Brick wall excursions

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(joint with Sergey Kirgizov and Michael Wallner)

Random walks: applications and interactions

January 22, 2026

Short random walks

- d is the dimension,
- $\nu = \dfrac{d}{2} - 1$,
- $m = \#$ steps,
- A_k is a random step, $|A_k| = 1$.

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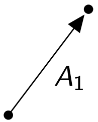

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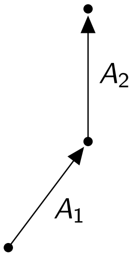

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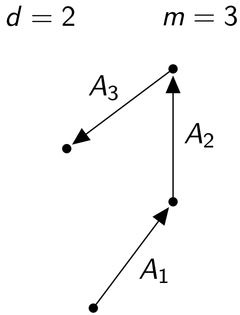

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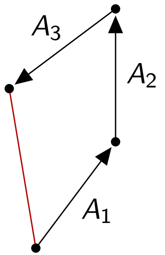

Key object: moments $W_m(\nu, n) = \mathbb{E}\Big(|A_1 + \ldots + A_m|^n\Big)$

- Fact: for any $m, n \in \mathbb{Z}_{\geqslant 0}$,
  $W_m(0, 2n)$ and $W_m(1, 2n)$ are integers. Interpretation?

Matrix form for $d = 2$ ($\nu = 0$). Example: $m = 2$

Fact:

$$W_m(0, 2n) = \sum_{\ell=0}^{n} [M^{m-1}]_{n\ell}\,, \qquad \text{where} \quad M = \left(\binom{i}{j}^2 \right)_{i,j \geqslant 0}$$

Example: $m = 2$, $\quad W_2(0, 2n) = \binom{2n}{n}$

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 1 & 4 & 1 & 0 & \cdots \\ 1 & 9 & 9 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{matrix} \to & 1 \\ \to & 2 \\ \to & 6 \\ \to & 20 \\ \vdots & \vdots \end{matrix}$$

$W_2(0, 4) = 6$:

UUDD
UDUD
UDDU
DUUD
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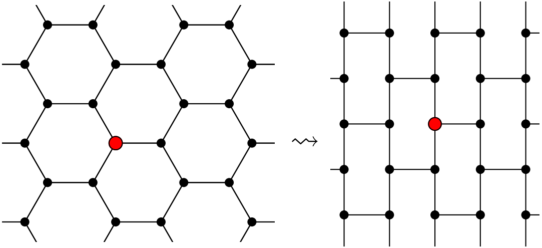

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Interpretation for $d = 2$ ($\nu = 0$)

Let $A_k \in \mathbb{C}, \quad |A_k| = 1 \qquad (k = 1, \ldots, m)$.

$$W_m(0, 2n) \quad = \quad \mathbb{E}\left|A_1 + \ldots + A_m\right|^{2n}$$

$$\stackrel{(1)}{=} \quad \mathbb{E}\left(\left(A_1 + \ldots + A_m\right)\left(A_1^{-1} + \ldots + A_m^{-1}\right)\right)^n$$

$$\stackrel{(2)}{=} \quad [A_1^0 \ldots A_m^0]\left(\left(A_1 + \ldots + A_m\right)\left(A_1^{-1} + \ldots + A_m^{-1}\right)\right)^n$$

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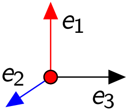
2 $\mathbb{E}\left(A_1 A_2^{-1} A_3 A_1^{-1} A_3 A_2^{-1}\right) = \mathbb{E}\left(A_2^{-2} A_3^2\right) = \mathbb{E}\left(A_2^{-2}\right) \cdot \mathbb{E}\left(A_3^2\right) = 0$

Interpretation for $d = 2$ ($\nu = 0$) and $m = 3$

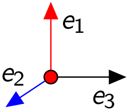
$$\mathbb{E}\Big(\big(A_1 + A_2 + A_3\big)\big(A_1^{-1} + A_2^{-1} + A_3^{-1}\big)\Big)^n$$

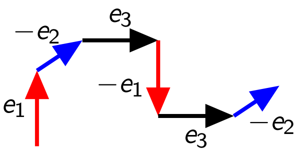
$$A_1 \to e_1,\quad A_2 \to e_2,\quad A_3 \to e_3,\quad A_1^{-1} \to -e_1,\quad A_2^{-1} \to -e_2,\quad A_3^{-1} \to -e_3$$



Interpretation for $d = 2$ ($\nu = 0$) and $m = 3$

$$\mathbb{E}\Big(\big(\underset{\substack{\downarrow \\ e_1}}{A_1} + \underset{\substack{\downarrow \\ e_2}}{A_2} + \underset{\substack{\downarrow \\ e_3}}{A_3} \big)\big(\underset{\substack{\downarrow \\ -e_1}}{A_1^{-1}} + \underset{\substack{\downarrow \\ -e_2}}{A_2^{-1}} + \underset{\substack{\downarrow \\ -e_3}}{A_3^{-1}} \big)\Big)^n$$

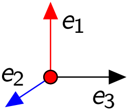




$$A_1 A_2^{-1} A_3 A_1^{-1} A_3 A_2^{-1}$$

Interpretation for $d = 2$ ($\nu = 0$) and $m = 3$

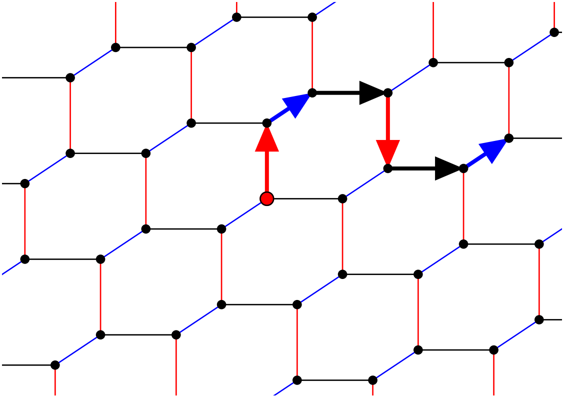
$$\mathbb{E}\Big(\big(\underset{\displaystyle e_1}{\underset{\downarrow}{A_1}} + \underset{\displaystyle e_2}{\underset{\downarrow}{A_2}} + \underset{\displaystyle e_3}{\underset{\downarrow}{A_3}}\big)\big(\underset{\displaystyle -e_1}{\underset{\downarrow}{A_1^{-1}}} + \underset{\displaystyle -e_2}{\underset{\downarrow}{A_2^{-1}}} + \underset{\displaystyle -e_3}{\underset{\downarrow}{A_3^{-1}}}\big)\Big)^n$$



terms

$\downarrow$

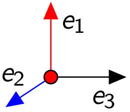
paths



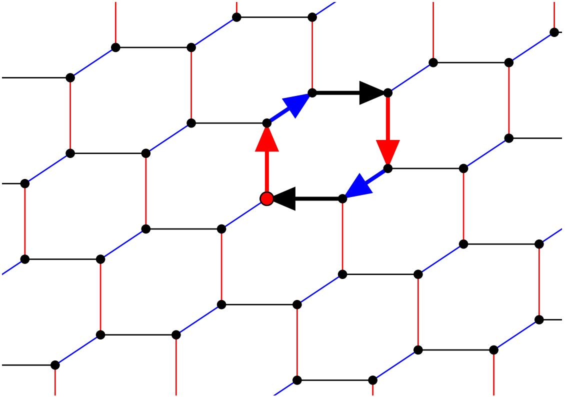
Interpretation for $d = 2$ ($\nu = 0$) and $m = 3$

$$\mathbb{E}\Big(\big(A_1 + A_2 + A_3 \big)\big(A_1^{-1} + A_2^{-1} + A_3^{-1} \big)\Big)^n$$

$A_1 \to e_1$, $A_2 \to e_2$, $A_3 \to e_3$, $A_1^{-1} \to -e_1$, $A_2^{-1} \to -e_2$, $A_3^{-1} \to -e_3$



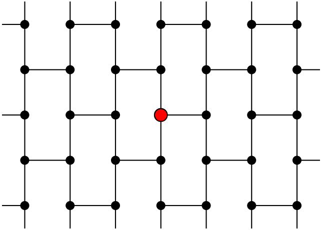
constant terms $\downarrow$ closed paths



Matrix form revisited for $d = 2$ ($\nu = 0$) and $m = 3$

Paths $\leftrightarrow$ words on $\{U, D, R, L\}$:

- R on odd positions,
- L on even positions,
- $2n = \#$ steps,



$$\#\{\text{closed paths of size } n\} = \sum_{k=0}^{n} \binom{n}{n-k}\binom{n}{n-k}\binom{2k}{k}$$

$$= \sum_{k=0}^{n} \binom{n}{k}^2 \sum_{\ell=0}^{k} \binom{k}{\ell}^2 = \sum_{\ell=0}^{n} [M^2]_{n\ell}$$

$n = 5$ ____ ____ ____ ____ ____ ____ ____ ____ ____ ____

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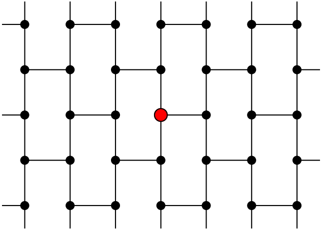

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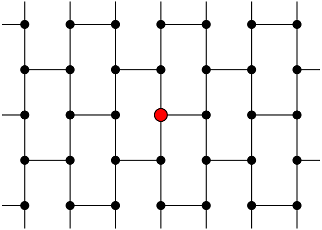

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$n = 5$ R R R $k = 2$

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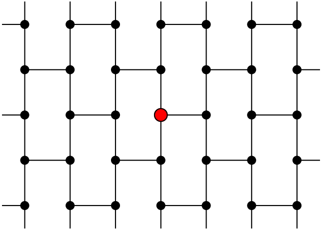

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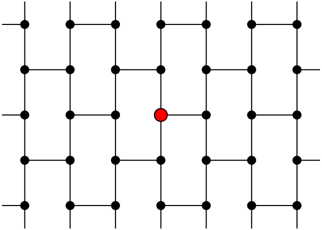

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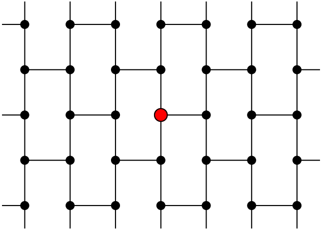

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$n=5$ R L D L U L R U R D $k=2$

Matrix form revisited for $d = 2$ ($\nu = 0$) and $m = 3$

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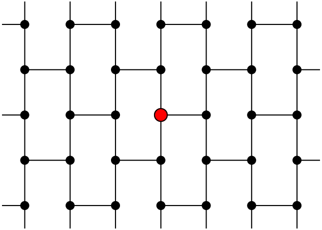

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$n = 5$ R L D L U L R U R D $k = 2$

General construction for $d = 2$ ($\nu = 0$)

Paths $\leftrightarrow$ words on $\{U, D, R_1, L_1, \ldots, R_{m-2}, L_{m-2}\}$:

- R_s on odd positions,
- L_s on even positions.

Let

- $2n = \#$ steps,
- $k_0 = \#U = \#D$,
- $k_s = \#R_s = \#L_s$.

$$\begin{aligned}
W_m(0, 2n) &= \sum_{k_0+\ldots+k_{m-2}=n} \binom{n}{k_{m-2}}^2 \binom{n-k_{m-2}}{k_{m-3}}^2 \ldots \binom{k_1+k_0}{k_0}^2 \binom{2k_0}{k_0} \\
&= \sum_{k_0+\ldots+k_{m-2}=n} \binom{k_0+\ldots+k_{m-2}}{k_0+\ldots+k_{m-3}}^2 \ldots \binom{k_0+k_1}{k_0}^2 \sum_{\ell=0}^{n} \binom{k_0}{\ell}^2 \\
&= \sum_{\ell, r_1, \ldots, r_{m-2}=0}^{n} M_{nr_{m-2}} \ldots M_{r_2 r_1} M_{r_1 \ell} = \sum_{\ell=0}^{n} [M^{m-1}]_{n\ell}
\end{aligned}$$

Summary for $d = 2$ ($\nu = 0$)

- We consider moments $W_m(0, 2n) = \mathbb{E}\Big(|A_1 + \ldots + A_m|^n\Big)$, where $A_k \in \mathbb{C}$, $|A_k| = 1$ $(k = 1, \ldots, m)$.
- $W_m(0, 2n)$ is the constant term in $\Big((A_1 + \ldots + A_m)(A_1^{-1} + \ldots + A_m^{-1})\Big)^n$.
- Thus, $W_m(0, 2n)$ can be interpreted as the number of closed paths of length $2n$ on a specific m-dimensional lattice.
- In particular,

$$W_m(0, 2n) = \sum_{\ell=0}^{n} [M^{m-1}]_{n\ell}\,, \qquad \text{where} \quad M = \left(\binom{i}{j}^2\right)_{i,j\geqslant 0}$$

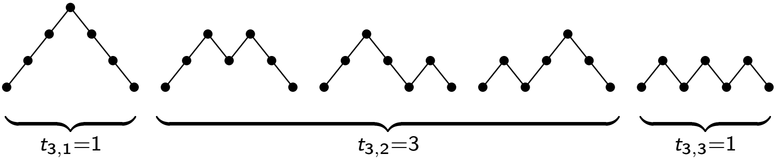
Matrix form for $d = 4$ ($\nu = 1$). Example: $m = 2$

Fact:

$$W_m(1, 2n) = \sum_{\ell=0}^{n} [N^{m-1}]_{n\ell}\,, \qquad \text{where} \quad N = \Big(t_{i+1,\,j+1}\Big)_{i,\,j \geqslant 0}$$

Here, $t_{i,j}$ are the **Narayana numbers**, *i.e.*

$$t_{i,j} = \#\{\text{Dyck paths of length } i \text{ with } j \text{ peaks}\}$$



Matrix form for $d = 4$ ($\nu = 1$). Example: $m = 2$

Fact:

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Example: $m = 2$, $W_2(1, 2n) = C_{n+1} = \frac{1}{n+2}\binom{2n+2}{n+1}$

$W_2(1, 4) = 5$:

$$N = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 1 & 3 & 1 & 0 & \cdots \\ 1 & 6 & 6 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{matrix} \to & 1 \\ \to & 2 \\ \to & 5 \\ \to & 14 \\ \vdots & \vdots \end{matrix}$$

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Matrix form for $d = 4$ $(\nu = 1)$. Example: $m = 2$

Fact:

$$W_m(1, 2n) = \sum_{\ell=0}^{n} [N^{m-1}]_{n\ell}, \qquad \text{where} \quad N = \left(t_{i+1, j+1}\right)_{i, j \geqslant 0}$$

Example: $m = 2$, $W_2(1, 2n) = C_{n+1} = \frac{1}{n+2}\binom{2n+2}{n+1}$

$$N = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 1 & 3 & 1 & 0 & \cdots \\ 1 & 6 & 6 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \begin{matrix} \to & 1 \\ \to & 2 \\ \to & 5 \\ \to & 14 \\ \vdots & \vdots \end{matrix}$$

$W_2(1, 4) = 5$:

UUUDDD
UUDUDD
UUDDUD
UDUUDD
UDUDUD

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$W_2(1, 4) = 5$:

UUUDDD
UUDUDD
UUDDUD
UDUUDD
UDUDUD

Bijective lemma

D_n is words on $\{R, L, O\}$ such that:

- R on odd positions,
- L on even positions,
- $\#R = \#L$,
- in each prefix, $\#R \geqslant \#L$,
- $2n = \#$ letters.

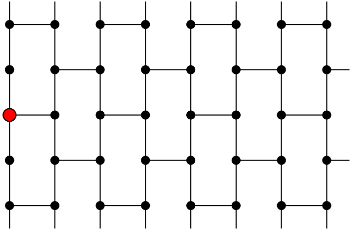
$$D_n = \sum_{k=0}^{n} t_{n+1,\, k+1}$$

R O O L O O R L $\quad 2k = \#O$

Bijective lemma and its consequences

Paths, half-plane:

- R on odd positions,
- L on even positions,
- $\#R = \#L$,
- in each prefix, $\#R \geqslant \#L$,
- $2n = \#$ steps.



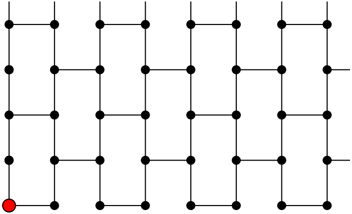
$$\#\{\text{closed paths}\} = \sum_{k=0}^{n} t_{n+1,\,k+1} \binom{2k}{k}$$

$O \rightsquigarrow U, D$ R D U L U D R L $k = \#U$ $k = \#D$

Bijective lemma and its consequences

Paths, quarter-plane:

- R on odd positions,
- L on even positions,
- $\#R = \#L$,
- in each prefix, $\#R \geqslant \#L$,
- $2n = \#$ steps.



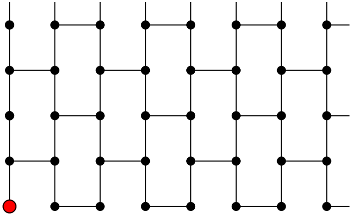
$$\#\{\text{closed paths}\} = \sum_{k=0}^{n} t_{n+1,\,k+1} C_k$$

$O \rightsquigarrow U, D$ R U D L U D R L $k = \#U$ $k = \#D$

Bijective lemma and its consequences

Paths, shifted quarter-plane:

- R on even positions,
- L on odd positions,
- $\#R = \#L$,
- in each prefix, $\#R \geqslant \#L$,
- $2n + 2 = \#$ steps.



$$\#\{\text{closed paths}\} = \sum_{k=0}^{n} t_{n+1,\,k+1}\, C_{k+1}$$

U R U D L U D R L D

Bijective lemma and its consequences

Paths, shifted quarter-plane:

- R on even positions,
- L on odd positions,
- $\#R = \#L$,
- in each prefix, $\#R \geqslant \#L$,
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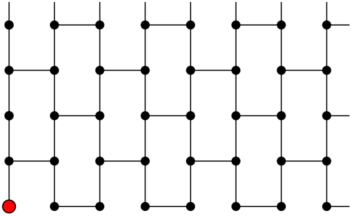

$$\#\{\text{closed paths}\} = \sum_{k=0}^{n} t_{n+1,\,k+1} C_{k+1} = \sum_{k=0}^{n} t_{n+1,\,k+1} \sum_{\ell=0}^{k} t_{k+1,\,\ell+1} = \sum_{\ell=0}^{n} [N^2]_{n\ell}$$

U R U D L U D R L D

General construction for $d=4$ $(\nu=1)$

Paths $\leftrightarrow$ words on $\{U, D, R_1, L_1, \ldots, R_{m-2}, L_{m-2}\}$:

- R_s on even positions (after removing all R_t and L_t, $t>s$),
- L_s on odd positions (after removing all R_t and L_t, $t>s$),
- in each prefix, $\#U \geqslant \#D$ and $\#R_s \geqslant \#L_s$.

Let

- $2n = \#$ steps,
- $k_0 = \#U = \#D$,
- $k_s = \#R_s = \#L_s$.

$$
\begin{aligned}
W_4(0,2n) &= \sum_{k_0+k_1+k_2=n} t_{n+1,\,k_1+k_0+1} \cdot t_{n-k_2+1,\,k_0+1} \cdot C_{k_0+1} \\
&= \sum_{k_0+k_1+k_2=n} t_{n+1,\,k_1+k_0+1} \cdot t_{k_1+k_0+1,\,k_0+1} \sum_{\ell=0}^{n} t_{k_0+1,\,\ell+1} \\
&= \sum_{\ell,r_1,r_2=0}^{n} N_{nr_2} N_{r_2 r_1} N_{r_1 \ell} = \sum_{\ell=0}^{n} [N^3]_{n\ell}
\end{aligned}
$$

Summary for $d = 4$ $(\nu = 1)$

- We consider moments $W_m(1, 2n) = \mathbb{E}\Big(|A_1 + \ldots + A_m|^n\Big)$, where $A_k \in \mathbb{R}^4$, $|A_k| = 1$ $(k = 1, \ldots, m)$.

- It is known that,

$$W_m(1, 2n) = \sum_{\ell=0}^{n} [N^{m-1}]_{n\ell}\,, \qquad \text{where} \quad N = \big(t_{i+1, j+1}\big)_{i, j \geqslant 0}$$

- Thus, $W_m(1, 2n)$ can be interpreted as the number of closed paths of length $2n$ on a specific m-dimensional lattice.

- Question. Can we obtain the above result directly? (one could expect the use of quaternions)

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- Question. Can we obtain the above result directly? (one could expect the use of quaternions)

Thank you for your attention!

Literature

Borwein J.M., Straub A., Vignat C.
Densities of short uniform random walks in higher dimensions
J. Math. Anal. Appl., 437(1): pp. 668–707, 2016.

Bijective lemma

The Narayana number

$$t_{n+1,\,k+1} = \frac{1}{n+1}\binom{n+1}{k}\binom{n+1}{k+1}$$

counts:

words on $\{R, L, O\}$ such that

- R on odd positions,
- L on even positions,
- in each prefix, $\#R \geqslant \#L$,
- $\#R = \#L = n - k$,
- $\#O = 2k$.

R L R O R O O L O L

Bijective lemma

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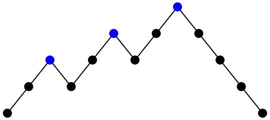
R L R O R O O L O L

Dyck paths:

- of size $(n+1)$,
- with $(k+1)$ peaks.



Bijective lemma

The Narayana number

$$t_{n+1,\, k+1} = \frac{1}{n+1}\binom{n+1}{k}\binom{n+1}{k+1} = t_{n+1,\, n-k+1}$$

counts:

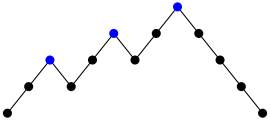
words on $\{R, L, O\}$ such that

- R on odd positions,
- L on even positions,
- in each prefix, $\#R \geqslant \#L$,
- $\#R = \#L = k$,
- $\#O = 2(n-k)$.

Dyck paths:

- of size $(n+1)$,
- with $(k+1)$ peaks.

O O R O R O O L O L $\rightsquigarrow$



Bijection, part I

1. Take a word.

R O O O R O O L O L O O R O R O O L O L

Bijection, part I

1. Take a word.
2. Identify
   - external R's and L's,

R O O O R O O L O L O O R O R O O L O L

Bijection, part I

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   - external R's and L's,
   - external O's.

R O O O R O O L O L O O R O R O O L O L

Bijection, part I

1. Take a word.
2. Identify
   - external R's and L's,
   - external O's.
3. Add one more R and L:
   - R: just before the first external O,
   - L: in the end.

R O O O R O O L O L R O O R O R O O L O L L

Bijection, part I

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   - external R's and L's,
   - external O's.

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4. Determine subwords $R\alpha_k L$.

R O O O R O O L O L R O O R O R O O L O L L

$R\alpha_1 L$ $R\alpha_2 L$

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1. Take a word.
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4. Determine subwords $R\alpha_k L$.
5. Treat each subword separately.

R O O O R O O L O L R O O R O R O O L O L L

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Bijection, part II

$$\mathrm{R}\ \underbrace{\mathrm{O}\ldots\mathrm{O}}_{x}\ \underline{\mathrm{R}\,\beta_1\,\mathrm{L}}\ \underbrace{\mathrm{O}\ \mathrm{O}\ldots\mathrm{O}\ \mathrm{O}}_{2i_1}\ \ \ldots\ \ \underbrace{\mathrm{O}\ \mathrm{O}\ldots\mathrm{O}\ \mathrm{O}}_{2i_{m-1}}\ \underline{\mathrm{R}\,\beta_m\,\mathrm{L}}\ \underbrace{\mathrm{O}\ldots\mathrm{O}}_{y}\ \mathrm{L}$$

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$$\mathrm{R}\ \underbrace{\mathrm{O}\ldots\mathrm{O}}_{x}\ \underline{\mathrm{R}\,\beta_1\,\mathrm{L}}\ \underbrace{\mathrm{O}\ \mathrm{O}\ldots\mathrm{O}\ \mathrm{O}}_{2i_1}\ \ldots\ \underbrace{\mathrm{O}\ \mathrm{O}\ldots\mathrm{O}\ \mathrm{O}}_{2i_{m-1}}\ \underline{\mathrm{R}\,\beta_m\,\mathrm{L}}\ \underbrace{\mathrm{O}\ldots\mathrm{O}}_{y}\ \mathrm{L}$$

$$2k = \begin{cases} x - y & \text{if } x > y \\ y - x + 2 & \text{if } x \leqslant y \end{cases} \qquad \downarrow \qquad j = \begin{cases} y & \text{if } x > y \\ x - 1 & \text{if } x \leqslant y \end{cases}$$

$$\mathrm{R}\ \underbrace{\mathrm{O}\ \mathrm{O}\ \mathrm{O}\ldots\mathrm{O}\ \mathrm{O}\ \mathrm{O}}_{j+2k+i_1+\ldots+i_{m-1}}\ \underline{\mathrm{R}\,\beta_1\,\mathrm{L}}\ \underbrace{\mathrm{O}\ldots\mathrm{O}}_{i_1}\ \ldots\ \underbrace{\mathrm{O}\ldots\mathrm{O}}_{i_{m-1}}\ \underline{\mathrm{R}\,\beta_m\,\mathrm{L}}\ \underbrace{\mathrm{O}\ldots\mathrm{O}}_{j}\ \mathrm{L}$$

Bijection, part II

$$\mathrm{R}\ \underbrace{\mathrm{O}\ldots\mathrm{O}}_{x}\ \underline{\mathrm{R}\,\beta_1\,\mathrm{L}}\ \underbrace{\mathrm{O}\ \mathrm{O}\ldots\mathrm{O}\ \mathrm{O}}_{2i_1}\ \ldots\ \underbrace{\mathrm{O}\ \mathrm{O}\ldots\mathrm{O}\ \mathrm{O}}_{2i_{m-1}}\ \underline{\mathrm{R}\,\beta_m\,\mathrm{L}}\ \underbrace{\mathrm{O}\ldots\mathrm{O}}_{y}\ \mathrm{L}$$

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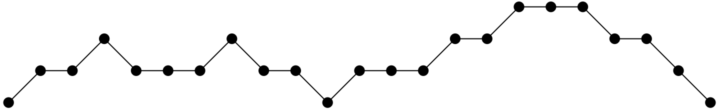
$$\downarrow$$

$$\mathrm{R}\ \underbrace{\mathrm{R}\ \mathrm{R}\ \mathrm{R}\ldots\mathrm{R}\ \mathrm{R}\ \mathrm{L}}_{j+2k+i_1+\ldots+i_{m-1}}\ \underline{\mathrm{R}\,\beta_1\,\mathrm{L}}\ \underbrace{\mathrm{L}\ldots\mathrm{L}}_{i_1}\ \ldots\ \underbrace{\mathrm{L}\ldots\mathrm{L}}_{i_{m-1}}\ \underline{\mathrm{R}\,\beta_m\,\mathrm{L}}\ \underbrace{\mathrm{L}\ldots\mathrm{L}}_{j}\ \mathrm{L}$$

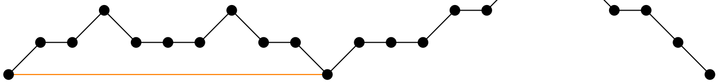
Bijection, example

R O R L O O R L O L R O O R O R O O L O L L



Bijection, example

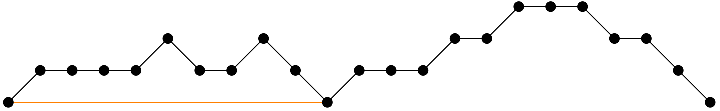
R O R L O O R L O L R O O R O R O O L O L L



Bijection, example

R O R L O O R L O L R O O R O R O O L O L L

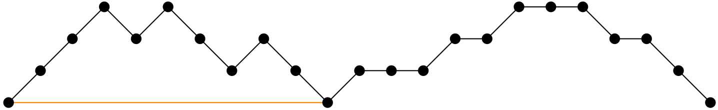
R O O O R L O R L L R O O R O R O O L O L L



Bijection, example

R O R L O O R L O L R O O R O R O O L O L L

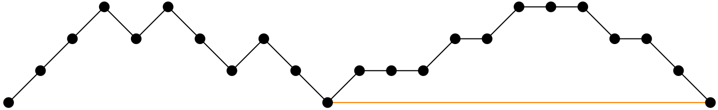
R R R L R L L R L L R O O R O R O O L O L L



Bijection, example

R O R L O O R L O L R O O R O R O O L O L L

R R R L R L L R L L R O O R O R O O L O L L

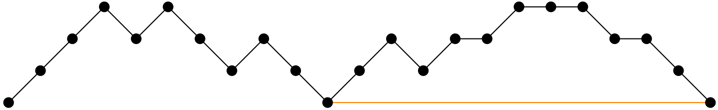


Bijection, example

R O R L O O R L O L R O O R O R O O L O L L

R R R L R L L R L L R O O R O R O O L O L L

R R R L R L L R L L R R L R O R O O L O L L

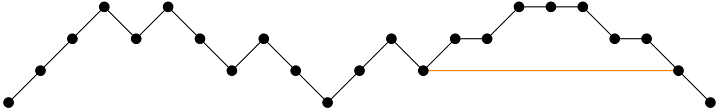


Bijection, example

R O R L O O R L O L R O O R O R O O L O L L

R R R L R L L R L L R O O R O R O O L O L L

R R R L R L L R L L R R L R O R O O L O L L



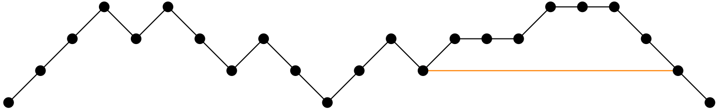
Bijection, example

R O R L O O R L O L R O O R O R O O L O L L

R R R L R L L R L L R O O R O R O O L O L L

R R R L R L L R L L R R L R O R O O L O L L

R R R L R L L R L L R R L R O O R O O L L L



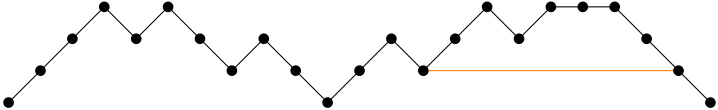
Bijection, example

R O R L O O R L O L R O O R O R O O L O L L

R R R L R L L R L L R O O R O R O O L O L L

R R R L R L L R L L R R L R O R O O L O L L

R R R L R L L R L L R R L R R L R O O L L L



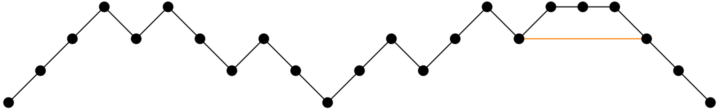
Bijection, example

R O R L O O R L O L R O O R O R O O L O L L

R R R L R L L R L L R O O R O R O O L O L L

R R R L R L L R L L R R L R O R O O L O L L

R R R L R L L R L L R R L R R L R O O L L L



Bijection, example

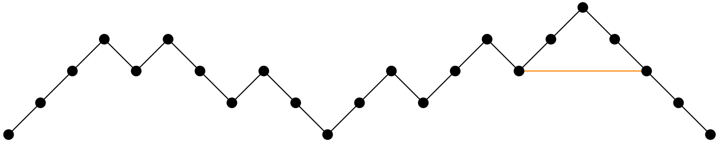
R O R L O O R L O L R O O R O R O O L O L L

R R R L R L L R L L R O O R O R O O L O L L

R R R L R L L R L L R R L R O R O O L O L L

R R R L R L L R L L R R L R R L R O O L L L

R R R L R L L R L L R R L R R L R R L L L L



Bijection, example

R O R L O O R L O L R O O R O R O O L O L L

R R R L R L L R L L R O O R O R O O L O L L

R R R L R L L R L L R R L R O R O O L O L L

R R R L R L L R L L R R L R R L R O O L L L

R R R L R L L R L L R R L R R L R R L L L L