Generalized critical values and solving polynomial inequalities

Mohab SAFEY EL DIN
LIP6, INRIA/LIP6 SALSA Project
Université Pierre et Marie Curie
4 place Jussieu
75005 Paris
France
Mohab.Safey@lip6.fr

Abstract

Consider $f \in \mathbb{Q}[X_1, \ldots, X_n]$ which is identified to the polynomial mapping sending $x \in \mathbb{C}^n$ to $f(x)$. The set $K(f)$ of *generalized critical values* of f is a finite set of points in $\mathbb{C}$ such that f realizes a locally trivial fibration on $\mathbb{C}^n \setminus f^{-1}(K(f))$. The core of this work is twofold. First, we provide an algorithm computing the *generalized critical values* of the polynomial mapping f. Secondly, we show how to use it in order to compute sampling points of a semi-algebraic set defined by a single polynomial inequality. The complexity of our algorithms is asymptotically optimal and an implementation is already provided in the RAGLib package (see `http://www-calfor.lip6.fr/~safey/RAGLib`).

Introduction. We provide an algorithm computing at least one point in each connected component of a semi-algebraic set defined by a single polynomial inequality $f > 0$ (or $f \neq 0$) where $f \in \mathbb{Q}[X_1, \ldots, X_n]$ of degree bounded by D. Some methods provide such routines with a complexity in D^n (see [2] and references therein) which improves Collins'Cylindrical Algebraic Decomposition. These are based on an infinitesimal deformation: each connected component of the solution set of $f > 0$ embedded in $\mathbb{R}\langle\varepsilon\rangle^n$ intersects at least one connected component of the hypersurface of $\mathbb{R}^n$ defined by $f - \varepsilon = 0$. Hence, the problem is reduced to study the hypersurface defined by $f - \varepsilon = 0$.
Instead of computing over $\mathbb{Q}(\varepsilon)$, our strategy consists in considering the problem as a problem of algebraic optimization. Consider $\mathcal{E}(f)$ the set of local extrema of f restricted to each connected component of the solution set of $f \neq 0$. Thus, for each connected component S of $f > 0$ and $0 < e < \min(B(f) \cap \mathbb{R}_+)$, the hypersurface defined by $f - e = 0$ has at least one connected component $D \subset S$. The set of *generalized critical values* of f defined in [3] (see below) contains $\mathcal{E}(f)$. Our contribution consists in designing an algorithm computing the set of generalized critical values of f with a worst-case complexity which is polynomial in D^{n+1} (where D denotes the degree of f) and use it to compute at least one

point in each connected component of the solution set of $f > 0$ with the same complexity.

Definition and properties of generalized critical values. Consider $f \in \mathbb{Q}[X_1, \ldots, X_n]$ of degree bounded by D. The set of *generalized critical values* of the mapping sending $x \in \mathbb{C}^n$ to $f(x)$ is the set:

$$K(f) = \{y \in \mathbb{C} \mid \exists (x_\ell)_{\ell \in \mathbb{N}}, f(x_\ell) \to y \text{ and } ||x_\ell||.||d_{x_\ell}(f)|| \to 0\}$$

The set of *asymptotic critical values* is the set:

$$K_\infty(f) = \{y \in \mathbb{C} \mid \exists (x_\ell)_{\ell \in \mathbb{N}}, f(x_\ell) \to y,\ ||x_\ell||.||d_{x_\ell}(f)|| \to 0 \text{ and } ||x_\ell|| \to \infty\}.$$

Obviously, $K(f) = K_\infty(f) \cup K_0(f)$ where $K_0(f)$ is the set of critical values of f. One can prove that $K(f)$ contains the local extrema of f on each connected component of the solution set of $f \neq 0$. Also, performing a linear change of variables does change neither the set of critical values nor the set of asymptotic critical values. Moreover, for $e \in \mathbb{Q}$, if $c \in K_\infty(f - e)$ then $c + e \in K_\infty(f)$.
In [3], the authors prove that the set of generalized critical values is Zariski closed. The following degree bound is provided in [3]: $D.\sharp K_\infty(f) + \sharp K_0(f) \leq D^n - 1$. Thus, providing an algorithm computing $K(f)$ whose worst-case complexity is polynomial in D^n is a relevant question.
In [3], the polynomial mapping $\phi : \mathbb{C}^n \to \mathbb{C} \times \mathbb{C}^n \times \mathbb{C}^{n^2}$ whose coordinates are: $(f, \frac{\partial f}{\partial X_1}, \ldots, \frac{\partial f}{\partial X_n}, X_1.\frac{\partial f}{\partial X_1}, \ldots, X_1.\frac{\partial f}{\partial X_n}, \ldots, X_n.\frac{\partial f}{\partial X_1}, \ldots, X_n.\frac{\partial f}{\partial X_n})$ is considered to characterize $K(f)$ as the intersection of the Zariski-closure of the image of Φ with the first axis of the target space.
They design an algorithm from this as follows. By introducing new variables t_i (for $i \in \{1, \ldots, n\}$, $t_{i,j}$ (for $(i,j) \in \{1, \ldots, n\}^2$) and T consider the ideal I generated by: $f - T, \left(\frac{\partial f}{\partial X_i} - t_i\right)_{i \in \{1,\ldots,n\}}, \left(X_i \frac{\partial f}{\partial X_j} - t_{i,j}\right)_{(i,j) \in \{1,\ldots,n\}^2}$; eliminate the variables $X_1, \ldots, X_n$ and put the variables t_i and $t_{i,j}$ to 0. Then, compute the gcd of the obtained univariate polynomials.

Summary of the results. Our contribution allows to improve dramatically the above algorithm. Our algorithm takes advantage of properness properties of *generic* projections to compute the set of asymptotic critical values of f.
Given a new variable T, denote by $\mathcal{H} \subset \mathbb{C}^{n+1}$ the hypersurface defined by $f - T = 0$. We consider in the sequel $\mathbf{A} \in GL_n(\mathbb{C})$, and denote by $f^{\mathbf{A}}$ the polynomial $f(\mathbf{A}.\mathbf{X})$, and by $\mathcal{H}^{\mathbf{A}}$ the hypersurface defined by $f^{\mathbf{A}} - T = 0$. At last, denote by $I^{\mathbf{A}}_{n-i+2}(f)$ the ideal generated by $f^{\mathbf{A}} - T = 0, \frac{\partial f^{\mathbf{A}}}{\partial X_n} = 0, \ldots, \frac{\partial f^{\mathbf{A}}}{\partial X_i} = 0$ and by $\mathcal{W}^{\mathbf{A}}_{n-i+2}(f) \subset \mathbb{C}^{n+1}$ its associated algebraic variety. Remark that denoting by $\pi_i : \mathbb{C}^{n+1} \to \mathbb{C}^i$ (for $i = 1, \ldots, n$) the canonical projection sending $(T, x_1, \ldots, x_n)$ on $(T, x_1, \ldots, x_{i-1})$, $\mathcal{W}^{\mathbf{A}}_{n-i+2}(f)$ is the critical locus of π_i restricted to $\mathcal{H}^{\mathbf{A}}$. We are ready to state our first result.

Theorem 1 *Let $e \in \mathbb{Q} \setminus K_0(f)$. There exists a proper Zariski-closed set $\mathcal{Z}$ in $GL_n(\mathbb{C})$ such that if $\mathcal{A} \notin \mathcal{Z}$ the set of asymptotic critical values $K_\infty(f - e)$ of $f - e$ is contained in the set of non-properness of π_1 restricted to $W^{\mathbf{A}}_{n-1}(f - e)$.*

From this result, one can retrieve easily the asymptotic critical values of f. We give here the main ingredients of the proof. The proof of [Proposition 3, 1] can be extended in our case to prove that for $i = 1, \dots, n$, $I^{\mathbf{A}}_{n-i+2}(f-e)$ is radical and equidimensional, and $\mathcal{W}^{\mathbf{A}}_{n-i+2}(f-e)$ is smooth. Moreover, in [4], the authors show that outside a proper Zariski-closed set of $GL_n(\mathbb{C})$, for any $i = 1, \dots, n$ the projection π_i restricted to $\mathcal{W}^{\mathbf{A}}_{n-i+1}$ is proper. Then, using this geometric property, Lojasiewicz's inequality and the algebraic characterization of asymptotic critical values of projections restricted to smooth and equidimensional varieties given in [3], we prove by descending induction on i that if $c \in K_\infty(f-e)$, then c belongs to the set of asymptotic critical values of π_1 restricted to $W^{\mathbf{A}}_{n-i+1}(f-e)$. Thus, computing the set of generalized critical values is done by retrieving the set of non-properness of π_1 restricted to $\mathcal{W}^{\mathbf{A}}_n$. This can be done via Gröbner bases computations and interpolation techniques; or via geometric resolutions and Hensel lifting in a complexity which is polynomial in D^{n+1} operations over $\mathbb{Q}$. Computing the set of critical values can be done with the same complexity. This complexity dominates the one of finding $e \in \mathbb{Q}$ such that $0 < e < \min(K(f) \cap \mathbb{R}_+)$ and studying the hypersurface defined by $f - e = 0$ which can be done using [4]. This yields the following result.

Theorem 2 *There exists an algorithm computing at least one point in each connected component of the semi-algebraic set defined by $f \neq 0$ whose complexity is polynomial in D^{n+1} operations in $\mathbb{Q}$.*

Some experiments show the efficiency of our approach compared to techniques using infinitesimal deformations and sharper complexity estimates justify it. Compared to CAD, the output size of our algorithm is smaller even on some problems with 3 variables.

References:

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Keywords: Polynomial system solving, inequalities, effective real algebraic geometry.