.

Then
$$W_3(0,2n) = \sum_{k=0}^{n} \binom{n}{n-k}\binom{n}{n-k}\binom{2k}{k} = \sum_{k=0}^{n} \binom{n}{k}^2 \sum_{\ell=0}^{k} \binom{k}{\ell}^2$$

Matrix form revisited for $d = 2$ ($\nu = 0$) and $m = 3$

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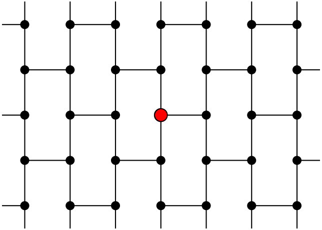

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Then
$$W_3(0, 2n) = \sum_{k=0}^{n} \binom{n}{n-k}\binom{n}{n-k}\binom{2k}{k}$$
$$= \sum_{k=0}^{n} \binom{n}{k}^2 \sum_{\ell=0}^{k} \binom{k}{\ell}^2$$
$$= \sum_{k,\ell=0}^{n} M_{nk} M_{k\ell} = \sum_{\ell=0}^{n} [M^2]_{n\ell}$$

General construction for $d = 2$ ($\nu = 0$)

Paths $\leftrightarrow$ words on $\{U, D, R_1, L_1, \dots, R_{m-2}, L_{m-2}\}$:

- R_s on odd positions,
- L_s on even positions.

Let

- $2n = \#$ steps,
- $k_0 = \#U = \#D$,
- $k_s = \#R_s = \#L_s$.

$$\begin{aligned}
W_m(0,2n) &= \sum_{k_0+\ldots+k_{m-2}=n} \binom{n}{k_{m-2}}^2 \binom{n-k_{m-2}}{k_{m-3}}^2 \ldots \binom{k_1+k_0}{k_0}^2 \binom{2k_0}{k_0} \\
&= \sum_{k_0+\ldots+k_{m-2}=n} \binom{k_0+\ldots+k_{m-2}}{k_0+\ldots+k_{m-3}}^2 \ldots \binom{k_0+k_1}{k_0}^2 \sum_{\ell=0}^{n} \binom{k_0}{\ell}^2 \\
&= \sum_{\ell, r_1, \ldots, r_{m-2}=0}^{n} M_{nr_{m-2}} \ldots M_{r_2 r_1} M_{r_1 \ell} = \sum_{\ell=0}^{n} [M^{m-1}]_{n\ell}
\end{aligned}$$

Summary for $d = 2$ ($\nu = 0$)

- We consider moments $W_m(0,2n) = \mathbb{E}\Big(|A_1 + \ldots + A_m|^n\Big)$, where $A_k \in \mathbb{C}$, $|A_k| = 1$ $(k = 1, \ldots, m)$.
- $W_m(0,2n)$ is the constant term in $\Big((A_1 + \ldots + A_m)(A_1^{-1} + \ldots + A_m^{-1})\Big)^n$.
- Thus, $W_m(0,2n)$ can be interpreted as the number of closed paths of length $2n$ on a specific m-dimensional lattice.
- In particular,

$$W_m(0,2n) = \sum_{\ell=0}^{n} [M^{m-1}]_{n\ell}\,, \qquad \text{where} \quad M = \left(\binom{i}{j}^2\right)_{i,j\geqslant 0}$$

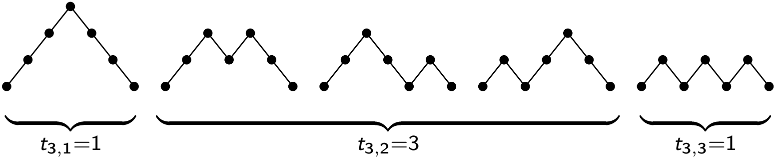
Matrix form for $d=4$ ($\nu=1$). Example: $m=2$

Fact:

$$W_m(1,2n) = \sum_{\ell=0}^{n} [N^{m-1}]_{n\ell}\,, \qquad \text{where} \quad N = \left(t_{i+1,j+1}\right)_{i,j\geqslant 0}$$

Here, $t_{i,j}$ are the **Narayana numbers**, *i.e.*

$$t_{i,j} = \#\{\text{Dyck paths of length } i \text{ with } j \text{ peaks}\}$$



Matrix form for $d = 4$ ($\nu = 1$). Example: $m = 2$

Fact:

$$W_m(1, 2n) = \sum_{\ell=0}^{n} [N^{m-1}]_{n\ell}\,, \qquad \text{where} \quad N = \left(t_{i+1,j+1}\right)_{i,j\geqslant 0}$$

Example: $m = 2$,

$$N = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 1 & 3 & 1 & 0 & \cdots \\ 1 & 6 & 6 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Matrix form for $d = 4$ ($\nu = 1$). Example: $m = 2$

Fact:

$$W_m(1, 2n) = \sum_{\ell=0}^{n} [N^{m-1}]_{n\ell}\,, \qquad \text{where} \quad N = \left(t_{i+1,\,j+1}\right)_{i,j\geqslant 0}$$

Example: $m = 2$, $W_2(1, 2n) = \frac{1}{n+2}\binom{2n+2}{n+1}$

$$N = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 1 & 3 & 1 & 0 & \cdots \\ 1 & 6 & 6 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \begin{matrix} \to & 1 \\ \to & 2 \\ \to & 5 \\ \to & 14 \\ \vdots & \vdots \end{matrix}$$

Matrix form for $d = 4$ ($\nu = 1$). Example: $m = 2$

Fact:

$$W_m(1, 2n) = \sum_{\ell=0}^{n} [N^{m-1}]_{n\ell}\,, \qquad \text{where} \quad N = \Big(t_{i+1,j+1} \Big)_{i,j \geqslant 0}$$

Example: $m = 2$, $W_2(1, 2n) = \frac{1}{n+2}\binom{2n+2}{n+1}$

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$W_2(1, 4) = 5$:

$UUUDDD$
$UUDUDD$
$UUDDUD$
$UDUUDD$
$UDUDUD$

Bijective lemma

Consider words on $\{R, L, O\}$ such that:

- R on odd positions,
- L on even positions,
- $\#R = \#L$,
- in each prefix, $\#R \geqslant \#L$.

Then the number D_n of such words of size $2n$ is

$$D_n = \sum_{k=0}^{n} t_{n+1,\, k+1} \,,$$

where

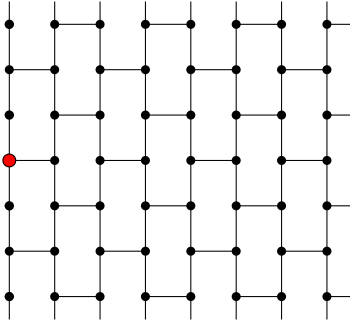
- $2k = \#O$,
- $n - k = \#R = \#L$.

Applications: closed path counting

Let us count closed paths with $2n$ steps.

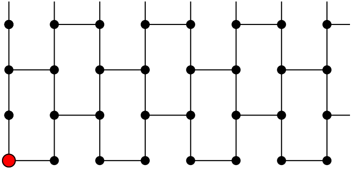
Half-plane:

$$\sum_{k=0}^{n} \binom{2k}{k} t_{n+1,\,k+1}$$



Quarter-plane:

$$\sum_{k=0}^{n} C_k t_{n+1,\,k+1}$$



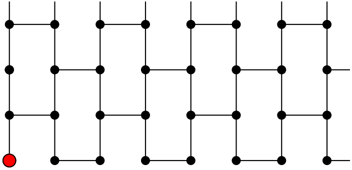
Hint: apply lemma with

$$O \rightsquigarrow U, D$$

Matrix form revisited for $d = 4$ ($\nu = 1$) and $m = 3$

Shifted quarter-plane:

- $2n = \#$ steps,
- R on even positions,
- L on odd positions,
- in each prefix, $\#U \geqslant \#D$,
- in each prefix, $\#R \geqslant \#L$.



Then $\quad \#\{\text{closed paths}\} = \sum_{k=0}^{n} C_{k+1} t_{n+1,\,k+1}$

(here, $\quad k = \#U = \#D \quad$ and $\quad n - k = \#R = \#L \quad$)

Matrix form revisited for $d = 4$ $(\nu = 1)$ and $m = 3$

Shifted quarter-plane:

- $2n = \#$ steps,
- R on even positions,
- L on odd positions,
- in each prefix, $\#U \geqslant \#D$,
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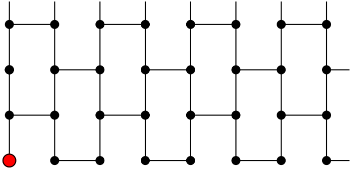


Then

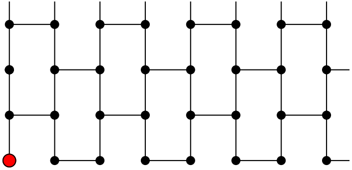
$$\#\{\text{closed paths}\} = \sum_{k=0}^{n} C_{k+1} t_{n+1,\,k+1} = \sum_{k=0}^{n} t_{n+1,\,k+1} \sum_{\ell=0}^{k} t_{k+1,\,\ell+1}$$

(here, $k = \#U = \#D$ and $n - k = \#R = \#L$)

Matrix form revisited for $d = 4$ ($\nu = 1$) and $m = 3$

Shifted quarter-plane:

- $2n = \#$ steps,
- R on even positions,
- L on odd positions,
- in each prefix, $\#U \geqslant \#D$,
- in each prefix, $\#R \geqslant \#L$.



Then

$$\#\{\text{closed paths}\} = \sum_{k=0}^{n} C_{k+1} t_{n+1,\,k+1} = \sum_{k=0}^{n} t_{n+1,\,k+1} \sum_{\ell=0}^{k} t_{k+1,\,\ell+1} = \sum_{k,\ell=0}^{n} N_{nk} N_{k\ell} = \sum_{\ell=0}^{n} [N^2]_{n\ell}$$

(here, $k = \#U = \#D$ and $n - k = \#R = \#L$)

General construction for $d = 4$ ($\nu = 1$)

Paths $\leftrightarrow$ words on $\{U, D, R_1, L_1, \ldots, R_{m-2}, L_{m-2}\}$:

- R_s on even positions (after removing all R_t and L_t, $t > s$),
- L_s on odd positions (after removing all R_t and L_t, $t > s$),
- in each prefix, $\#U \geqslant \#D$ and $\#R_s \geqslant \#L_s$.

Let

- $2n = \#$ steps,
- $k_0 = \#U = \#D$,
- $k_s = \#R_s = \#L_s$.

$$\begin{aligned} W_4(0, 2n) &= \sum_{k_0+k_1+k_2=n} t_{n+1,\, k_1+k_0+1} \cdot t_{n-k_2+1,\, k_0+1} \cdot C_{k_0+1} \\ &= \sum_{k_0+k_1+k_2=n} t_{n+1,\, k_1+k_0+1} \cdot t_{k_1+k_0+1,\, k_0+1} \sum_{\ell=0}^{n} t_{k_0+1,\, \ell+1} \\ &= \sum_{\ell, r_1, r_2=0}^{n} N_{nr_2} N_{r_2 r_1} N_{r_1 \ell} = \sum_{\ell=0}^{n} [N^3]_{n\ell} \end{aligned}$$

Summary for $d = 4$ ($\nu = 1$)

- We consider moments $W_m(1, 2n) = \mathbb{E}\Big(|A_1 + \ldots + A_m|^n\Big)$, where $A_k \in \mathbb{R}^4$, $|A_k| = 1$ $(k = 1, \ldots, m)$.

- It is known that,

$$W_m(1, 2n) = \sum_{\ell=0}^{n} [N^{m-1}]_{n\ell}\,, \qquad \text{where} \quad N = \big(t_{i+1,j+1}\big)_{i,j\geqslant 0}$$

- Thus, $W_m(1, 2n)$ can be interpreted as the number of closed paths of length $2n$ on a specific m-dimensional lattice.

- Question. Can we obtain the above result directly? (one could expect the use of quaternions)

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Thank you for your attention!

Literature

Borwein J.M., Straub A., Vignat C.
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