Positive dimensional parametric polynomial systems, connectivity queries and applications in robotics

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Abstract

In this paper we introduce methods and algorithms that will help us solve connectivity queries of parameterized semi-algebraic sets. Answering these connectivity queries is applied in the design of robotic structures having similar kinematic properties (e.g. topology of the kinematic-singularity-free space). From these algorithms one also obtain solutions to connectivity queries of a specific parameter which is in turn related to kinematic-singularity free path-planning of a specific manipulator belonging to the family of robots with these properties; i.e. we obtain paths joining two given singularity free configurations lying in the same connected component of the singularity-free space.

We prove in the paper how one reduces the problems related to connectivity queries of parameterized semi-algebraic sets to closed and bounded semi-algebraic sets. We then design an algorithm using computer-algebra methods for "solving" positive dimensional polynomial system depending on parameters. The meaning of solving here means partitioning the parameter space into semi-algebraic components over which the number of connected components of the semi-algebraic set defined by the input system is invariant. The complexity of this algorithm is singly exponential in the dimension of the ambient space. The algorithm scales enough to analyze automatically the family of UR-series robots.

Finally we provide manual analysis of the family of UR-series robots, proving that the number of connected components of the complementary of kinematic singularity set of a generic UR-robot is eight.

Keywords: Polynomial systems, Semi-algebraic sets, Roadmaps, Kinematic singularity

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1. Introduction

Problem statements and motivation. Polynomial system solving over the reals arises in many engineering sciences, in particular in robotics. In this paper, we investigate an algorithmic problem in computational real algebraic geometry which is motivated by applications in robotics, especially the kinematic singularity analysis mechanisms such as *serial robots* (also known as *manipulators*).

The algorithmic problem we look at can be stated as follows. Let $F = (f_1, \ldots, f_p)$ and $G = (g_1, \ldots, g_q)$ in $\mathbb{Q}[\boldsymbol{x}, \boldsymbol{y}]$ with $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_t)$, $\sigma = (\sigma_1, \ldots, \sigma_q)$ with $\sigma_i \in \{>, \geq\}$ for $1 \leq i \leq q$ and $S \subset \mathbb{R}^n \times \mathbb{R}^t$ be the semi-algebraic set defined by

$$f_1 = \cdots = f_p = 0, \quad g_1 \ \sigma_1 \ 0, \ldots, g_s \ \sigma_s \ 0$$

Further, we also say that S is defined by (F, G, σ) (or by (F, G) when σ is clear from the context).

Consider the canonical projection $\pi : (\boldsymbol{x}, \boldsymbol{y}) \mapsto \boldsymbol{y}$; we will call *parameter space* the $\boldsymbol{y}$-space. As a matter of fact the number of connected components of $S \cap \pi^{-1}(\eta)$ may change when η ranges over $\mathbb{R}^t$.

Example 1.1. *Consider the algebraic set S defined by $x_1^2+x_2^2+y_1^2-1 = 0$, then $S\cap\pi^{-1}(\eta)$ has one connected component (level sets of the sphere) for any η in the interval $[-1, 1]$ and no component for η outside this interval.*

When η ranges over $\mathbb{R}^t$, the set of numbers of connected components of $S \cap \pi^{-1}(\eta)$ is finite. Actually, we will see that there exists a finite partition of the parameter space into semi-algebraic connected components $U_1, \ldots, U_k$ such that, for a given $i \in \{1, \ldots, k\}$ the number of connected components of $S \cap \pi^{-1}(\eta)$ is invariant when η ranges over U_i.

We aim at computing a finite sequence of pairs

$$(b_1, \eta_1), \ldots, (b_r, \eta_r)$$

with $(b_i, \eta_i) \in \mathbb{N} \times \mathbb{R}^t$ such that b_i is the number of connected components of $S \cap \pi^{-1}(\eta_i)$ and $\eta_i \in U_i$ for $1 \leq i \leq k$. Moreover, we would like to guarantee that if b is the number of connected components of $S \cap \pi^{-1}(\eta)$ for some η, then one or more element of the form (b, η') appears in the finite sequence.

Example 1.2. *Consider the real torus S centered at the origin given by the zero set of the equation*

$$(y_1^2 + x_1^2 + x_2^2 + R^2 - r^2)^2 - 4R^2(y_1^2 + x_1^2) = 0$$

where R is the radius of the set of center points of the torus tube and r is the radius of the tube with $r < R$. Let the parameter be the y_1-coordinate, i.e. the projection we consider is $\pi : (x_1, x_2, y_1) \mapsto y_1$. Then

- *For η in the open interval $]-R+r, R-r[$ the set $S \cap \pi^{-1}(\eta)$ has two components,*
- *For $R - r \leq |\eta| \leq R + r$ the set $S \cap \pi^{-1}(\eta)$ has one component,*
- *For $|\eta| > R + r$ the set $S \cap \pi^{-1}(\eta)$ has no component.*

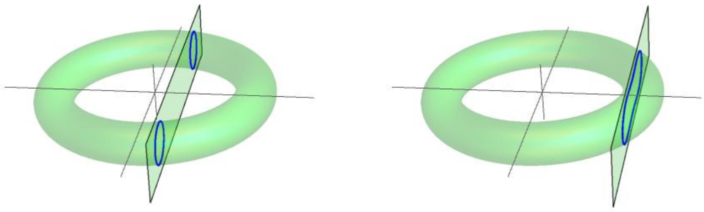

Figure 1: $S \cap \pi^{-1}(\eta)$ as blue curves; left for $-R+r<\eta<R-r$, right for $R-r \leq |\eta| \leq R+r$

We can thus choose our sequence to be

$$(0,-R-2r),(1,-R+r/2),(2,R),(1,R-r/2),(0,R+2r)$$

and the corresponding partitions $U_1,\ldots,U_5$ *are*

$$]-\infty,-R-r[,[-R-r,-R+r],]-R+r,R-r[,[R-r,R+r],]R+r,\infty[$$

A weak variant of this algorithmic problem would be to compute only the couples (b_i,η_i) as above but such that there exists an *open neighborhood* $\mathcal{U} \in \mathbb{R}^t$ of η_i such that for any $\eta \in \mathcal{U}$, the number of connected component of $S \cap \pi^{-1}(\eta)$ equals b_i.

Problem A. *Given* $F=(f_1,\ldots,f_p)$ *and* $G=(g_1,\ldots,g_q)$ *in* $\mathbb{Q}[\boldsymbol{x},\boldsymbol{y}]$ *which define the semi-algebraic set* $S \subset \mathbb{R}^n \times \mathbb{R}^t$ *as above, compute a finite sequence* $(b_i,\eta_i) \in \mathbb{N} \times \mathbb{R}^t$ *for* $1 \leq i \leq r$ *such that*

- b_i *is the number of connected components of* $S \cap \pi^{-1}(\eta_i)$*;*
- *there exists an open neighborhood* $\mathcal{U} \in \mathbb{R}^t$ *of* η_i *such that for any* $\eta \in \mathcal{U}$*, the number of connected component of* $S \cap \pi^{-1}(\eta)$ *equals* b_i*;*
- *the set* $\{\eta_1,\ldots,\eta_r\}$ *meets all the connected components* U *with non-empty interior of the parameters space such that the number of connected components of* $S \cap \pi^{-1}(\eta)$ *is invariant when* η *ranges over* U*.*

Furthermore, if b *is the number of connected components of* $S \cap \pi^{-1}(\eta)$ *for some* η*, then there should exist an element* (b,η') *in the finite sequence.*

Further, we say that an algorithm solves over the reals a parametric polynomial system of equations and inequalities when it solves Problem A.

Problem B. *Let* S *be the semi-algebraic set defined by* (F,G) *as in Problem A and suppose*

$$(b_i,\eta_i) \quad i=1,\ldots,r$$

be the sequence of pairs computed for Problem A. In this problem we want a program that, on an input $\eta_0 \in \pi(S)$*, outputs a path* $\gamma:[0,1] \to \pi(S)$ *from* η_0 *to* η_j *for some* $j=1,\ldots,r$ *with the property that*

$$\forall t \in [0,1] \qquad \text{the number of connected components of } S \cap \pi^{-1}(\gamma(t)) \text{ is } b_j$$

These problems arises naturally when one aims at analyzing the kinematic singularities of manipulators. The kinematic map of a manipulator is a differentiable map, from the configuration space (a manifold whose dimension is generally the same as the number of joints) to SE(3) which is a manifold of dimension 6. With proper parameterizations of the configuration space, the critical points of this map is an algebraic hypersurface in the configuration space and is also known as the kinematic singularities of the manipulator. Engineers prefer to plan a robot movements avoiding kinematic singularities mainly because controling a robot in a singular or near a singular configuration is rather difficult: If an end effector velocity or force for a robot in the vicinity of a singularity is desired, then the necessary joint velocity or torque is either not defined or very large (see [35] §4.3 and [42] §5.9). Furthermore, industrial controllers are based on Newton's method for the incremental solution of the inverse problem, and this method is not guaranteed to converge if it is used with a starting point close to the singularities.

Avoiding the singularities amounts to planning a path in the semi-algebraic set defined by the complement of the hypersurface that describes the singularities. This translates to solving the roadmap of a specific parameter (in robotic one uses the Denavit/Hartenberg-parameters which describes the kinematic structure of a given manipulator), i.e. solving the roadmap of the fibers which is inherent in both Problems A and B. Problems A and B are however more general. In applications, this would mean that we know a family of manipulators that share kinematic properties (e.g. inverse kinematics and topology of the singularity-free configurations). Problem B is more interesting in application, because this would mean that the manufacturers can vary the parameters and be able to design various robots having similar kinematic properties.

Main results. Our first contribution is an algorithm which solves Problem A under some genericity assumptions which are made explicit below. Hence, as above we consider $F = (f_1, \ldots, f_p)$, $G = (g_1, \ldots, g_q)$ in $\mathbb{Q}[\boldsymbol{x}, \boldsymbol{y}]$ and $\sigma \in \{>, \geq\}^q$; we let $S \subset \mathbb{R}^n \times \mathbb{R}^t$ be the semi-algebraic set defined by (F, G, σ). We say that (F, G) satisfy the property (R) when the following holds:

- the ideal $\langle f_1, \ldots, f_p \rangle$ is radical, equidimensional (of codimension p if not equal to $\langle 1 \rangle$) and the singular locus of its associated algebraic set has dimension less than t;
- for any $\{i_1, \ldots, i_s\} \subset \{1, \ldots, q\}$, the ideal $\langle f_1, \ldots, f_p, g_{i_1}, \ldots, g_{i_s} \rangle$ is radical, equidimensional of codimension $p + s$ if not equaled to $\langle 1 \rangle$ and its associated algebraic set has a singular locus whose dimension less than t.

We use in this paper the arithmetic complexity model, i.e. we count only arithmetic operations in the base field, which here is $\mathbb{Q}$. We use the classical big-O notation, i.e. $O(\varphi(a))$ where φ is a non-negative real valued function stands for the class of real valued functions such that, at infinity, they are dominated by φ up to a multiplicative constant.

The main algorithmic result of this paper is the following.

Theorem 1.3. *Let $F = (f_1, \ldots, f_p)$ and $G = (g_1, \ldots, g_q)$ be sequences in $\mathbb{Q}[\boldsymbol{x}, \boldsymbol{y}]$ with $D = \max(\deg(f_i), \deg(g_j), 1 \leq i \leq p, 1 \leq j \leq s)$, and $\sigma \in \{>, \geq\}^q$.*

Assume that (F, G) satisfies assumption (R). There exists a randomized algorithm which on input (F, G, σ) as above solves Problems A and B using

$$(\max(2^q, nq^n)nD)^{O(nt+n^2)}$$

arithmetic operations in $\mathbb{Q}$.

To achieve this result, our algorithm builds upon semi-algebraic variants of Thom's isotopy lemma [16] to compute an algebraic set $\mathscr{D}$ in the parameter space such that the number of connected components of $S \cap \pi^{-1}(\eta)$ remains invariant when η ranges over a connected component of $\mathbb{R}^t - \mathscr{D}$. We will see that doing so boils down to describing critical loci of well-chosen polynomial maps which are then projected to the parameter space. We will also see that these critical loci have degree bounded by $\max(2^q, nq^n)(nD)^n$. Algebraically, this leads to solving polynomial systems which encode the vanishing of some of the f_i's and the g_j's and maximal minors of some Jacobian matrices. A key algorithmic ingredient is then the use of algebraic elimination algorithms to manipulate and project these critical loci.

Next, we compute sample points per connected components in $\mathbb{R}^t - \mathscr{D}$ using a variant of the critical point method running in time which is singly exponential in the number of parameters (but with input polynomials which have degree exponential in n). We will show that this allows us to obtain the points $\eta_1, \ldots, \eta_r$ which are part of the output and it remains to compute the number of connected components of $S \cap \pi^{-1}(\eta_i)$ for $1 \leq i \leq r$. This is done using algorithms for computing roadmaps of semi-algebraic sets which allow us to count the number of connected components the solution set to a system of polynomial equations and inequalities.

We have implemented our algorithm for solving parametric polynomial systems of equations and inequalities using the Maple computer algebra system. The algebraic elimination algorithms we use are based on Gröbner bases computations. To perform these computations, we use the FGb library for computing Gröbner bases, which is implemented in C by J.-C. Faugère [22]. The computation of sample points per connected components in a semi-algebraic set defined as the complementary of a given algebraic set is done using the library RAGlib which is implemented within Maple by M. Safey El Din [37].

It turns out that our implementation can tackle kinematic singularities of some families of manipulators (e.g. the UR series).

In this paper we have specifically analysed the singularity-free space of the UR-series robot. This family of robot have the same kinematic structure as the UR5 robot: there are three consecutive joint axes that are parallel (so the second and third distances are non-zero) and otherwise the consecutive joint axes have right-angle twists and finally there are only two offset for after the third and fourth joint. By analysing the polynomial describing the kinematic singularity of this robot, we conclude that a generic UR robot will have 8 connected components in its singularity-free configuration space.

Prior works. Computing a Cylindrical Algebraic Decomposition (CAD) [15] of the semi-algebraic set defined by (F, G) provides a decomposition of the parameter space into cells yielding the sample points η_i for Problems A and B (up to our knowledge this is not formally proven in the literature). Next, computing a CAD adapted to the polynomials obtained after specializing $\boldsymbol{y}$ to η_i in (F, G) and adjacency relations between the cells of those CAD provides the corresponding b_i. Recall that CAD has a doubly exponential complexity in $n + t$ [12, 18]. Algorithms for answering connectivity queries in semi-algebraic sets go back to Canny's work, in particular [13] which has then been made deterministic through a series of work (see [8, Chap. 14] and references therein). It should be noted that, recently, important improvements on the complexity of computing roadmaps of real algebraic sets have been developed thanks to a new approach introduced in [39] and developed in [9, 10] which culminates with fast algorithms given in [40] for smooth and bounded real algebraic sets. We also mention an alternative symbolic-numeric technique in [30] which makes use of gradient vector fields and Morse theory to decide if two given points lie in the same connected component of a semi-algebraic set defined by the non-vanishing of a given multivariate polynomial.

Solving parametric systems of polynomial equations has mainly been studied in the special case where $V(f_1, \ldots, f_p) \cap \pi^{-1}(\eta)$ is finite where η is a generic point of $\mathbb{C}^t$. We call such systems *zero-dimensional* parametric polynomial systems. Solving them over the reals boils down to classifying the number of real roots with respect to parameter values. We refer to [31, 47] and references therein for works on this topic which culminate with [32] which provide faster and quasi-optimal degree bounds for formulæ defining the regions of the parameter space over which the number of real roots remains invariant.

Note that in this paper, we go beyond this problem since we consider parametric polynomial systems of *positive* dimension. By "solving", we do not mean "computing the solutions explicitly", but "constructing a classification of the real solutions". Up to our knowledge, the definition of *solving over the reals* positive dimensional polynomial systems with parameters is done first in [14]. The algorithm given in [14] follows the same pattern as the one we give here and this paper is an extended version of [14].

Remark however that this paper extends in the following ways our previous results:

- the assumptions needed here are weaker than the ones needed in [14] where we were requesting that $V(f_1, \ldots, f_p)$ and its intersection with all algebraic sets $V(g_{i_1}, \ldots, g_{i_\ell})$ are smooth;
- we provide a new complexity analysis of this kind of algorithms and show that our algorithm runs in singly exponential time with respect to n and t and polynomial time with respect to D;
- we revisit algorithms for answering connectivity queries in semi-algebraic sets and show how to obtain an efficient reduction to closed and bounded semi-algebraic sets without using infinitesimals when some regularity assumptions are satisfied by the input ;
- we reformulate and correct the theoretical analysis of the UR-series in [14] so that now it is a consequence of a general mathematical fact.

Organization of the paper. The paper is organized as follows. Section 2 recalls some basic notions and results of semi-algebraic geometry and proves that Problem A makes sense and is decidable. We also recall some basics on robotics, formulating the problems for analysis of kinematic singularity of 6-jointed serial manipulators. Section 3 investigates, under some regularity properties, how we are able to compute roadmap of semi-algebraic sets and even reduce the problem to the case of closed and bounded semi-algebraic sets. The section that describes our novel algorithm for parameteric systems is in Section 4. We describe in the section the three subroutines we need (all under some regularity assumptions) and we prove the correctness of the algorithm. Finally, in the section we prove Theorem 1.3 and provide a complexity analysis of our algorithm. In Section

5, we prove (without the algorithm) that a generic robot in the UR-series, will have eight singularity-free connected components. In the last section we compare our manual analysis with the result of the algorithm using the UR-series as input and we show that the results are indeed similar.

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2. Preliminaries

2.1. Basic results of real algebraic geometry

We start by recalling some basic results of real algebraic geometry which are important to show that, actually, Problems A and B make sense and are decidable. The only new result in this section is Proposition 2.3 which establishes the requested finiteness properties.

The very first well-known basic fact is about the finiteness of the number of connected components of a given semi-algebraic set.

Proposition 2.1. *[11, Theorem 2.4.4] Let $S \subset \mathbb{R}^n$ be a semi-algebraic set. Then, S is the union of finitely many connected components which are themselves semi-algebraic sets.*

We also recall Hardt's theorem. To do that, we need to introduce a few basic notions. We let X, Y' and Y be semi-algebraic sets with $Y' \subset Y$ and we consider a continuous semi-algebraic map $\varphi : X \to Y$. A semi-algebraic trivialization of φ over Y' with fiber F is a semi-algebraic *homeomorphism* $\vartheta : Y' \times F \to \varphi^{-1}(Y')$ such that the following diagram commutes:

$$\begin{tikzcd} Y' \times F \arrow[r, "\vartheta"] \arrow[dr, "\mathrm{proj}_1"'] & \varphi^{-1}(Y') \arrow[d, "\varphi"] \\ & Y' \end{tikzcd}$$

where proj_1 is the projection $(\eta, e) \in Y' \times F \to \eta \in Y'$. We say that ϑ is compatible with a subset X' of X if there is $F' \subset F$ such that $\vartheta(Y' \times F') = X' \cap \varphi^{-1}(Y')$.

Theorem 2.2. *[11, Theorem 9.3.2][28] Let X and Y be semi-algebraic sets and $\varphi : X \to Y$ be a continuous semi-algebraic map. Consider a finite family of semi-algebraic subsets $(X_i)_{1\le i\le k}$ of X.*

There exists a finite partition of Y into semi-algebraic sets $Y = \cup_{1\le j\le \ell} Y_j$ and, for $1 \le j \le \ell$, a trivialization $\vartheta_j : Y_j \times F_j \to \varphi^{-1}(Y_j)$ of φ over Y_j (where F_j is a fiber), compatible with X_i for $1 \le i \le k$.

Using Hardt's semi-algebraic triviality theorem, we can now prove that solving Problem A and B makes sense.

Proposition 2.3. *Let $S \subset \mathbb{R}^n \times \mathbb{R}^t$ be a semi-algebraic set and π be the canonical projection*

$$(x_1, \ldots, x_n, y_1, \ldots, y_t) \to (y_1, \ldots, y_t).$$

There exist semi-algebraic sets $Y_1, \ldots, Y_r$ in $\mathbb{R}^t$ such that

- $\mathbb{R}^t = Y_1 \cup \cdots \cup Y_r$,
- *there exists $b_i \in \mathbb{N}$ such that for any $y \in Y_i$, the number of connected components of $\pi^{-1}(y) \cap S$ is b_i.*

Proof. Observe that the restriction of π to S is semi-algebraically continuous. From Hardt's semi-algebraic triviality theorem [11, Theorem 9.3.2], there exists a finite partition of $\mathbb{R}^t$ into semi-algebraic sets $Y_1, \ldots, Y_r$ and for each $1 \leq i \leq r$, a trivialization $\vartheta_i : Y_i \times E_i \to \pi^{-1}(Y_i) \cap S$ (where E_i is a fiber $\pi^{-1}(y) \cap S$ for some $y \in Y_i$). Fix i and choose an arbitrary point $y' \in Y_i$. Observe that we are done once we have proved that $\pi^{-1}(y') \cap S$ and E_i have the same number of connected components. Recall that, by definition of a trivialization (see [11, Definition 9.3.1]), $\theta_i : Y_i \times E_i \to \pi^{-1}(Y_i) \cap S$ is a semi-algebraic homeomorphism and for any $(y', x) \in Y_i \times E_i$, $\pi \circ \theta_i(y', x) = y'$. Hence, we deduce that E_i is homeomorphic to $\pi^{-1}(y') \cap S$. As a consequence, they both have the same number of connected components which is finite by Proposition 2.1. $\square$

We finish this section by recalling a semi-algebraic version of Thom's isotopy lemma. This is a key ingredient to prove the correctness of our algorithm solving Problems A and B. Recall that, for a real closed field $\mathbf{R}$, a Nash mapping on an open semi-algebraic subset $U \subset \mathbf{R}^t$ is a $\mathcal{C}^\infty$ semi-algebraic map (see [11, Chap. 2]).

Also a map φ from a metric space X to a metric space Y is said to be proper if the preimage of any closed and bounded set of Y by φ is closed and bounded in X.

Theorem 2.4. *[16, Theorem 1] Let $\mathbf{R}$ be a real closed field and $X \subset \mathbf{R}^n$ be a locally closed semi-algebraic set together with a Whitney stratification $\mathscr{X}$ of X. Denote by $X^{(i)}$ the union of the strata in $\mathscr{X}$ of dimension i.*

Let φ be a Nash mapping from a neighborhood of $U \supset X$ in $\mathbf{R}^n$ to an open connected semi-algebraic subset Y of $\mathbf{R}^t$ such that

- *the restriction of φ to X is proper and*
- *the restriction of φ to $X^{(i)}$ is a submersion*

Then, for $y \in Y$, there is a semi-algebraic trivialization of φ over $\mathbb{R}^t$ which is compatible with $X^{(i)}$

$$\vartheta : X \to (X \cap \varphi^{-1}(y)) \times Y$$

such that ϑ restricted to $X^{(i)}$ is a Nash diffeomorphism onto $(X \cap \varphi^{-1}(y)) \times Y$ for $1 \leq i \leq n$.

2.2. Robotics problem formulation

We define a *manipulator* or *robot* as follows: we have finite ordered rigid bodies called *links* which are connected by n revolute *joints* that are also ordered. To each joint we associate a coordinate system or a *frame*. The links are connected in a serial manner i.e.

if we consider the robot as a graph such that the vertices are joints and the edges are links then this graph is a path (the first and last joint has degree 1 and all other joints have degree 2) and the joints allow rotation about its axes, so that if a joint rotates then all other subsequent links rotate about the axes of this joint. A reference coordinate system is chosen for the final joint which is called the *end-effector*[1]. See Fig. 2 for an illustration.

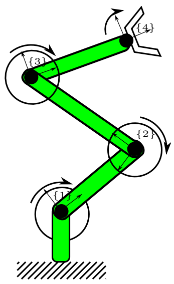

Figure 2: Green objects are links. Joints 1,2,3,4 and their corresponding frames are shown. The frame of joint 4 is also the end-effector frame

In theoretical kinematics one may forget that the links are rigid bodies so that *collision* between links are disregarded. In this case we may as well think of a robot as a differentiable map $F : \mathrm{SO}(2)^n \longrightarrow \mathrm{SE}(3)$ where SO(2) is the one-dimensional group of rotations around a fixed line, parameterised by the rotation angle, and SE(3) is the six-dimensional group of Euclidean congruence transformations. This map is defined in the following way:

- The i-th coordinate of an element in $\mathrm{SO}(2)^n$ is associated to the i-th (revolute) joint parameter.
- For joint values $\boldsymbol{\theta} := (\theta_1, \ldots, \theta_n)$ in $\mathrm{SO}(2)^n$, the image $F(\boldsymbol{\theta})$ is the transformation of the end-effector from the initial position corresponding to all angles being zero to the final position obtained by composing the n rotations.

The map F itself is called the *kinematic map* (of the robot). Its domain is called the *configuration space/set*, while its image is called the *work-space* or the *kinematic image*.

We use the Denavit-Hartenberg (DH) convention when describing relations between two joint frames. It is standard in robotics ; its advantages are discussed in e.g. [42, §3.2], [1, §4.2]. The transformation between the frames is given by the following rule:

- The z-axis of the reference frame will be the axis of rotation of the joint.
- To obtain the next frame, one starts with a rotation about the z-axis of the reference frame, called the *rotation*, followed by
- a translation along the z-axis of the reference frame, called the *offset*, followed by

[1] this is usually another frame, but this is just an additional fixed transformation in SE(3) and, without loss of generality, we may assume that the final distance and twist are 0

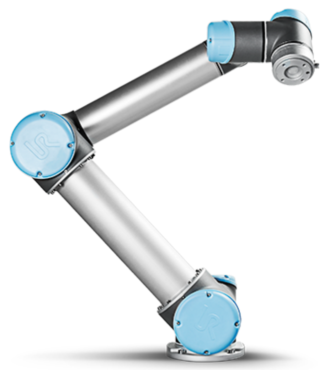

Figure 3: The UR5 manipulator by Universal Robots (seated in Denmark) is a lightweight, adaptable collaborative industrial robot that tackles medium-duty applications.

- a translation along the x-axis, called the *distance*, followed by
- a rotation about the x-axis, called the *twist*.

The transformation between frame i to frame $i+1$ is

$$R_z(\theta_i)T_z(d_i)T_x(a_i)R_x(\alpha_i)$$

where R_z, T_z, T_x, R_x are rotations or translations with respect to z- or x-axis parameterised by the angle of rotation θ_i (the i-th joint parameter), the offset d_i, the distance a_i and the angle of twist α_i of the i-th frame. For a given robot with n joints all DH parameters except for the rotation are fixed values. So that image of F for given joint values (the rotations) $(\theta_1, \ldots, \theta_n)$ is just the multiplication of these transformations in SE(3). The parameters d_1, d_n, a_n, α_n are assumed to be 0. This is not a loss of generality, because we can freely choose the frame at the base and at the end-effector and this will not affect the rank of the Jacobian matrix of the kinematic map (we will later see that this describes the kinematic singularities of the robot). More detailed discussion on the DH-parameters can be seen in [42].

Example 2.5. *The UR5 robot – see Figure 3 – has the following DH parameters:*

distances (m.)

$$(a_1, \ldots, a_6) := (0, -\frac{17}{40}, -\frac{1569}{4000}, 0, 0, 0)$$

offsets (m.)

$$(d_1, \ldots, d_6) := (0, 0, 0, \frac{2183}{20000}, \frac{1893}{20000}, 0)$$

twist angles (rad.)

$$(\alpha_1, \ldots, \alpha_6) := (\frac{\pi}{2}, 0, 0, \frac{\pi}{2}, -\frac{\pi}{2}, 0)$$

For example, the following joint angles (rotations, in rad.)

$$(\theta_1, \dots, \theta_6) := \left(\frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right)$$

leads to the following transformation in $(R, \mathbf{t}) \in \mathrm{SE}(3)$ *(represented as elements in* $\mathrm{SO}(3) \rtimes \mathbb{R}^3$*) where:*

$$R \approx \begin{pmatrix} 0.047 & -0.977 & -0.209 \\ -0.393 & 0.174 & -0.903 \\ 0.918 & 0.123 & -0.376 \end{pmatrix}, \ \mathbf{t} \approx (-6.768, -1.7784, -3.336).$$

Definition 2.6. *Given the kinematic map of a manipulator* $F : \mathrm{SO}(2)^n \to \mathrm{SE}(3)$*, the* kinematic singularities *in the configuration space are the points* $P \in \mathrm{SO}(2)^n$ *such that the Jacobian of* F *at* P *is rank-deficient. Elements in* $\mathrm{SO}(2)^n$ *are also known as* configurations.

In this paper, we will only deal with 6-jointed manipulators. Therefore the kinematic map is a differentiable map from the 6-dimensional configuration space $(\mathbb{P}^1)^6$ (we identify $\mathbb{P}^1$ with $\mathrm{SO}(2)$) to the group $\mathrm{SE}(3)$, which is also 6-dimensional.

In practice, the robots have kinematic map F such that there is a point $P \in (\mathbb{P}^1)^6$ for which the Jacobian of F at P is full-rank (in our case, rank 6). Thus, at these points (i.e. points that are not kinematic singularities), the Jacobian is non-singular and so F is a local homeomorphism. When the context is clear, we often write *singularity* instead of *kinematic singularity.* Furthermore, in applications, we are interested in the real configurations so that we may restrict the domain of F to $\mathbb{P}^1(\mathbb{R})$. Here is a well-known geometric description of kinematic singularities.

Theorem 2.7. *Let* $F : (\mathbb{P}^1(\mathbb{R}))^6 \to \mathrm{SE}(3)$ *be the kinematic map of a robot with 6 joints and suppose that the Jacobian of* F *achieves full-rank at a configuration in* $(\mathbb{P}^1(\mathbb{R}))^6$*. Let* $P \in (\mathbb{P}^1(\mathbb{R}))^6$*, then the following are equivalent.*

1. *P is a kinematic singularity.*
2. *The Jacobian of F at P is singular.*
3. *If $P_1, \dots, P_6 \in \mathbb{P}^5(\mathbb{R})$ are the Plücker representation of the axes (lines in $\mathbb{P}^3$) of the joints of the robot at the configuration point P then the matrix consisting of the Plücker coordinates $(p_{i,j})_{i,j \le 6}$ ($P_i = (p_{i,1} : p_{i,2} : \cdots : p_{i,6})$ for $i = 1, \dots, 6$) is singular.*

Proof. The equivalence of the first two items is clear by definition. The equivalence of the first and the third item is found in [41, §4.5.1], [35, §4.1] or [1, §4.5.1.] □

Assume that we have two non-singular points in the configuration set. As explained earlier, we want to decide if these two configurations can be connected by a curve in configuration space which avoids the singular hypersurface (see [45] §1.2 for some history on this question). If yes, then an explicit construction of such a curve is also of interest. In order to tackle these problems, we choose parameters for $\mathrm{SO}(2)$ so that the equation of the hypersurface becomes a polynomial. This is not the case when we use the angles $\theta_1, \dots, \theta_n$, because the Jacobian contains trigonometric functions in these angles. One well-known strategy is to parametrize by points on a unit circle, i.e. by two parameters satisfying the equation of the unit circle. This has a clear disadvantage: the number of variables increases, and the singular set has co-dimension greater than one. Another well-known strategy is to replace θ_i by $v_i = \tan\frac{\theta_i}{2}$. The variable v_i ranges over the projective line, and the angle π corresponds to the point at infinity. If we set $v_i = \tan\frac{\theta_i}{2}$ for $i = 1, \ldots, n$, then we obtain, in general, a polynomial in $v_2, \ldots, v_5$. More precisely, the degree is 2 in v_2 and v_5 and degree 4 in v_3 and in v_4. The Jacobian does not depend on the joint angles θ_1 and θ_6. This is clear from the third characterization of singularities in Theorem 2.7: only the position of the axes are relevant, and a rotation along the first or the last axis does not change the position of any axis.

3. Roadmap algorithms in the context of kinematic singularity analysis

A key ingredient of our algorithm is the design of a subroutine which on input a polynomial system of equations and inequalities defining a semi-algebraic set S computes a *roadmap* $\mathscr{R}$ of S, i.e. a semi-algebraic curve which has a non-empty and connected intersection with all connected components of S.

We explain the roadmap algorithm for the special case where the semi-algebraic set S is defined by (F, G, σ) with $F = (f_1, \ldots, f_p)$, $G = (g_1, \ldots, g_q)$ in $\mathbb{Q}[x_1, \ldots, x_N]$ and $\sigma = (\neq, \ldots, \neq)$.

In the whole section, we assume that the ideal generated by F is radical and its associated algebraic set $V(F)$ is smooth equidimensional of co-dimension p.

For manipulators, the variables for the joint configurations are actually half-angle tangents parameterizing $\mathbb{P}^1(\mathbb{R})$ (see Subsection 2.2). These variables are used to define G in the algorithm. However, we may lose information at infinity (i.e. at configurations involving half-revolutions) when dealing with this parameterization. In order to account for the whole projective line we can reparametrize using the sines and cosines as variables e.g. for the i-th joint we replace the half-angle tangent v_i with s_i/c_i and multiply the determinant of the Jacobian by appropriate powers of c_i to obtain polynomials in s_i and c_i and finally we include the polynomials $c_i^2 + s_i^2 - 1$ into the system of equations F in our algorithm. Thus, the roadmap algorithm is sufficient for our application in robotics.

The goal of this section is to show how one can compute roadmaps for such semi-algebraic sets using $\max(2^q, Nq^N)(ND)^{O(N^2)}$ arithmetic operations in $\mathbb{Q}$. This complexity bound is not new but the algorithm designed below is. Its main originality is that it avoids the computationally expensive use of infinitesimals as in [8] while its complexity lies in the best known class for computing roadmaps of semi-algebraic sets and allows us to perform practical computations on the aforementioned robotics applications.

3.1. Regularity properties

We start with an easy consequence of Sard's theorem.

Lemma 3.1. *For $r \in \mathbb{C}$, we denote by β_r the polynomial $x_1^2 + \cdots + x_N^2 - r$.*

There exists a non-empty Zariski open set $O \subset \mathbb{C}$ such that for $r \in O$ the ideal generated by F, β_r is radical and its associated algebraic set is either empty or smooth equidimensional of codimension $p+1$.

Proof. We consider the restriction of the map

$$\nu : \boldsymbol{x} = (x_1, \ldots, x_N) \to x_1^2 + \cdots + x_N^2$$

to $V(F)$. Since F generates a radical ideal of codimension p whose associated algebraic set is smooth and equidimensional, the critical locus of ν is defined by the vanishing of F and the maximal minors of the Jacobian matrix associated to F, ν. By Sard's Theorem (see e.g. [40, App. B.1] for an algebraic version), the set of critical values of ν is finite in $\mathbb{C}$. Let O be the complementary of this set.

Then, for $r \in O$, the Jacobian matrix associated to F, β_r has full rank at all points of $V(F, \beta_r)$. Our conclusion then follows from a direct application of the Jacobian criterion [19, Theorem 16.19]. $\square$

We state now a slight generalization of [23, Lemma 1] which will allow us to perform some deformation in order to ensure some regularity properties.

Proposition 3.2. *There exists a non-empty Zariski open set $\Omega \subset \mathbb{C}^q \times \mathbb{C}$ such that for any $\boldsymbol{a}, e \in \Omega$, the following holds. For any $\boldsymbol{i} = \{i_1, \ldots, i_s\} \subset \{1, \ldots, q\}$, the ideal generated by*

$$f_1, \ldots, f_p, g_{i_1} - a_{i_1} e, \ldots, g_{i_s} - a_{i_s} e$$

is radical and its associated algebraic set is either empty or smooth equidimensional of codimension $p + s$.

Proof. We adapt the proof of [23, Lemma 1].

Take $\boldsymbol{i} = \{i_1, \ldots, i_s\} \subset \{1, \ldots, q\}$. We prove that there exists a non-empty Zariski open set $\Omega_{\boldsymbol{i}} \subset \mathbb{C}^q$ such that for $\boldsymbol{a} = (a_1, \ldots, a_q) \in \Omega_{\boldsymbol{i}}$ the ideal generated by F and $\frac{g_{i_1}}{a_{i_1}}, \ldots, \frac{g_{i_s}}{a_{i_s}}$ is radical and its associated algebraic set is either empty or smooth equidimensional of co-dimension $p + s$.

We consider the map $\Psi_{\boldsymbol{i}}$

$$\Psi_{\boldsymbol{i}} : (\boldsymbol{x}, \boldsymbol{a}, e) \in V(F) \times \mathbb{C}^q \times (\mathbb{C} - \{0\}) \to (g_{i_1}(\boldsymbol{x}) - a_{i_1} e, \ldots, g_{i_s}(\boldsymbol{x}) - a_{i_s} e).$$

Since F generates a radical ideal whose algebraic set is smooth of codimension p when it is not empty, the Jacobian criterion [19, Theorem 16.19] implies that the Jacobian of F has maximal rank at any point of $V(F)$.

This implies that the Jacobian of $\Psi_{\boldsymbol{i}}$ has maximal rank at any point of all points of $\Psi_{\boldsymbol{i}}^{-1}(\mathbf{0})$. Then, applying an algebraic version of Thom's weak transversality theorem (see e.g. [40, App. B.1]), we deduce that there exists a non-empty Zariski open set $\Omega_{\boldsymbol{i}} \subset \mathbb{C}^q \times \mathbb{C}$ such that for any $\boldsymbol{a} = (a_1, \ldots, a_q), e$ in $\Omega_{\boldsymbol{i}}$, $\mathbf{0}$ is a regular value for the restricted map

$$\Psi_{\boldsymbol{a},e} : \boldsymbol{x} \in V(F) \to (g_{i_1}(\boldsymbol{x}) - a_{i_1} e, \ldots, g_{i_s}(\boldsymbol{x}) - a_{i_s} e).$$

In other words, the Jacobian matrix associated to F and $(g_{i_1}(\boldsymbol{x}) - a_{i_1} e, \ldots, g_{i_s}(\boldsymbol{x}) - a_{i_s} e)$ has maximal rank at any point of the set of common complex solutions to these polynomials.

Hence, applying again the Jacobian criterion we deduce that the ideal generated by

$$(f_1, \ldots, f_p, g_{i_1}(\boldsymbol{x}) - a_{i_1} e, \ldots, g_{i_s}(\boldsymbol{x}) - a_{i_s} e)$$

is radical and its associated algebraic set is either empty or is smooth equidimensional of codimension $p + s$.

We defined Ω as the intersection of the finitely many $\Omega_{\boldsymbol{i}}$; hence it is also a non-empty Zariski open set and our statement follows. $\square$

Let ε be an infinitesimal and $\mathbb{R}\langle\varepsilon\rangle$ be the field of Puiseux series in ε with coefficients in $\mathbb{R}$. By [8, Chap. 2], $\mathbb{R}\langle\varepsilon\rangle$ is a real closed field and one can define semi-algebraic sets over $\mathbb{R}\langle\varepsilon\rangle^{n+t}$. In particular, the solution set in $\mathbb{R}\langle\varepsilon\rangle^{n+t}$ to the system defining S is a semi-algebraic set which we denote by $\mathrm{ext}(S, \mathbb{R}\langle\varepsilon\rangle)$. We refer to [8] for properties of real Puiseux series fields and semi-algebraic sets defined over such field. We make use of the notions of bounded points of $\mathbb{R}\langle\varepsilon\rangle^n$ over $\mathbb{R}$ (those whose all coordinates have non-negative valuation) and their limits in $\mathbb{R}$ (when $\varepsilon \to 0$). We denote by $\lim_0$ the operator taking the limits of such points. These notions and notations extend to Puiseux series with coefficients in $\mathbb{C}$. Note that if A is a non-empty Zariski open set of $\mathbb{C}^N$ then $\mathrm{ext}(A, \mathbb{C}\langle\varepsilon\rangle)$ is a non-empty Zariski open set of $\mathbb{C}\langle\varepsilon\rangle^N$.

Observe that, from the proof of Proposition 3.2, the complementary of Ω is defined with polynomials with coefficients in $\mathbb{Q}$. Hence, given $\boldsymbol{a}, e$ in Ω, one deduces that $\boldsymbol{a}, \varepsilon$ lies in $\mathrm{ext}(\Omega, \mathbb{C}\langle\varepsilon\rangle)$ (since ε is transcendental it cannot be a root of a non-zero polynomial with coefficients in $\mathbb{C}$).

The following corollary is immediate since ε is transcendental over $\mathbb{R}$.

Corollary 3.3. *Let Ω be as in Proposition 3.2. For any $\boldsymbol{a}, \varepsilon \in \mathrm{ext}(\Omega, \mathbb{C}\langle\varepsilon\rangle)$, the following holds. For any $\boldsymbol{i} = \{i_1, \dots, i_s\} \subset \{1, \dots, q\}$, the ideal generated by*

$$f_1, \dots, f_p, g_{i_1} - a_{i_1}\varepsilon, \dots, g_{i_s} - a_{i_s}\varepsilon$$

is radical and its associated algebraic set is either empty or smooth equidimensional of codimension $p + s$.

3.2. Reduction to the closed and bounded case

We explain now how to perform our reduction to bounded and closed semi-algebraic sets.

Firstly, we reduce our problem to one where the semi-algebraic set we consider is bounded. For $r > 0$, we denote by $B_r \subset \mathbb{R}^N$ the ball defined by

$$x_1^2 + \cdots + x_N^2 \leq r.$$

A first ingredient to the reduction to the bounded case is the following lemma.

Lemma 3.4. *Let $S \subset \mathbb{R}^N$ be a semi-algebraic set. There exists a real number $r > 0$ large enough such that the connected components of S are in one-to-one correspondence with those of $S \cap B_r$ and a roadmap of $S \cap B_r$ is a roadmap of S.*

Proof. By [11, Corollary 9.3.7], there exists r such that $S \cap B_r$ is a semi-algebraic deformation retract of S. In other words, there exists a continuous semi-algebraic map $h : [0, 1] \times S \to S$ such that for any $x \in S$, $h(0, x) = x$ and $h(1, x) \in S \cap B_r$ and for any $\vartheta \in (0, 1)$ and $x \in S \cap B_r$, $h(\vartheta, x) = x$.

Now consider x and y in S. Note that if there does not exist a semi-algebraic path in S which connects them, then there is no semi-algebraic path connecting them in $S \cap B_r$

(since $S \cap B_r \subset S$). Now, assume that there exists a semi-algebraic path $\gamma \subset S$ which connects them. Since h is semi-algebraic continuous, $h(1, \gamma)$ is a semi-algebraic path in $S \cap B_r$ connecting $h(1, x)$ to $h(1, y)$ which both lie in $S \cap B_r$. Hence, we deduce that the connected components of S are in one-to-one correspondence with those of $S \cap B_r$.

Since $S \cap B_r \subset S$ trivially holds, we deduce that any roadmap of $S \cap B_r$ has a non-empty and connected intersection with the connected components of S and then is also a roadmap of S. This ends the proof. $\square$

Further, we say that a semi-algebraic set $S' \subset S \subset \mathbb{R}^N$ satisfies $\mathsf{C}(S)$ if the connected components of S' are in one-to-one correspondence with those of S and if a roadmap of S' is a roadmap of S.

A second key ingredient is to show how to compute such a large enough real number r. For a smooth equidimensional algebraic set $V \subset \mathbb{K}^N$ where $\mathbb{K}$ is an algebraically closed field, and a map $\varphi : V \to \mathbb{K}$, we denote by $\mathrm{crit}(\varphi, V)$ the set of critical points of the restriction of φ to V. We refer to [40, App. A] for properties of polynomial systems defining $\mathrm{crit}(\varphi, V)$ given a system of generators of the ideal associated to V.

For $\boldsymbol{i} = \{i_1, \dots, i_s\} \subset \{1, \dots, q\}$, we denote by $G_{\boldsymbol{i}}$ the subset $(g_{i_1}, \dots, g_{i_s})$ of G. Also, given $\boldsymbol{a} = (a_1, \dots, a_q) \subset \mathbb{C}^q$ we denote by $G_{\boldsymbol{i}}^{\boldsymbol{a}}$ the set $\left(\frac{g_{i_1}}{a_{i_1}}, \dots, \frac{g_{i_s}}{a_{i_s}}\right)$. When $\boldsymbol{i} = \{1, \dots, q\}$, we simply denote $G_{\boldsymbol{i}}^{\boldsymbol{a}}$ by $G^{\boldsymbol{a}}$. Finally, for $\tau = (\tau_1, \dots, \tau_q) \in \{-1, 1\}^q$, we denote by $G_{\boldsymbol{i},\tau}^{\boldsymbol{a},\varepsilon}$ the sequence of polynomials

$$g_{i_1} + (-1)^{\tau_{i_1}} a_{i_1} \varepsilon, \quad \dots, \quad g_{i_s} + (-1)^{\tau_{i_s}} a_{i_s} \varepsilon.$$

We denote by ν the following map:

$$\nu : (x_1, \dots, x_N) \to x_1^2 + \cdots + x_N^2.$$

Proposition 3.5. *Let $\Omega \subset \mathbb{C}^q \times \mathbb{C}$ be the non-empty Zariski open set defined in Proposition 3.2 and $\boldsymbol{a}, \varepsilon$ in $\mathrm{ext}(\Omega, \mathbb{C}\langle\varepsilon\rangle) \cap \mathbb{R}^q \times \mathbb{R}\langle\varepsilon\rangle$.*

Let $\mathscr{K}_\varepsilon$ be the union of the sets $crit(\nu, V(F) \cap V(G_{\boldsymbol{i},\tau}^{\boldsymbol{a},\varepsilon}))$ where $\boldsymbol{i}$ and τ range on all their (finitely many) possible values.

Let $\mathscr{K}_0$ be the limits of the bounded points in $\mathscr{K}_\varepsilon$ and $r^\star$ be the maximum of the values of the restriction of ν to $\mathscr{K}_0$.

Then, for any $r > r^\star$, $S \cap B_r$ satisfies $\mathsf{C}(S)$.

Proof. For $\iota = (\iota_1, \dots, \iota_q) \in \{>, <\}^q$, i we denote by S_ι the semi-algebraic set defined by

$$f_1 = \cdots = f_p = 0, \qquad \frac{g_1}{a_1} \;\iota_1\; 0, \quad \dots, \quad \frac{g_q}{a_q} \;\iota_q\; 0.$$

It follows immediately that S is the disjoint union of the semi-algebraic sets S_ι when ι ranges over all its (finitely many) possible values. Hence, it suffices to prove that for $r > r^\star$ in the statement, and for any ι, $S_\iota \cap B_r$ satisfies $\mathsf{C}(S_\iota)$.

We prove this for $\iota = \{>, \dots, >\}$; extending this to arbitrary ι is immediate. Abusing slightly with notation, we will also denote by S, S_ι their extensions to $\mathbb{R}\langle\varepsilon\rangle^N$. We also denote by S_ε the semi-algebraic set defined by

$$f_1 = \cdots = f_p = 0, \qquad \frac{g_1}{a_1} \geq \varepsilon, \quad \dots, \quad \frac{g_q}{a_q} \geq \varepsilon.$$

Note that to prove that $S_\iota \cap B_r$ satisfies $\mathsf{C}(S_\iota)$, it suffices to prove the following. Take $\boldsymbol{x}$ and $\boldsymbol{y}$ in S_ι and $\gamma : [0,1] \to S_\iota$ a semi-algebraic continuous function such that $\gamma(0) = \boldsymbol{x}$ and $\gamma(1) = \boldsymbol{y}$. Then, there exists $\boldsymbol{x}'$ and $\boldsymbol{y}'$ in $S_\iota \cap B_r$ such that:

- there exists a semi-algebraic continuous path $\alpha : [0,1] \to S_\iota$ such that $\alpha(0) = \boldsymbol{x}$ and $\alpha(1) = \boldsymbol{x}'$.
- there exists a semi-algebraic continuous path $\beta : [0,1] \to S_\iota$ such that $\beta(0) = \boldsymbol{y}$ and $\beta(1) = \boldsymbol{y}'$.
- there exists a continuous semi-algebraic path $\gamma' : [0,1] \to S_\iota \cap B_r$ such that $\gamma'(0) = \boldsymbol{x}'$ and $\gamma'(1) = \boldsymbol{y}'$.

Of course, when both $\boldsymbol{x}$ and $\boldsymbol{y}$ lie in B_r and $\gamma([0,1]) \subset B_r$, the conclusion immediately holds.

Assume now that $\boldsymbol{x}$ and $\boldsymbol{y}$ lie in B_r but there exists $a \in]0,1[$ such that $\gamma(a) \notin B_r$. Since B_r is closed and γ is semi-algebraic and continuous, there exist finitely many open sub-intervals $I_1, \ldots, I_\ell$ of $[0,1]$ such that their image by γ lie outside B_r. We will prove that, for $1 \leq i \leq \ell$ there exists a continuous semi-algebraic function ϑ_i such that $\vartheta_i \circ \gamma(I_i)$ lie in a connected components of $S_\iota \cap B_r$, which is then enough to conclude the lemma.

Now, we take one such interval I_i, which we denote by I to keep simple notations. Observe that $\gamma(I)$ is contained in a ball $B_R \subset \mathbb{R}^N$ and in S_ε.

Choose r' such that $R > r' > r$. Note that, since $\boldsymbol{a}, \varepsilon$ lies in $\mathrm{ext}(\Omega, \mathbb{C}\langle\varepsilon\rangle)$, the maps

$$\boldsymbol{x} \to \left(f_1(\boldsymbol{x}), \ldots, f_p(\boldsymbol{x}), \frac{g_{i_1}}{a_{i_1}} - \varepsilon, \ldots, \frac{g_{i_s}}{a_{i_s}} - \varepsilon \right)$$

are transverse at the origin (Corollary 3.3). Hence, by [13, 43], one obtains a Whitney stratification of S_ε by fixing the signs of its defining polynomials. Note also that since $r \in \mathbb{R}$ and $r > r^\star$, the interval $[r', R]$ does not contain a critical value of the restriction of ν to S_ε Hence, one can apply Theorem 2.4 to $X = S_\varepsilon \cap B_R$, $Y =]r', R[$ and ν'. Hence, there exists a trivialization

$$X \cap \nu^{-1}(]r', R[) \to (X \cap \nu^{-1}(r)) \times]r', R[$$

Hence, one can use this trivialization to deform continuously $\gamma(I)$ to a semi-algebraic curve lying in $S_\varepsilon \cap B_r$. Since this curve is bounded over $\mathbb{R}$, its limit exists when ε tends to 0 and is connected in $S_\iota \cap B_r$. Performing such a deformation to all curves $\gamma(I_i)$ yields a semi-algebraic curve connecting $\boldsymbol{x}$ to $\boldsymbol{y}$ in $S_\iota \cap B_r$, which allows us to conclude.

It remains to study the cases where at least one of the points $\boldsymbol{x}$ or $\boldsymbol{y}$ lies outside B_r (and then part of $\gamma(0,1)$ also lies outside B_r). We deal with this case in a very similar way to the previous one. As above, one can note that there exists a ball B_R centered at the origin of radius R which contains $\gamma([0,1])$. Again, one can deduce that there are finitely many sub-intervals of $[0,1]$ such that there images by γ lies outside B_r. Then, using Theorem 2.4 as above, one obtains trivializations which allow to deform continuously these images to semi-algebraic curves lying in $S_\varepsilon \cap B_r$. Taking again the limit, we obtain points $\boldsymbol{x}'$ and $\boldsymbol{y}'$ as well as semi-algebraic curves lying in $S \cap B_r$ which connect $\boldsymbol{x}$ to $\boldsymbol{x}'$, $\boldsymbol{y}$ to $\boldsymbol{y}'$ and $\boldsymbol{x}'$ to $\boldsymbol{y}'$. This ends the proof. $\square$

The last ingredient, which is given by the lemma below, allows us a final reduction to closed and bounded semi-algebraic sets. Further, for $r \in \mathbb{C}$, we denote by β_r the polynomial $x_1^2 + \cdots + x_N^2 - r$. Note that by Lemma 3.1, for a generic choice of r, the ideal generated by F, β_r is radical and its associated algebraic set is either empty or is smooth equidimensional of codimension $p+1$.

Below, we take r in the non-empty Zariski open set O defined in Lemma 3.1 and as in Proposition 3.5. We choose $\boldsymbol{a}, e$ in $\Omega \cap \mathbb{R}^q \times \mathbb{R}$ (with Ω being defined in Proposition 3.2) with $e > 0$ smaller than the minimum of the absolute values of the critical values of the maps $\boldsymbol{x} \to G_{\boldsymbol{i}}^{\boldsymbol{a}}$ and their restrictions to the union of the algebraic sets $V(F)$, $V(\beta_r)$, $V(F, \beta_r)$.

For $\tau = (\tau_1, \ldots, \tau_q) \in \{-1, 1\}^q$, we denote by $S_{\tau,e}^{\boldsymbol{a},r}$ the semi-algebraic set defined by

$$f_1 = \cdots = f_p = 0, \qquad \beta_r \leq 0, \qquad g_1 \ + (-1)^{\tau_1}\ a_1 e \geq 0, \quad \ldots, \quad g_q \ + (-1)^{\tau_q}\ a_q e \geq 0.$$

Below, we define $\iota = (\iota_1, \ldots, \iota_q) \in \{>, <\}^q$ such that $\iota_i => $ if $\tau_i = 1$ else $\iota_i =<$.

Finally, we denote by S_ι^r the semi-algebraic set defined by

$$f_1 = \cdots = f_p = 0, \qquad \beta_r \leq 0, \qquad \frac{g_1}{a_1}\ \iota_1\ 0, \ldots, \frac{g_q}{a_q}\ \iota_q\ 0.$$

The following is immediate from our definitions, Lemma 3.1 and Proposition 3.5.

Proposition 3.6. *Let r be as in Proposition 3.5 and e be as above. For any $\boldsymbol{i} = \{i_1, \ldots, i_s\} \subset \{1, \ldots, q\}$, the ideals generated by $F, G_{\boldsymbol{i},\tau}^{\boldsymbol{a},e}$ and $F, \beta_r, G_{\boldsymbol{i},\tau}^{\boldsymbol{a},e}$ are radical and their associated algebraic sets are either empty or equidimensional of respective codimension $p+s$ and $p+s+1$.*

Moreover, $S_{\tau,e}^{\boldsymbol{a},r} \cap B_r$ satisfies $\mathsf{C}(S_\iota^r)$.

3.3. Algorithm NumberOfConnectedComponents

Finally, let D be the maximum of the degrees of the polynomials in F and G. Combining Lemma 3.4, Propositions 3.5 and 3.6 one reduces our initial problem to the one of computing a roadmap of a closed and bounded semi-algebraic set by:

- computing a rational number r ; this is done by computing $\lim_{\varepsilon \to 0} \mathscr{K}_\varepsilon$ using the algorithms given in [5, 33] with real root isolation ; note that this is done using $\max(2^q, Nq^N)(ND)^{O(N)}$ arithmetic operations in $\mathbb{Q}$;
- we compute e as in Proposition 3.6 by computing the critical values of the maps $\boldsymbol{x} \in V(F) \to G_{\boldsymbol{i}}^{\boldsymbol{a}}(\boldsymbol{x})$ and $\boldsymbol{x} \in V(F, \beta_r) \to G_{\boldsymbol{i}}^{\boldsymbol{a}}(\boldsymbol{x})$ using e.g. the geometric resolution algorithm of [26] as in [27] ; this is done within $\max(2^q, Nq^N)D^{O(N)}$ arithmetic operations in $\mathbb{Q}$.

Moreover, by Proposition 3.6, the transversality of the intersections of the zero sets of the defining polynomials of the closed and bounded semi-algebraic set $S_{\tau,e}^{\boldsymbol{a},r}$ implies that a Whitney stratification of this set is obtained by fixing the signs of its defining polynomials. (see [13, 43]).

Note also that the reduction to computing a roadmap to such semi-algebraic sets uses $\max(2^q, Nq^N)(ND)^{O(N)}$ arithmetic operations in $\mathbb{Q}$. To compute a roadmap for such semi-algebraic sets we are led to use a slight modification of the roadmap algorithm given in [13].

The algorithm in [13] then takes as input a polynomial system defining a closed and bounded semi-algebraic set S for which one obtains a Whitney stratification by fixing the signs of its defining polynomials. It proceeds as follows. The core idea is to start by computing a curve $\mathscr{C}$ which has a non-empty intersection with each connected component of S. That curve will be typically the critical locus on the (x_1, x_2)-plane when one is in generic coordinates (else, one just needs to change linearly generically the coordinate system).

A few remarks are in order here. To define the critical locus of the projection on the (x_1, x_2)-plane restricted to S one takes the union of the critical loci of that projection restricted to the real algebraic sets defined by the vanishing of a subset of the inequalities as we do in the previous paragraph. (see [13]). Following [13], this is done with $\max(2^q, Nq^N)(ND)^{O(N)}$ arithmetic operations in $\mathbb{Q}$.

That way, one obtains curves that intersect all connected components of S but these intersections may not be connected. To repair these connectivity failures, Canny's algorithm finds appropriate slices of S. Let π_1 be the canonical projection $(x_1, \ldots, x_N) \to x_1$. This basically consists in finding $\alpha_1 < \ldots < \alpha_k$ in $\mathbb{R}$ such that the union of $\cup_{i=1}^{k} S \cap \pi_1^{-1}(\alpha_i)$ with the critical curve $\mathscr{C}$ has a non-empty and connected intersection with each connected component of S.

The way Canny proposes to find those α_i's is to compute the critical values of the restriction of π_1 to $\mathscr{C}$. By the algebraic Sard's theorem (see e.g. [40, Appendix B]), these values are in finite number and Canny proposes to take $\alpha_1, \cdots, \alpha_k$ as those critical values. This leads to compute with real algebraic numbers which can be encoded with their minimal polynomials and isolating intervals. Since these minimal polynomials may have large degrees (singly exponential in N), that step can be prohibitive for practical computations. We use then the technique introduced in [34] which consists in replacing $\alpha_1 < \cdots < \alpha_k$ with rational numbers $\rho_1 < \cdots < \rho_{k-1}$ with $\alpha_i < \rho_i < \alpha_{i+1}$. We refer to [34] for the rationale justifying this trick. All in all, one obtains a recursive algorithm with a decreasing number of variables at each recursive call. Again, combining [13] with [34] and [40], all this is done using $\max(2^q, Nq^N)(ND)^{O(N)}$ arithmetic operations in $\mathbb{Q}$ and there are $\max(2^q, Nq^N)(ND)^{O(N)}$ fibers to consider.

Since, when slicing S with the fibers $\pi^{-1}(\alpha_i)$ for $1 \leq i \leq k$, the dimension decreases by at least one, the depth of the recursion is $\simeq N$. Hence, all in all, computing a roadmap for such a semi-algebraic set can be done with $\max(2^q, Nq^N)\,(ND)^{O(N^2)}$ arithmetic operations in $\mathbb{Q}$ (see also [8] for more general algorithms).

Finally, note that the computations we described above can be performed with any algebraic elimination framework such as Gröbner bases. Hence, combined with efficient Gröbner bases engines, these can be performed efficiently enough to tackle practical problems. Note also that answering connectivity queries over roadmaps is done in time which is polynomial in the degree of their Zariski closures which does not increase the complexity results stated above (see [39]). We illustrate in Section 6 how this algorithm (with the modifications introduced above) can be used in practice to answer connectivity queries in semi-algebraic sets in the concrete robotics applications which we have introduced in the previous section.

4. Parametric polynomial systems

We describe now our algorithm for solving parametric systems and prove Theorem 1.3.

4.1. Overview

Let $F = (f_1, \dots, f_p)$ and $G = (g_1, \dots, g_q)$ in $\mathbb{Q}[\boldsymbol{x}, \boldsymbol{y}]$ with $\boldsymbol{x} = (x_1, \dots, x_n)$ and $\boldsymbol{y} = (y_1, \dots, y_t)$. We consider further $\boldsymbol{y}$ as a sequence of parameters and the polynomial system

$$f_1 = \cdots = f_p = 0, \qquad g_1 \ \sigma_1 \ 0, \ \dots, \ g_q \ \sigma_q \ 0$$

with $\sigma_i \in \{>, \geq\}$. We let $S \subset \mathbb{R}^n \times \mathbb{R}^t$ be the semi-algebraic set defined by this system. For $y \in \mathbb{R}^t$, we denote by F_y and G_y the sequences of polynomials obtained after instantiating $\boldsymbol{y}$ to y in F and G respectively. Also, we denote by $S_y \subset \mathbb{R}^n$ the semi-algebraic set defined by the above system when $\boldsymbol{y}$ is specialized to y. The algebraic set defined by the simultaneous vanishing of the entries of F (resp. F_y) is denoted by $V(F) \subset \mathbb{C}^{n+t}$ (resp. $V(F_y) \subset \mathbb{C}^n$).

Our algorithm for solving such a parametric polynomial system does *not* assume that for a *generic* point y in $\mathbb{C}^t$, $V(F_y)$ is finite. In our context, solving such a parametric polynomial system may consist in partitioning the parameter space $\mathbb{R}^t$ into semi-algebraic sets $Y_1, \dots, Y_r$ such that, for $1 \leq i \leq r$, the number of connected components of S_y is invariant for any choice of y in Y_i. Proposition 2.3 shows that this makes sense.

Instead of computing a partition of the parameter space into semi-algebraic sets $Y_1, \dots, Y_r$ as above, we tackle Problems A and B. Hence, one considers non-empty *disjoint* open semi-algebraic sets $U_1, \dots, U_\ell$ in $\mathbb{R}^t$ such that the complement of $U_1 \dot\cup \cdots \dot\cup U_\ell$ in $\mathbb{R}^t$ is a semi-algebraic set of dimension less than t and such that for $1 \leq i \leq t$, there exists $b_i \in \mathbb{N}$ such that b_i is the number of connected components of S_y for any $y \in U_i$. For instance, one can take $U_1, \dots, U_\ell$ as the non-empty interiors (for the Euclidean topology) of $Y_1, \dots, Y_r$.

Let $\mathscr{D} \subset \mathbb{C}^t$ be an algebraic set containing the boundary of $U_1 \dot\cup \cdots \dot\cup U_\ell$. Our goal is then to output a sequence of pairs

$$(b_1, \eta_1), \dots, (b_r, \eta_r)$$

such that for $1 \leq i \leq \ell$, the set $\{\eta_1, \dots, \eta_r\}$ has a non-empty intersection with U_i as well as a *program* which on input $\eta \in \mathbb{R}^t - \mathscr{D}$ can output a semi-algebraic curve γ connecting η to some η_j and such that for any $\vartheta \in \gamma$, the number of connected components of $S \cap \pi^{-1}(\vartheta)$ equals b_j.

To solve this problem, one first computes a polynomial Δ in $\mathbb{Q}[\boldsymbol{y}] - \{0\}$ defining a Zariski closed set $\mathscr{D} \subset \mathbb{C}^t$ such that $\mathscr{D}$ contains $\mathbb{R}^t - \Big(U_1 \dot\cup \cdots \dot\cup U_\ell\Big)$.

Lemma 4.1. *Let $\mathscr{E} \subset \mathbb{R}^t$ be a finite set of points which has a non-empty intersection with any of the connected components of the semi-algebraic set defined by $\Delta \neq 0$. For $1 \leq i \leq \ell$, $\mathscr{E} \cap U_i$ is not empty.*

Proof. Recall that the U_i's are open semi-algebraic sets in $\mathbb{R}^t$ and that $\mathscr{D}$ has co-dimension 1 (since $\Delta \neq 0$) and contains $\mathbb{R}^t - (U_1 \dot\cup \cdots \dot\cup U_\ell)$. Note that to establish the above statement it suffices to prove that for any U_i, there exists a (non-empty) connected component C of $\mathbb{R}^t - \mathscr{D}$ such that $C \subset U_i$.

Let C be a connected component of $\mathbb{R}^t - \mathscr{D}$. Since $C \subset \mathbb{R}^t - \mathscr{D}$ and $\mathscr{D}$ contains $\mathbb{R}^t - (U_1 \dot\cup \cdots \dot\cup U_\ell)$, there exists $1 \leq i \leq \ell$ such that $C \cap U_i \neq \emptyset$; further we let $\eta \in C \cap U_i$. Assume by contradiction that there exists $\eta' \in C - U_i$. Then, since C is connected there exists a semi-algebraic path $\gamma \subset C$ which connects η to η' and then there exists $\vartheta \in \gamma$ in the boundary of U_i. Hence, we have $\vartheta \in C$ and $\Delta(\vartheta) = 0$. This is a contradiction since C is a connected component of the semi-algebraic set defined by $\Delta \neq 0$.

To finish the proof, it remains to establish that for any $1 \leq i \leq \ell$, there exists a connected component C of $\mathbb{R}^t - \mathscr{D}$ such that $C \cap U_i \neq \emptyset$. Hence we pick $1 \leq i \leq \ell$ and consider U_i. Since U_i is open, it has an empty intersection with $\mathscr{D}$ which implies that it has a non-empty intersection with some connected component of $\mathbb{R}^t - \mathscr{D}$. $\square$

Hence, computing sample points in each connected component of the set defined by $\Delta \neq 0$ (e.g. using the algorithm in [38] applied to the set defined by $z\Delta - 1 = 0$ where z is a new variable) is enough to obtain at least one point per connected component of $U_1, \ldots, U_\ell$. Finally, for each such a point y, it remains to count the number of connected components of the set S_y by using a roadmap algorithm.

We call *partial semi-algebraic resolution* of (F, G) the data $(b_1, \eta_1), \ldots, (b_k, \eta_k)$ where b_i is the number of connected components of S_{η_i} and $\{\eta_1, \ldots, \eta_k\}$ has a non-empty intersection with each connected component of $U_1 \dot\cup \cdots \dot\cup U_\ell$.

4.2. Algorithm description

Our algorithm relies on three subroutines.

- $\mathsf{Eliminate}$ takes as input F and G, as well as $\boldsymbol{x}$ and $\boldsymbol{y}$ and outputs $\Delta \in \mathbb{Q}[\boldsymbol{y}]$ as above ; we let $\mathscr{D} = V(\Delta)$.
- $\mathsf{SamplePoints}$ takes as input Δ and outputs a finite set of sample points $\{\eta_1, \ldots, \eta_k\}$ (with $\eta_i \in \mathbb{Q}^t$) which meets each connected component of $\mathbb{R}^t - \mathscr{D}$.
- $\mathsf{NumberOfConnectedComponents}$ takes as input F_η, G_η σ, for some $\eta \in \mathbb{Q}^t$, and computes the number of connected components of the semi-algebraic set S_η.

The algorithm is described hereafter.

Algorithm 1: $\mathsf{ParametricSolve}(F, G, \boldsymbol{x}, \boldsymbol{y})$

Data: Finite sequences F and G in $\mathbb{Q}[\boldsymbol{x}, \boldsymbol{y}]$ with $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_t)$, and $\sigma \in \{>, \geq\}^q$.
Assumes that assumptions (R) holds.
Result: a partial semi-algebraic resolution of (F, G)

$\Delta \leftarrow \mathsf{Eliminate}(F, G, \boldsymbol{x}, \boldsymbol{y})$
$\{\eta_1, \ldots, \eta_k\} \leftarrow \mathsf{SamplePoints}(\Delta \neq 0)$
for i *from* 1 *to* k **do**
    $b_i = \mathsf{NumberOfConnectedComponents}(F_{\eta_i}, G_{\eta_i}, \sigma)$
end
return $\{(b_1, \eta_1), \ldots, (b_k, \eta_k)\}$.

Note that Algorithm ParametricSolve solves Problem A. Note also that in order to solve Problem B, it suffices to combine its output with NumberOfConnectedComponents. Hence, in the sequel, we focus on detailing how ParametricSolve is designed and analyze its complexity in order to prove Theorem 1.3.

While the rationale of algorithm ParametricSolve is mostly straightforward, detailing each of its subroutines is less. The easiest ones are SamplePoints and NumberOfConnectedComponents. The first one relies on the critical point method [6, 8] combined with polar varieties [3, 4, 36, 38]. We refer to [32, Section 3] for a practical algorithm which meets the best known complexity bounds. For computing roadmaps, we refer to [7, 10, 39, 40] and the techniques described in Section 3.

The most difficult one to describe is subroutine Eliminate. We provide a detailed description of it under the following regularity assumption. Recall that we say that (F, G) satisfies assumption (R)

- the ideal $\langle f_1, \ldots, f_p \rangle$ is radical, equidimensional (of codimension p if not equal to $\langle 1 \rangle$) and the real trace of the singular locus of its associated algebraic set has dimension less than t;
- for any $\{i_1, \ldots, i_s\} \subset \{1, \ldots, q\}$, the ideal $\langle f_1, \ldots, f_p, g_{i_1}, \ldots, g_{i_s} \rangle$ is radical, equidimensional of codimension $p+s$ if not equalled to $\langle 1 \rangle$ and its associated algebraic set has a singular locus whose real trace has dimension less than t.

Note that using the Jacobian criterion [19, Chap. 16], it is easy to decide whether (R) holds. Note also that it holds generically.

For $\boldsymbol{i} = \{i_1, \ldots, i_s\} \subset \{1, \ldots, q\}$, under assumption (R), the algebraic set $V_{\boldsymbol{i}} \subset \mathbb{C}^{n+t}$ defined by

$$f_1 = \cdots = f_p = g_{i_1} = \cdots = g_{i_s} = 0.$$

are smooth and equidimensional and these systems generate radical ideals (applying the Jacobian criterion [19, Theorem 16.19]). Besides, the tangent space to $z \in V_{\boldsymbol{i}}$ coincides with the the (left) kernel of the Jacobian matrices associated to $(f_1, \ldots, f_p, g_{i_1}, \ldots, g_{i_s})$ at z.

Let I be the ideal generated by $(f_1, \ldots, f_p, g_{i_1}, \ldots, g_{i_s})$ and the maximal minors of the truncated Jacobian matrix associated to $(f_1, \ldots, f_p, g_{i_1}, \ldots, g_{i_s})$ obtained by removing the columns corresponding to the partial derivatives with respect to the $\boldsymbol{y}$-variables. *Under assumption* (R)*, one can compute the union of the set of critical values of the restriction of the projection* π *to the algebraic set* $V_{\boldsymbol{i}}$ *with the image by* π *of the singular locus of* $V_{\boldsymbol{i}}$ *by eliminating the variables* $\boldsymbol{x}$ *from* I. This is why we need this assumption.

Hence, using elimination algorithms, which include Gröbner bases [20, 21] with elimination monomial orderings, or triangular sets (see e.g. [2, 44]) or geometric resolution algorithms [24–26], one can compute a polynomial $\Delta_{\boldsymbol{i}} \in \mathbb{Q}[\boldsymbol{y}]$ whose vanishing set is the set of critical values of the restriction of π to $V_{\boldsymbol{i}}$. By the algebraic Sard's theorem (see e.g. [40, App. A]), $\Delta_{\boldsymbol{i}}$ is not identically zero (the critical values are contained in a Zariski closed subset of $\mathbb{C}^t$).

Under assumption (R), we define the set of critical points (resp. values) of the restriction of π to the Euclidean closure of S as the union of the set of critical points (resp. values) of the restriction of π to $V_{\boldsymbol{i}} \cap \mathbb{R}^{n+t}$ when $\boldsymbol{i}$ ranges over the subsets of $\{1, \ldots, q\}$. We denote the Euclidean closure of S by $\overline{S}$, the set of critical points (resp. values) of the restriction of π to $\overline{S}$ by $\mathscr{W}(\pi, \overline{S})$ (resp. $\mathscr{D}(\pi, \overline{S})$).

We say that S satisfies a *properness* assumption (P) if:

(P) the restriction of π to $\overline{S}$ is proper ($\forall y \in \mathbb{R}^t$, there exists a ball $B \ni y$ s.t. $\pi^{-1}(B) \cap \overline{S}$ is closed and bounded).

Under assumption (P), we can apply the semi-algebraic version of Thom's isotopy lemma [16] (see Theorem 2.4). Hence, consider an open semi-algebraic subset $U \subset \mathbb{R}^t$ which does not meet the union of the sets of critical values of the restriction of π to $\overline{S}$ with the images by π of the singular loci of the sets $V_{\boldsymbol{i}}$ for $\boldsymbol{i} \subset \{1, \ldots, q\}$. Let $y \in U$ and $E = \pi^{-1}(y) \cap S$, there exists a semi-algebraic trivialization $\vartheta : U \times E \to \pi^{-1}(U) \cap S$.

We deduce that $\cup_{\boldsymbol{i} \subset \{1,\ldots,q\}} \mathscr{D}(\pi, V_{\boldsymbol{i}})$ contains the boundaries of the open disjoint semi-algebraic set $U_1, \ldots, U_\ell$. Note that since (F, G) satisfies (R) one can restrict to consider subsets $\boldsymbol{i}$ of cardinality $n - p + 1$. Recall that by Sard's theorem it has co-dimension ≥ 1. This leads to algorithm EliminateProper whose correctness is now immediate.

In the description of EliminateProper below we assume that the input polynomials have coefficients in a real field $\mathbf{Q}$ (such as $\mathbb{Q}(\varepsilon)$ where ε is an infinitesimal). Indeed, we will see further that in order to drop Assumption (P), we will need to work over such a real field. We will denote by $\mathbf{R}$ a real closure of $\mathbf{Q}$.

The algorithm EliminateProper uses a subroutine AlgebraicElimination which has the following specifications:

- it takes as input a polynomial sequence $H \subset \mathbf{Q}[\boldsymbol{x}, \boldsymbol{y}]$ and a subset $\boldsymbol{x}$ of the indeterminates;
- it outputs a non-zero polynomial in the intersection of the ideal generated by H with the subring $\mathbf{Q}[\boldsymbol{y}]$ when this intersection is non-zero else it outputs 0.

Note that AlgebraicElimination can be implemented using many algebraic algorithms for eliminating variables in polynomial ideals such as Gröbner bases (see e.g. [17]), regular chains (see e.g. [2]) or, under some genericity assumptions the geometric resolution algorithm [26].

Algorithm 2: EliminateProper$(F, G, \boldsymbol{x}, \boldsymbol{y})$

Data: Finite sequences F and G in $\mathbf{Q}[\boldsymbol{x}, \boldsymbol{y}]$ with $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_t)$, defining a semi-algebraic set $S \subset \mathbb{R}^n \times \mathbb{R}^t$.
Assumes that assumptions (R) and (P) hold.
Result: $\Delta \in \mathbf{Q}[\boldsymbol{y}]$ such that the number of connected components of S_y is invariant when y ranges over a connected component of $\mathbf{R}^t - \{\Delta = 0\}$

```
for all subsets i in {1, ..., q} of cardinality ≤ n − p + 1 do
    M_i ← maximal minors of jac([F, G_i], x)
    Δ_i ← AlgebraicElimination([F, G_i, M_i], x)
end
Δ ← ∏_i Δ_i.
return Δ.
```

For some applications, deciding if (P) holds is easy (e.g. when the inequalities in G define a box). However, in general, one needs to generalize EliminateProper to situations where (P) does not hold.

For $a = (a_1, \ldots, a_n)$, we consider the intersection of $\mathrm{ext}(S, \mathbb{R}\langle\varepsilon\rangle)$ with the semi-algebraic set defined by

$$\Phi^{(a)} = a_1 x_1^2 + \cdots + a_n x_n^2 - 1/\varepsilon \leq 0$$

where $a_i > 0$ in $\mathbb{R}$ for $1 \leq i \leq n$. We denote by S_ϵ this intersection. Since $a_i > 0$ for all $1 \leq i \leq n$, S_ϵ satisfies (P).

Lemma 4.2. *Assume that (F, G) satisfies (R). There exists a non-empty Zariski open set $\mathscr{A} \subset \mathbb{C}^n$ such that for any choice of $a = (a_1, \ldots, a_n) \in \mathscr{A}$, $(F, G^{(a)})$ satisfies (R) with $G^{(a)} = G \cup \{\Phi^{(a)}\}$.*

Proof. Let $\boldsymbol{i} = \{i_1, \ldots, i_s\} \subset \{1, \ldots, q\}$. We prove below that there exists a non-empty Zariski open set $\mathscr{A}_{\boldsymbol{i}} \subset \mathbb{C}^n$ such that for $(a_1, \ldots, a_n) \in \mathscr{A}_{\boldsymbol{i}}$, the following property $(\mathsf{A})_{\boldsymbol{i}}$ holds. Denoting by $G^{(a),\boldsymbol{i}}$ the sequence $(g_{i_1}, \ldots, g_{i_s}, \Phi^{(a)})$, the Jacobian matrix of $(F, G^{(a),\boldsymbol{i}})$ has maximal rank at any point of $V(F, G^{(a),\boldsymbol{i}})$. Taking the intersection of the (finitely many) $\mathscr{A}_{\boldsymbol{i}}$'s is then enough to define $\mathscr{A}$.

Consider new indeterminates $\alpha_1, \ldots, \alpha_n$ and the polynomial

$$\Phi^{(\alpha)} = \alpha_1 x_1^2 + \cdots + \alpha_n x_n^2 - 1/\varepsilon.$$

Let Ψ be the map

$$\Psi : (x, a) \to F(x), g_{i_1}(x), \ldots, g_{i_s}(x), \Phi^{(a)}(x)$$

Observe that $\mathbf{0}$ is a regular value for Ψ since (F, G) satisfies (R). Hence, Thom's weak transversality theorem (see e.g. [40, App. B]) implies that there exists $\mathscr{A}_{\boldsymbol{i}}$ such that $(\mathsf{A})_{\boldsymbol{i}}$ for any $a \in \mathscr{A}_{\boldsymbol{i}}$. ☐

Assume for the moment that $(F, G^{(a)})$ satisfies assumption (R). Observe that the coefficients of F and $G^{(a)}$ lie in $\mathbb{Q}(\varepsilon)$. Hence, applying the subroutine EliminateProper to $(F, G^{(a)})$ and the above inequality will output a polynomial $\Delta_\varepsilon \in \mathbb{Q}(\varepsilon)[\boldsymbol{y}]$ such that the restriction of π to $\overline{S'_\varepsilon}$ realizes a trivialization over each connected component of $\mathbb{R}\langle\varepsilon\rangle^t - \{\Delta_\varepsilon = 0\}$. Without loss of generality, one can assume that $\Delta_\varepsilon \in \mathbb{Q}[\varepsilon][\boldsymbol{y}]$ and has content 1. In other words, one can write $\Delta_\varepsilon = \Delta_0 + \varepsilon\tilde{\Delta}$ with $\Delta_0 \in \mathbb{Q}[\boldsymbol{y}]$ and $\tilde{\Delta} \in \mathbb{Q}[\varepsilon][\boldsymbol{y}]$.

Lemma 4.3. *Let U be a connected component of $\mathbb{R}^t - \{\Delta_0 = 0\}$. Then, there exists a semi-algebraically connected component U_ε of $\mathbb{R}\langle\varepsilon\rangle^t - \{\Delta_\varepsilon = 0\}$ such that $\mathrm{ext}(U, \mathbb{R}\langle\varepsilon\rangle) \subset U_\varepsilon$.*

Proof. Let y and y' be two distinct points in U. Since U is a semi-algebraically connected component of $\mathbb{R}^t - \{\Delta_0 = 0\}$, there exists a semi-algebraic continuous function $\gamma : [0, 1] \to U$ with $\gamma(0) = y$ and $\gamma(1) = y'$ such that Δ_0 is sign invariant over $\gamma([0, 1])$ (assume, without loss of generality that it is positive). Note also for all $t \in [0, 1]$, $\Delta_0(\gamma(t)) \in \mathbb{R}$.

We will deduce that $\Delta_\varepsilon(\gamma(\vartheta)) > 0$ for all $\vartheta \in \mathrm{ext}([0, 1], \mathbb{R}\langle\varepsilon\rangle)$.

Take $\vartheta \in \mathrm{ext}([0, 1], \mathbb{R}\langle\varepsilon\rangle)$. Observe that ϑ is bounded over $\mathbb{R}$ and $\lim_0 \vartheta$ exists and lies in $[0, 1]$. We deduce that $\Delta_\varepsilon(\gamma(\lim_0 \vartheta)) > 0$ and its limit when $\varepsilon \to 0$ is $\Delta_0(\gamma(\lim_0 \vartheta)) > 0$ in $\mathbb{R}$. Thus, $\Delta_\varepsilon(\gamma(\vartheta)) > 0$.

Hence, Δ_ε is sign invariant over $\mathrm{ext}(\gamma([0,1]), \mathbb{R}\langle\varepsilon\rangle)$ and y and y' both lie in the same semi-algebraically connected component of $\mathbb{R}\langle\varepsilon\rangle^t - \{\Delta_\varepsilon = 0\}$. ☐

We deduce that there exists $b \in \mathbb{N}$ such that for all $y \in U$, the number of semi-algebraically connected components of $S_\varepsilon \cap \pi^{-1}(y)$ is b. Using the transfer principle as in [10], we deduce that there exists $e' \in \mathbb{R}$ positive and small enough such that, the following holds: There exists $b \in \mathbb{N}$ such that for all $e \in]0, e'[$ the number of connected components of $S \cap \{a_1x_1^2 + \cdots + a_nx_n^2 \leq \frac{1}{e}\} \cap \pi^{-1}(y)$ is b when y ranges over U. This proves the following lemma.

Lemma 4.4. *Let U be the connected component of $\mathbb{R}^t - \{\Delta_0 = 0\}$ as above. Then the number of connected components of S_y is invariant when y ranges over U.*

Finally, we can describe the subroutine Eliminate whose correctness follows from the previous lemmas. It consists in calling EliminateProper with input F and $G \cup \{g\}$ with $g = \frac{1}{\varepsilon} - \left(a_1x_1^2 + \cdots + a_nx_n^2\right))$ where the a_i's are randomly chosen positive rational numbers and the sequences of variables $\boldsymbol{x}$ and $\boldsymbol{y}$. Observe that here ε is considered as a parameter and then lies in the ground field. Hence the properness assumption P is satisfied thanks to g. One then obtains a non-zero polynomial Δ_ε in $\mathbb{Q}(\varepsilon)[\boldsymbol{y}]$. Multiplying it by the least common multiple of the denominators of its coefficients and factoring out the highest power of ε which divides it, one obtains a polynomial in $\mathbb{Q}[\varepsilon][\boldsymbol{y}]$; instantiating ε to 0 in this polynomial yields the output polynomial Δ_0 which is non-zero. The routine performing these last operations is called Normalize.

Algorithm 3: Eliminate$(F, G, \boldsymbol{x}, \boldsymbol{y})$

Data: Finite sequences F and G in $\mathbb{Q}[\boldsymbol{x}, \boldsymbol{y}]$ with $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_t)$, defining a semi-algebraic set $S \subset \mathbb{R}^n \times \mathbb{R}^t$.
Assumes that (F, G) satisfies assumption (R).
Result: $\Delta \in \mathbb{Q}[\boldsymbol{y}]$ such that the number of connected components of S_y is invariant when y ranges over a connected component of $\mathbb{R}^t - \{\Delta = 0\}$

Choose $a_1 > 0, \ldots, a_n > 0$ in $\mathbb{Q}$ randomly and let $g \leftarrow \frac{1}{\varepsilon} - \left(a_1x_1^2 + \cdots + a_nx_n^2\right)$
$\Delta_\varepsilon \leftarrow$ EliminateProper$(F, G \cup g, \boldsymbol{x}, \boldsymbol{y})$
$\Delta_0 \leftarrow$ Normalize(Δ_ε)
return Δ_0.

4.3. Complexity analysis and proof of Theorem 1.3

We can now finish the proof of Theorem 1.3 and establish that the Algorithm ParametricSolve uses $(\max(2^q, nq^n)(nD))^{O(nt+n^2)}$ arithmetic operations in $\mathbb{Q}$.

Recall that ParametricSolve consists in calling first Eliminate to obtain Δ (step 1). Next, it calls SamplePoints with Δ as input (step 2) and NumberOfConnectedComponents with input the specialization of (F, G) at all points output by SamplePoints (step 4).

Analysis of Eliminate. We start with the analysis of the routine Eliminate. We claim that the total cost of this step uses $\max(2^q, nq^n)(nD)^{O(nt)}$ arithmetic operations in $\mathbb{Q}$ and outputs a non-zero polynomial of degree bounded by $\max(2^q, nq^n)(nD)^n$. This is a consequence of the following lemma since Eliminate relies on EliminateProper which is called with input polynomials with coefficients in $\mathbb{Q}[\varepsilon]$.

Lemma 4.5. *Let $F' = (f_1, \dots, f_p)$ and $G' = (g_1, \dots, g_q)$ in $\mathbb{Q}[\varepsilon][\boldsymbol{x}, \boldsymbol{y}]$ such that all entries have total degree bounded by D and degree 1 in ε. Let $S_\varepsilon \subset \mathbb{R}\langle\varepsilon\rangle^{n+t}$ be the semi-algebraic set defined by (F', G').*

Assume that (F', G') satisfies assumptions (R) *and* (P)*. Then, there exists $\Delta_\varepsilon \in \mathbb{Q}[\varepsilon][\boldsymbol{y}]$ of total degree less than or equalled to* $\max(2^q, nq^{n-p})(nD)^n$ *such that the number of connected components of S_y is invariant when y ranges over a connected component of $\mathbb{R}\langle\varepsilon\rangle^t - \{\Delta_\varepsilon = 0\}$.*

Moreover there exists an algorithm EliminateProper *which on input $F', G', \boldsymbol{x}, \boldsymbol{y}$ computes Δ_ε using* $\max(2^q, nq^{n-p})(nD)^{O(nt)}$ *arithmetic operations in $\mathbb{Q}$.*

Proof. Take $\boldsymbol{i} = \{i_1, \dots, i_s\}$ in $\{1, \dots, q\}$ of cardinality $\leq n - p + 1$ and consider the system of polynomial equations

$$f_1 = \cdots = f_p = 0, \qquad g_{i_1} = \cdots = g_{i_s} = e$$

where e is a new variable. We denote by $W_{\boldsymbol{i}} \subset \mathbb{C}\langle\varepsilon\rangle^{n+t+1}$ the Zariski closure of the solutions to this system at which its Jacobian has maximal rank. Since (R) holds, it is either empty or equidimensional of codimension $p + s$. Also, $V_{\boldsymbol{i},\varepsilon} \subset \mathbb{C}\langle\varepsilon\rangle^{n+t}$ is the algebraic set obtained by taking the projection on the $(\boldsymbol{x}, \boldsymbol{y})$-space of the intersection of $W_{\boldsymbol{i}}$ with the hyperplane defined by $e = \varepsilon$; since (R) holds it is the set of solutions of the above system where e is instantiated to ε.

By using Heintz-Bézout's theorem [29], the degree of the intersection of $W_{\boldsymbol{i}}$ with the maximal minors $\mathcal{M}_{\boldsymbol{i}}$ considered at step 2 of EliminateProper is bounded above by $D^{p+s}((p+s)D)^{n-(p+s)}$. This implies that the degree of the Zariski closure of its projection on the $(\boldsymbol{y}, e)$-space is contained in a hypersurface of degree bounded by $D^{p+s}((p+s)(D-1))^{n-(p+s)} \leq D^p(n(D-1))^{n-p}$. We let $\Delta_{\boldsymbol{i},e}$ be a polynomial of minimal degree defining this hypersurface ; hence it has degree $\leq D^p(n(D-1))^{n-p}$.

Note that specializing e to ε in $\Delta_{\boldsymbol{i},e}$ yields a polynomial $\Delta_{\boldsymbol{i},\varepsilon}$ containing the Zariski closure of the projection on the $\boldsymbol{y}$-space of the intersection of $V_{\boldsymbol{i},\varepsilon}$ with the zero set of $\mathcal{M}_{\boldsymbol{i}}$.

The polynomial Δ_ε in our statement is then obtained by taking the product of all the polynomials $\Delta_{\boldsymbol{i},\varepsilon}$. Note that the number of subsets $\boldsymbol{i}$ of cardinality $\leq n-p$ in $\{1, \dots, q\}$ is

$$\sum_{s=0}^{\min q, n-p+1} \binom{q}{s} \leq \max(2^q, (n-p+1)q^{n-p})$$

Our degree bound follows.

It remains to investigate the cost of computing such a polynomial Δ_ε. To do that we will compute separately the polynomials $\Delta_{\boldsymbol{i},e}$ for all subsets of indices of cardinality less than or equalled to $n-p+1$. To do that we rely on on a classical evaluation/interpolation scheme as follows. We consider an affine linear space E defined by the instantiation of the $y_2, \dots, y_t, e$ variables to random values and compute a univariate polynomial defining the projection on the y_1 axis of the intersection of $W_{\boldsymbol{i},e}$ with E and the zero set of $\mathcal{M}_{\boldsymbol{i}}$. To do that we follow [27, Lemma 6.4] which relies itself on [26] ; this is done using $(D^p(nD)^{n-p})^{O(1)}$ arithmetic operations in $\mathbb{Q}$ (since $0 \leq s \leq n-p+1$). We repeat this with as many points as needed to interpolate a polynomial of degree bounded by $D^p(nD)^{n-p}$ involving $t+1$ variables. Hence, we need $(D^p(nD)^{n-p})^{O(t)}$ points. Taking into account the number of subsets $\boldsymbol{i}$ to consider, our complexity estimates follow. $\square$

Analysis of the call to SamplePoints. Next, we need to analyze the cost of the call to SamplePoints using the algorithm designed in [32, Section 3].

Since this algorithm is called with input a polynomial of degree $\leq \max(2^q, nq^n)(nD)^n$ involving t variables, this algorithm runs using $(\max(2^q, nq^n)(nD))^{O(nt)}$ arithmetic operations in $\mathbb{Q}$. Besides it outputs a finite set of points with rational coordinates of cardinality $(\max(2^q, nq^n)(nD))^{O(nt)}$ (see [32, Theorem III and Corollary 4]).

Analysis of calls to NumberOfConnectedComponents. Finally, one calls NumberOfConnectedComponents on the points computed by SamplePoints. We just established that the number of these points lies in $(\max(2^q, nq^n)(nD))^{O(nt)}$.

Let y be one of these points ; we need now to estimate the complexity of computing the number of connected components of the semi-algebraic set S_y (which is defined by the system obtained by substituting $\boldsymbol{y}$ by y).

By Section 3, this is done using $\max(2^q, nq^n)(nD)^{O(n^2)}$ operations in $\mathbb{Q}$.

Summing up all these costs, we deduce that ParametricSolve uses

$$(\max(2^q, nq^n)(nD))^{O(nt+n^2)}$$

arithmetic operations in $\mathbb{Q}$.

Finally, note that given $\eta \in \mathbb{R}^t - V(\Delta_0)$, one can use NumberOfConnectedComponents to compute a roadmap connecting η to one of the points of the parameters space which is output by ParametricSolve within a runtime lies in the above complexity bound.

5. UR series

We define the *UR Family* to be robots having a similar DH-parameter as the known UR robots (UR5, UR10, etc.). Such *UR robots* are parameterised by the following DH parameters

distances (m.)

$$(a_1, \ldots, a_6) := (0, a_2, a_3, 0, 0, 0)$$

offsets (m.)

$$(d_1, \ldots, d_6) := (0, 0, 0, d_4, d_5, 0)$$

twist angles (rad.)

$$(\alpha_1, \ldots, \alpha_6) := (\frac{\pi}{2}, 0, 0, \frac{\pi}{2}, -\frac{\pi}{2}, 0)$$

i.e. these robots are parameterised by 4 parameters: a_2, a_3, d_4, d_5. For these robots, the determinant of the Jacobian (see [46]), expressed as a polynomial in $v_2, \ldots, v_5$, is $A = 512a_2a_3Bv_3v_5(v_4^2+1)$ with

$$\begin{aligned}B = {}& a_2v_2^2v_3^2v_4^2 - a_3v_2^2v_3^2v_4^2 - 2d_5v_2^2v_3^2v_4 - 2d_5v_2^2v_3v_4^2 - 2d_5v_2v_3^2v_4^2 + a_2v_2^2v_3^2 + a_2v_2^2v_4^2 - a_2v_3^2v_4^2 - a_3v_2^2v_3^2 \\ & + a_3v_2^2v_4^2 + 4a_3v_2v_3v_4^2 + a_3v_3^2v_4^2 + 2d_5v_2^2v_3 + 2d_5v_2^2v_4 + 2d_5v_2v_3^2 + 8d_5v_2v_3v_4 + 2d_5v_2v_4^2 + 2d_5v_3^2v_4 \\ & + 2d_5v_3v_4^2 + a_2v_2^2 - a_2v_3^2 - a_2v_4^2 + a_3v_2^2 + 4a_3v_2v_3 + a_3v_3^2 - a_3v_4^2 - 2d_5v_2 - 2d_5v_3 - 2d_5v_4 - a_2 - a_3\end{aligned}$$

Note that d_4 does not affect the singularity of the robot. Also note that the degrees of A in the variables (v_2, v_3, v_4, v_5) are not equal to $(2, 4, 4, 2)$ as predicted by the general analysis at the end of Section 2: they are $(2, 3, 4, 1)$. The reason for this is that the $v_i = \tan\frac{\theta_i}{2}$ may assume the value infinity, and in case of v_3 and v_5 the equation of the hyperplane where v_i is infinity is a factor of B. This can be computationally verified when we recompute B in terms of $w_i = \cot\frac{\theta_i}{2}$.

The discriminant g of B with respect to v_2 is a product of three sums of two squares:

$$\begin{aligned}g = ((&-a_2v_3v_4 + a_3v_3v_4 + d_5v_3 + d_5v_4 + a_2 + a_3)^2 \\ &+ (-d_5v_3v_4 + a_2v_3 + a_2v_4 - a_3v_3 + a_3v_4 + d_5)^2)(v_3^2+1)(v_4^2+1)\end{aligned}$$

We want to count the connected components of the complement of A in (v_2, v_3, v_4, v_5) space, where each of the four variables is also allowed to assume the value ∞. The above observations and the following general fact are helpful.

Lemma 5.1. *Let M be a connected differential manifold. Let $F_0, F_1, F_2 \in C^\infty(M)$ be smooth functions. Let $F := F_0x_0^2 + F_1x_0x_1 + F_2x_1^2$ be a bivariate quadratic form with coefficients F_0, F_1, F_2. Assume that the zero set of the discriminant $G = F_1^2 - 4F_0F_2$ has codimension 2 or higher, and that G is nonnegative in M. Then the complement of the zero set of F in $M \times \mathbb{P}^1$ has two components, and these components can be distinguished by the sign of the value of F.*

Proof. Note that the form F is not a function in $C^\infty(M \times \mathbb{P}^1)$: any point has coordinates $(u, (x_0 : x_1))$ with $(x_0, x_1) \neq (0, 0)$, but the number $F_0(u)x_0^2 + F_1x_0x_1 + F_2x_1^2$ depends on the choice of the projective coordinates. However, any other choice of coordinates is equal to $(u, (\lambda x_0 : \lambda x_1))$ for some $\lambda \neq 0$, and therefore the sign of F is well defined. Let X_+, X_0, X_- be the subsets of $M \times \mathbb{P}^1$ where F is positive, respectively zero, respectively negative. Suppose furthermore that $\pi : M \times \mathbb{P}^1 \to M$ be the canonical projection.

Let $U \subset M$ be the subset defined by $G > 0$. Because $M \setminus U$ has codimension 2 or higher, it follows that U is connected. For any point $u \in U$, there are exactly two distinct real zeroes of F in $\mathbb{P}^1$. Since the solution of a quadratic equation with positive discriminant depends continuously on the coefficients, there are continuous functions $s_1, s_2 : U \to \mathbb{P}^1$ such that $F(w, s_1(w)) = F(w, s_2(w)) = 0$ for all $w \in U$. For all $u \in U$, the fiber $(\pi|X_0)^{-1}(u)$ has exactly two points and the fiber $(\pi|X_+)^{-1}(u)$ is homeomorphic to an open interval. Let now $U_+ := \pi^{-1}(U) \cap X_+$. We claim that the projection $p : U_+ \to U$ has a continuous section $\sigma : U \to U_+$.

To prove this claim, we use an homeomorphism $\tau : \mathbb{P}^1 \to (\mathbb{R}/\mathbb{Z})$ – for instance, $(s : t) \mapsto \frac{1}{\pi}\arctan\left(\frac{s}{t}\right)$ extended by $(1 : 0) \mapsto \frac{1}{2}$. For each $u \in U$, we set $x_1 := \tau(s_1(u))$ and $x_2 := \tau(s_2(u))$. There are exactly two $y_1(u), y_2(u) \in \mathbb{R}/\mathbb{Z}$ such that $\tau(s_1(u)) + \tau(s_2(u)) = 2y_1(u) = 2y_2(u)$. Without loss of generality, let $y_1(u)$ be such that $(u, \tau^{-1}(y_1(u))) \in U_+$, then this defines the image $\sigma(u)$ and proves our claim.

To connect any two points in $p, q \in U_+$, we start with a path in the fiber that connects p to $\sigma(\pi(p))$. Then we take a path $c : [0,1] \to U$ connecting $\pi(p)$ to $\pi(q)$ and lift it to a path $c' : [0,1] \to U_+$, $c'(t) := \sigma(c(t))$. Finally, we connect $\sigma(\pi(q))$ to q by a path in the fiber of q.

This shows that U_+ is connected. The set X_+ is contained in the closure of U_+ and is therefore also connected. Similarity, it follows that X_- is connected. □

Proposition 5.2. *If $a_2 a_3 \neq 0$, then the complementary of A has 8 components. These components can be distinguished by the signs of the value of the three factors B, v_3, v_5.*

Proof. In order to reduce to projective situation in Lemma 5.1, we need to homogenize B with respect to the variable v_2. This is done by substituting $v_2 = x/y$ and then multiply by y^2 to clear the denominator. The manifold M is defined as a connected component of the zero set of $v_3 v_5$ – apparently, there are four such components. Over each of these components, the discriminant G is a product of sums of squares. This implies that G is nonnegative. Moreover, the zero set of G is the common zero set of the two polynomials under the two squares in the first factor:

$$-a_2v_3v_4 + a_3v_3v_4 + d_5v_3 + d_5v_4 + a_2 + a_3 = -d_5v_3v_4 + a_2v_3 + a_2v_4 - a_3v_3 + a_3v_4 + d_5 = 0.$$

Because these two polynomials do have a common factor for any choice of (a_2, a_3, d_5) other than $(0, 0, 0)$, it follows that the codimension of the zero set is at least 2. By Lemma 5.1, we have exactly two components lying over each component of the zero set of $v_3 v_5$, and they can be distinguished by the sign of B. □

If $a_2 a_3 = 0$, then the determinant of the Jacobian A is identically zero. In this case, two consecutive joints are coaxial, so we would essentially be dealing with a 5R robot – we do not treat this case here.

Remark 5.3. *We need to point out a mistake we made in [14, Section 4]: in the special case $a_2^2 - a_3^2 = d_5 = 0$, two of the factors of g accidentally coincide. We overlooked that and were lead to the wrong conclusion that the number of connected components is larger for the special case. This is wrong: Lemma 5.1 still applies, and therefore the number of connected components is as stated in Corollary 5.2.*

6. Computations

We have implemented several variants of the roadmap algorithms sketched in Section 3 as well as variants of the algorithm ParametricSolve. To perform algebraic elimination, we use Gröbner bases implemented in the FGb library by J.-C. Faugère [22]. This is because we could not use an implementation of the geometric resolution algorithm on which the complexity analysis relies. The roadmap algorithm and the routines for computing sample points in semi-algebraic sets are implemented in the RAGlib library [37].

We have not directly applied the most general version of ParametricSolve to the polynomial B. Indeed, since its variables v_2, v_3, v_4 lie in the Cartesian product $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$ (which is compact), the projection on the parameter's space is proper and it suffices to compute critical loci of that projection. There is one technical (but easy) difficulty to overcome: polynomial B actually admits a positive dimensional singular locus. But an easy computation shows that this singular locus has one purely complex component (which satisfies v_4^2+1) which can then be forgotten. The other component has a projection on the paramaters'space which Zariski closed (it is contained in the set satisfied by $a_2a_3=0$). This way, we directly obtain the following polynomial for Δ by computing the critical locus and consider additionally the set defined by $a_2a_3=0$.

$$a_2a_3d_5\,(a_2+a_3+d_5)\,(a_2+a_3-d_5)$$

Computing Δ as above does not take more than 3 sec. on a standard laptop using FGb. Getting sample points in the set defined by $\Delta\neq 0$ is trivial. We obtain the following 10 sample points using RAGlib

$$\begin{gathered}\{a_2=-1,a_3=-3,d_5=3\},\{a_2=-1,a_3=-1,d_5=3\},\\ \{a_2=-1,a_3=2,d_5=3\},\{a_2=-1,a_3=5,d_5=3\},\\ \{a_2=-1,a_3=\frac{1}{2},d_5=3\},\{a_2=1,a_3=-120,d_5=118\},\\ \{a_2=1,a_3=-118,d_5=118\},\{a_2=1,a_3=1,d_5=118\},\\ \{a_2=1,a_3=118,d_5=118\},\{a_2=1,a_3=-1/2,d_5=118\}\end{gathered}$$

Our implementation allows us to compute a roadmap for one sample point within 20 minutes on a standard laptop. Analyzing the connectivity of these roadmaps is longer as it takes 40 min. All in all, approximately 10 hours are required to handle this positive dimensional parametric system. The data we computed are available at `http://ecarp.lip6.fr/papers/materials/issac20/`. These computations allow to retrieve the conclusions of our theoretical analysis of the UR family. They illustrate that prototype implementations of our algorithms are becoming efficient enough to tackle automated kinematic singularity analysis in robotics.

References

[1] Angeles, J., 2007. Fundamentals of Robotic Mechanical Systems, Theory, Methods, and Algorithms. Springer.

[2] Aubry, P., Lazard, D., Maza, M.M., 1999. On the theories of triangular sets. Journal of Symbolic Computation 28, 105–124.

[3] Bank, B., Giusti, M., Heintz, J., Mbakop, G., 1997. Polar varieties and efficient real equation eolving: the hypersurface case. J. of Complexity 13, 5–27.

[4] Bank, B., Giusti, M., Heintz, J., Safey El Din, M., Schost, E., 2010. On the geometry of polar varieties. Applicable Algebra in Engineering, Communication and Computing 21, 33–83.

[5] Bannwarth, I., Safey El Din, M., 2015. Probabilistic algorithm for computing the dimension of real algebraic sets, in: Proceedings of the 2015 ACM on International Symposium on Symbolic and Algebraic Computation, pp. 37–44.

[6] Basu, S., Pollack, R., Roy, M.F., 1997. On computing a set of points meeting every cell defined by a family of polynomials on a variety. Journal of Complexity 13, 28 – 37.

[7] Basu, S., Pollack, R., Roy, M.F., 2000. Computing roadmaps of semi-algebraic sets on a variety. Journal of the American Mathematical Society 13, 55–82.

[8] Basu, S., Pollack, R., Roy, M.F., 2003. Algorithms in real algebraic geometry. Springer-Verlag.

[9] Basu, S., Roy, M.F., 2014. Divide and conquer roadmap for algebraic sets. Discrete & Computational Geometry 52, 278–343.

[10] Basu, S., Roy, M.F., Safey El Din, M., Schost, É., 2014. A baby step–giant step roadmap algorithm for general algebraic sets. Foundations of Computational Mathematics 14, 1117–1172.

[11] Bochnak, J., Coste, M., Roy, M.F., 1998. Real Algebraic Geometry. Springer-Verlag.

[12] Brown, C.W., Davenport, J., 2007. The complexity of quantifier elimination and cylindrical algebraic decomposition, in: Proceedings of the 2007 international symposium on Symbolic and algebraic computation, pp. 54–60.

[13] Canny, J., 1993. Computing roadmaps of general semi-algebraic sets. The Computer Journal 36, 504–514.

[14] Capco, J., Safey El Din, M., Schicho, J., 2020. Robots, computer algebra and eight connected components, in: Emiris, I.Z., Zhi, L. (Eds.), ISSAC '20: International Symposium on Symbolic and Algebraic Computation, Kalamata, Greece, July 20-23, 2020, ACM. pp. 62–69. doi:10.1145/3373207.3404048.

[15] Collins, G., 1975. Quantifier elimination for real closed fields by cylindrical algebraic decompostion, in: Automata Theory and Formal Languages 2nd GI Conference Kaiserslautern, May 20–23, 1975, Springer. pp. 134–183.

[16] Coste, M., Shiota, M., 1995. Thom's first isotopy lemma: a semialgebraic version, with uniform bound, in: Real Analytic and Algebraic Geometry: Proc. of the International Conference, Trento, Walter de Gruyter. p. 83.

[17] Cox, D., Little, J., OShea, D., 2013. Ideals, varieties, and algorithms: an introduction to computational algebraic geometry and commutative algebra. Springer Science & Business Media.

[18] Davenport, J., Heintz, J., 1988. Real quantifier elimination is doubly exponential. Journal of Symbolic Computation 5, 29–35.

[19] Eisenbud, D., 2013. Commutative Algebra: with a view toward algebraic geometry. volume 150. Springer Science & Business Media.

[20] Faugère, J.C., 1999. A new efficient algorithm for computing gröbner bases (f4). Journal of pure and applied algebra 139, 61–88.

[21] Faugère, J.C., 2002. A new efficient algorithm for computing gröbner bases without reduction to zero (f 5), in: Proc. of the 2002 international symposium on Symbolic and algebraic computation, ACM. pp. 75–83.

[22] Faugère, J.C., 2010. Fgb: A library for computing gröbner bases, in: Fukuda, K., Hoeven, J., Joswig, M., Takayama, N. (Eds.), Mathematical Software - ICMS 2010, Springer, Berlin, Heidelberg. pp. 84–87.

[23] Faugère, J.C., Moroz, G., Rouillier, F., El Din, M.S., 2008. Classification of the perspective-three-point problem, discriminant variety and real solving polynomial systems of inequalities, in: Proceedings of the twenty-first international symposium on Symbolic and algebraic computation, pp. 79–86.

[24] Giusti, M., Heintz, J., Morais, J.E., Morgenstern, J., Pardo, L.M., 1998. Straight-line programs in geometric elimination theory. Journal of Pure and Applied Algebra 124, 101–146.

[25] Giusti, M., Heintz, J., Morais, J.E., Pardo, L.M., 1995. When polynomial equation systems can be solved fast?, in: AAECC-11, Springer. pp. 205–231.

[26] Giusti, M., Lecerf, G., Salvy, B., 2001. A gröbner free alternative for polynomial system solving. Journal of Complexity 17, 154 – 211.

[27] Greuet, A., Safey El Din, M., 2014. Probabilistic algorithm for polynomial optimization over a real algebraic set. SIAM Journal on Optimization 24, 1313–1343.

[28] Hardt, R.M., 1980. Semi-algebraic local-triviality in semi-algebraic mappings. Amer. J. Math. 102, 291–302. doi:10.2307/2374240.

[29] Heintz, J., Schnorr, C.P., 1980. Testing polynomials which are easy to compute, in: Proceedings of the twelfth annual ACM symposium on Theory of computing, pp. 262–272.

[30] Hong, H., Rohal, J., Safey El Din, M., Schost, E., 2020. Connectivity in semi-algebraic sets i. arXiv:2011.02162.

[31] Lazard, D., Rouillier, F., 2007. Solving parametric polynomial systems. Journal of Symbolic Computation 42, 636–667.

[32] Le, H.P., Safey El Din, M., 2020. Solving parametric systems of polynomial equations over the reals through Hermite matrices. URL: https://hal.archives-ouvertes.fr/hal-03029441. preprint.

[33] Le, H.P., Safey El Din, M., de Wolff, T., 2020. Computing the real isolated points of an algebraic hypersurface, in: Proceedings of the 45th International Symposium on Symbolic and Algebraic Computation, pp. 297–304.

[34] Mezzarobba, M., Safey El Din, M., 2006. Computing roadmaps in smooth real algebraic sets, in: Dumas, J.G. (Ed.), Proc. of Transgressive Computing, pp. 327–338.

[35] Murray, R., Li, Z., Sastry, S., 1994. A Mathematical Introduction to Robotic Manipulation. CRC Press Taylor & Francis Group.

[36] Safey El Din, M., 2005. Finding sampling points on real hypersurfaces is easier in singular situations. MEGA (Effective Methods in Algebraic Geometry) Electronic proceedings .

[37] Safey El Din, M., 2007. Real algebraic geometry library. available at http://www-polsys.lip6.fr/~safey.

[38] Safey El Din, M., Schost, E., 2003. Polar varieties and computation of one point in each connected component of a smooth real algebraic set, in: Proc. of the 2003 Int. Symp. on Symb. and Alg. Comp., ACM, NY, USA. pp. 224–231.
[39] Safey El Din, M., Schost, E., 2011. A baby steps/giant steps probabilistic algorithm for computing roadmaps in smooth bounded real hypersurface. Disc. Comput. Geom. 45, 181–220.
[40] Safey El Din, M., Schost, É., 2017. A nearly optimal algorithm for deciding connectivity queries in smooth and bounded real algebraic sets. Journal of the ACM (JACM) 63, 48.
[41] Selig, J., 2005. Geometric Fundamentals of Robotics. Monographs in Computer Science, Springer.
[42] Spong, M., Hutchinson, S., Vidyasagar, M., 2005. Robot Dynamics and Control. Monographs in Computer Science. 2nd ed., John Wiley & Sons.
[43] Trotman, D., 1991. On canny's roadmap algorithm: orienteering in semialgebraic sets (an application of singularity theory to theoretical robotics), in: Singularity Theory and its Applications. Springer, pp. 320–339.
[44] Wang, D., 2001. Elimination methods. Springer Science & Business Media.
[45] Wenger, P., 2019. Cuspidal robots, in: Müller A., Z.D. (Ed.), Singular Configurations of Mechanisms and Manipulators. Springer International Publishing, pp. 67–100.
[46] Weyrer, M., Brandstötter, M., Husty, M., 2019. Singularity avoidance control of a non-holonomic mobile manipulator for intuitive hand guidance. Robotics 8.
[47] Yang, L., Xia, B., 2005. Real solution classification for parametric semi-algebraic systems, in: Dolzmann, A., Seidl, A., Sturm, T. (Eds.), Algorithmic Algebra and Logic. Proceedings of the A3L 2005, April 3-6, Passau, Germany; Conference in Honor of the 60th Birthday of Volker Weispfenning, pp. 281–289.