Optimizing a Parametric Linear Function over a Non-compact Real Algebraic Variety

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ABSTRACT

We consider the problem of optimizing a parametric linear function over a non-compact real trace of an algebraic set $\mathcal{V}$. Our goal is to compute a representing polynomial which defines a hypersurface containing the graph of the optimal value function. Rostalski and Sturmfels showed that when $\mathcal{V}$ is irreducible and smooth with a compact real trace, then the least degree representing polynomial is given by the defining polynomial of the irreducible hypersurface dual to the projective closure of the $\mathcal{V}$.

First, we generalize this approach to non-compact situations. We prove that the graph of the opposite of the optimal value function is still contained in the affine cone over a dual variety similar to the one considered in compact case. In consequence, we present an algorithm for solving the considered parametric optimization problem for generic parameters' values. For some special parameters' values, the representing polynomials of the dual variety can be identically zero, which give no information on the optimal value. We design a dedicated algorithm that identifies those regions of the parameters' space and computes for each of these regions a new polynomial defining the optimal value over the considered region.

Categories and Subject Descriptors

I.1.2 [**Computing Methodologies**]: Symbolic and Algebraic Manipulation—*Algorithms*; G.1.6 [**Numerical Analysis**]: Global optimization

Keywords

Dual variety; polynomial optimization; recession pointed cone

1. INTRODUCTION

Parametric optimization problems widely arise in both theoretical problems and practical applications, like the maximum likelihood estimation and the model predictive control [5]. It is worthwhile to express the optimal value as an explicit or implicit function of the parameters in the region of interest.

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In this paper, we consider the problem of optimizing a parametric linear function over a real algebraic variety

$$c_0^* := \sup_{x\in\mathbb{R}^n} \quad \mathbf{c}^T x = \mathbf{c}_1 x_1 + \cdots + \mathbf{c}_n x_n \qquad \text{s.t.} \quad h_1(x) = \cdots = h_p(x) = 0, \tag{1}$$

where $h_1, \ldots, h_p \in \mathbb{R}[X_1, \ldots, X_n]$ are polynomials in the decision variables $(X_1, \ldots, X_n)$ and $\mathbf{c} = (\mathbf{c}_1, \ldots, \mathbf{c}_n)$ denotes unspecified parameters.

The optimal value c_0^* can be regarded as a function of the parameters $\mathbf{c}$, i.e. the *optimal value function*. Our goal is to compute a polynomial that defines a hypersurface in the parameters' space which contains the graph of this function.

Typically, the *cylindrical algebraic decomposition* (CAD) [8] can be applied to solve (1).

More precisely, by introducing the Boolean operators $\wedge$ (and), we associate (1) with a Boolean expression

$$(h_1(X) = 0) \wedge \cdots \wedge (h_p(X) = 0) \wedge (\mathbf{c}_0 - \mathbf{c}^T X \geq 0) \tag{2}$$

with $X = (X_1, \ldots, X_n)$.

Indeed, recall that a CAD can be used to describe the projection of a semi-algebraic set (which is equivalent to eliminating one block of quantifiers).

By computing the CAD of the semi-algebraic set in $\mathbb{R}^{2n+1}$ defined by (2) with an ordering where the X-variables are larger than the $\mathbf{c}$-variables, the *projection phase* provides us a set of polynomials in $\mathbb{R}[\mathbf{c}_0, \mathbf{c}, X]$, called *projection level factors*, which defines the boundaries of *cells* in the parameters' space $\mathbb{R}^n$. However, the complexity of CAD algorithms is doubly exponential in the number of variables which limits its practical application to nontrivial problems involving 4 variables at most and this general approach may return numerous irrelevant polynomials.

In the last decade, several approaches have been developed to design dedicated algebraic techniques for polynomial optimization (see [31, 17, 16, 18, 2, 34] and references therein). In the non-parametric case, they allow to compute polynomials defining the optimum of a polynomial optimization problem whose degrees are singly exponential in the number of decision variables.

Here, our goal is to extend these techniques to the parametric case. We denote by $\Phi \in \mathbb{Q}[\mathbf{c}_0, \mathbf{c}_1, \ldots, \mathbf{c}_n]$ a polynomial defining a hypersurface in the parameters' space which contains the graph of the optimal value function.

The smallest possible degree for Φ in the variable $\mathbf{c}_0$ is called the *algebraic degree* of the optimization problem (1). This number measures the complexity of (1). Therefore, a lot of interest has been attracted on finding the polynomial Φ and the algebraic degree [6, 15, 20, 24, 25, 27, 30].

We denote $\mathbf{h} = \{h_1, \ldots, h_p\}$ the sequence of polynomials appearing in (1) and let

$$\mathcal{V} = \{v \in \mathbb{C}^n \mid h_1(v) = \cdots = h_p(v) = 0\}. \tag{3}$$

We assume below that $\mathbf{h}$ generates a radical and equidimensional ideal. The *regular points* of $\mathcal{V}$ are those points at which the rank of Jacobian matrix associated to $\mathbf{h}$ is the codimension of $\mathcal{V}$.

We let $\mathcal{V}^*$ be the dual variety associated with $\mathcal{V}$, which is the Zariski closure of the vectors in the projective space tangent to the projective closure of $\mathcal{V}$ at its regular points. Its defining polynomials can be seen as polynomials with coefficients in $\mathbb{Q}[\mathbf{c}_0, \mathbf{c}_1, \ldots, \mathbf{c}_n]$. Rostalski and Sturmfels in [30, Theorem 5.23] show that, when $\mathcal{V}$ is irreducible, compact in $\mathbb{R}^n$ and smooth, the optimal value function Φ is represented by the defining polynomial of $\mathcal{V}^*$. Therefore, when $\mathcal{V}$ is compact in $\mathbb{R}^n$, the defining polynomial of $\mathcal{V}^*$ can fulfill our goal mentioned above. The compactness in the assumption is included to ensure that the optimum c_0^* is well-defined and achieved which are essential in the proof of [30, Theorem 5.23].

However, when $\mathcal{V} \cap \mathbb{R}^n$ is *non-compact*, the optimal value c_0^* for some $(\gamma_1, \ldots, \gamma_n) \in \mathbb{R}^n$ could be infinite or finite but can not be attained, i.e. c_0^* is an *asymptotic critical value* at infinity [21, 22, 26]. Hence, the proof of [30, Theorem 5.23] is not valid in this case. Another issue with the defining polynomials of $\mathcal{V}^*$ is that they might vanish on a Zariski closed set of parameters' values $(\gamma_1, \ldots, \gamma_n) \in \mathbb{R}^n$. In other words, they give no information about the optimal values for these parameters' values. We aim to explore the treatment of the above difficulties.

Main contributions. We consider the problem of optimizing a parametric linear function over a non-compact real trace of an algebraic set $\mathcal{V}$. Supposing $\mathcal{V}$ is smooth, we show that the graph of the opposite of the optimal value function is contained in the affine cone over a dual variety $\mathcal{V}^*$, i.e. $(-c_0^* : \gamma_1 : \cdots : \gamma_n) \in \mathcal{V}^*$ whenever the optimal value c_0^* is bounded at $(\gamma_1, \ldots, \gamma_n)$. We design an algorithm for solving the optimization problem (1) for generic parameters' values. It returns a set of two polynomials (Φ, Z) such that

- $\Phi \in \mathbb{Q}[\mathbf{c}_0, \mathbf{c}]$ and $Z \in \mathbb{Q}[\mathbf{c}]$;
- for any $\gamma = (\gamma_1, \ldots, \gamma_n) \notin \mathbf{V}(Z)$, if the associated optimum c_0^* of (1) is bounded, then $\Phi(\mathbf{c}_0, \gamma)$ is not zero and its set of roots contains the optimum c_0^* of (1).

If $\mathcal{V}$ is irreducible, smooth and the closure of the convex hull of $\mathcal{V} \cap \mathbb{R}^n$ contains no lines, then similar to [30, Theorem 5.23], we show that $\mathcal{V}^*$ is an irreducible hypersurface and its defining polynomial represents the optimal value function of (1).

When $\mathcal{V}$ is not smooth but its real trace is compact, we construct recursively a finite number of dual varieties such that $(-c_0^* : \gamma_1 : \cdots : \gamma_n)$ lies in the union of these dual varieties. We design an algorithm which returns a finite sequence of (Φ_i, Z_i) with the property that for any $\gamma = (\gamma_1, \ldots, \gamma_n)$ whose associated optimum c_0^* is bounded, there exists an i, if $Z_i(\gamma) \neq 0$, then c_0^* is contained in the roots of $\Phi_i(\mathbf{c}_0, \gamma)$.

It may happen that for some special parameters' values, the polynomials obtained with the above approach are identically 0. Then, they provide no information on the optimization problem when the parameters are instantiated to these values.

We design a parametric variant of [18] that solves this problem. Under the assumption that $\mathcal{V}$ is smooth and when the parameters are instantiated, the algorithm in [18] allows to obtain a polynomial of degree singly exponential in the number of decision variables X whose set of roots contains the global optimum of the instantiated polynomial optimization problem.

We use the algebraic nature of the algorithm in [18] to design a parametric variant that returns a list of triples

$$(\Phi_1, Z_1, \mathbf{P}_1), \ldots, (\Phi_k, Z_k, \mathbf{P}_k)$$

such that

- $\Phi_i \in \mathbb{Q}[\mathbf{c}_0, \mathbf{c}]$, $Z_i \in \mathbb{Q}[\mathbf{c}]$ and $\mathbf{P}_i \subset \mathbb{Q}[\mathbf{c}]$ generates a prime ideal for $1 \leq i \leq k$;
- $\cup_{i=1}^k \mathbf{V}(\mathbf{P}_i)$ is the whole parameters' space and $\mathbf{V}(\mathbf{P}_i) - \mathbf{V}(Z_i)$ is not empty for $1 \leq i \leq k$;
- for any $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{R}^n$ such that $\gamma \in \mathbf{V}(\mathbf{P}_i) - \mathbf{V}(Z_i)$, the set of roots of $\Phi(\mathbf{c}_0, \gamma_1, \ldots, \gamma_n)$ contains the global optimum of the polynomial optimization problem (1) when $\mathbf{c}_1, \ldots, \mathbf{c}_n$ are instantiated to $\gamma_1, \ldots, \gamma_n$.

This paper is organized as follows. In Section 2, we recall some background in convex analysis, algebraic geometry and dual varieties needed in this paper. In Section 3, we investigate the relation between the graph of the optimal value function and the dual variety $\mathcal{V}^*$ when the algebraic variety $\mathcal{V}$ is not compact in $\mathbb{R}^n$ or not smooth. In Section 4, we present the parametrized variant of [18].

2. PRELIMINARIES

2.1 Convex sets and cones

We first present some ingredients from convex analysis [28]. A non-empty subset $C \subseteq \mathbb{R}^n$ is said to be *convex* if $(1-\lambda)x + \lambda y \in C$ whenever $x \in C$, $y \in C$ and $0 < \lambda < 1$. We denote $\mathbf{cl}\,(C)$ and $\mathbf{int}\,(C)$ as the *closure* and *interior* of C, respectively. The *affine hull* of a convex set C, denoted by $\mathbf{aff}(C)$, is the unique smallest affine set containing C. The *relative interior* of a convex set $C \subseteq \mathbb{R}^n$, denoted by $\mathbf{ri}\,(C)$, is defined as the interior of C regarded as a subset of $\mathbf{aff}(C)$. For an arbitrary set $C \subseteq \mathbb{R}^n$, denote $\mathbf{co}\,(C)$ as its convex hall.

The *polar* of a non-empty convex set $C \subseteq \mathbb{R}^n$ is a closed convex set defined as

$$C^{\circ} = \{x \in \mathbb{R}^n \mid \forall y \in C, \langle x, y \rangle \leq 1\}.$$

We have $C^{\circ\circ} = \mathbf{cl}\,(\mathbf{co}\,(C \cup \{0\}))$.

A subset $K \subseteq \mathbb{R}^n$ is called a *cone* if it is closed under positive scalar multiplication, i.e. $\lambda x \in K$ for all $x \in K$ and $\lambda > 0$. A convex cone K is *pointed* if it is closed and $K \cap -K = \{0\}$. The *polar* of a non-empty convex cone K is defined as

$$K^{\circ} = \{x \in \mathbb{R}^n \mid \forall y \in K, \langle x, y \rangle \leq 0\}.$$

The *recession cone* 0^+C of a non-empty convex set C is the set including all vectors y satisfying $x + \lambda y \in C$ for every $\lambda > 0$ and $x \in C$. Importantly, a closed convex set $C \subseteq \mathbb{R}^n$ is bounded if and only if 0^+C consists of the zero vector alone. A closed and unbounded convex set C contains *no* lines if and only if 0^+C is pointed.

Let f be a function whose domain is a subset $S \subseteq \mathbb{R}^n$ and values are real or $\pm\infty$. The *epigraph* of f is defined as

$$\mathbf{epi}(f) = \{(x, \mu) \in \mathbb{R}^{n+1} \mid x \in S,\ \mu \in \mathbb{R},\ \mu \geq f(x)\}.$$

We say that f is a *convex function* on S if $\mathbf{epi}(f)$ is convex as a subset of $\mathbb{R}^{n+1}$. The effective domain of a convex function f on S is the projection of $\mathbf{epi}(f)$ on $\mathbb{R}^n$:

$$\begin{aligned}\mathbf{dom}(f) &= \{x \in \mathbb{R}^n \mid \exists \mu \in \mathbb{R} \text{ s.t. } (x, \mu) \in \mathbf{epi}(f)\} \\ &= \{x \in \mathbb{R}^n \mid f(x) < +\infty\}.\end{aligned}$$

THEOREM 2.1. [19, Theorem 1.2] *Let $C \subseteq \mathbb{R}^n$ be a closed and unbounded convex set, then*

1. *$(0^+C)^{\circ}$ is an n-dimensional convex set;*
2. *$\mathbf{int}\,((0^+C)^{\circ}) \subseteq \mathbf{dom}(c_0^*(\mathbf{c} \mid C)) \subseteq (0^+C)^{\circ}$. Moreover, we have $f(x) = a^T x$ attains its supremum on C for every $a \in \mathbf{int}\,((0^+C)^{\circ})$.*

2.2 Dual varieties

Denote $X = (X_1, \ldots, X_n)$. For any ideal (homogeneous ideal) I in $\mathbb{R}[X]$ ($\mathbb{R}[X_0, X]$), denote $\mathbf{V}(I)$ as the affine (projective) variety defined by I in $\mathbb{C}^n$ ($\mathbb{P}^n(\mathbb{C})$). Now let us review some background about dual varieties in $\mathbb{P}^n(\mathbb{C})$ [29, 30]. In the following, we abbreviate $\mathbb{P}^n(\mathbb{C})$ as $\mathbb{P}^n$ for convenience. Let $I = \langle f_1, \ldots, f_p \rangle$ be a homogeneous radical ideal in the polynomial ring $\mathbb{R}[X_0, X]$ and $V = \mathbf{V}(I) \subseteq \mathbb{P}^n$. The *singular locus* $\mathrm{sing}(V)$ is defined by the vanishing of the $c \times c$ minors of the $p \times (n+1)$ Jacobian matrix $\mathsf{Jac}(I) = (\partial f_i / \partial X_j)$, where $c = \mathrm{codim}(V)$. Let $V_{\mathrm{reg}} = V \backslash \mathrm{sing}(V)$ denote the set of regular points in V. The projective variety V is smooth if $V = V_{\mathrm{reg}}$.

A point $u = (u_0 : u_1 : \cdots : u_n)$ in the dual projective space $(\mathbb{P}^n)^*$ represents the hyperplane $\{x \in \mathbb{P}^n \mid \sum_{i=0}^n u_i x_i = 0\}$. We say that u is *tangent* to V at a regular point $x \in V_{\text{reg}}$ if x lies in the hyperplane $\sum_{i=0}^n u_i x_i = 0$ and its representing vector $(u_0, u_1, \ldots, u_n)$ lies in the row space of the Jacobian matrix $\mathsf{Jac}(I)$ at the point x. The *conormal variety* $\mathrm{CN}(V)$ is the closure of the set

$$\{(x, u) \in \mathbb{P}^n \times (\mathbb{P}^n)^* \mid x \in V_{\text{reg}} \text{ and } u \text{ is tangent to } V \text{ at } x\}.$$

The *dual variety* V^* is the projection of $\mathrm{CN}(V)$ onto the second factor. More precisely, the dual variety is the closure of the set

$$\{u \in (\mathbb{P}^n)^* \mid \; u \text{ is tangent to } V \text{ at some regular point}\}.$$

2.3 Generalized critical values

For a vector $v \in \mathbb{R}^n$, $\|v\|$ denotes the standard Euclidean norm of v. Let V be a smooth affine variety, and let $f : V \to \mathbb{R}$ be a polynomial dominant mapping. Denote $K_0(f, V)$ as the critical values of f on V. The set of *asymptotic critical values* at infinity [21, 22, 26] of f on V is defined as

$$K_\infty(f, V) = \left\{ \; y \in \mathbb{R} \;\middle|\; \begin{array}{l} \exists\, x^{(k)} \in V, \text{ s.t. } x^{(k)} \to \infty, \\ f(x^{(k)}) \to y, \; \|x^{(k)}\| \nu(d_{x^{(k)}} f) \to 0 \end{array} \right\},$$

where $d_{x^{(k)}} f$ stands for the differential of f evaluated at $x^{(k)}$ and ν stands for the distance of $d_{x^{(k)}} f$ to the space of degenerate linear maps on the tangent space to V at $x^{(k)}$. The set of generalized critical values of f is defined as

$$K(f, V) = K_0(f, V) \cup K_\infty(f, V).$$

It has been shown in [22, Theorem 3.1] and [21, Theorem 3.3, Corollary 4.1] that $K(f, V)$ is a finite set.

THEOREM 2.2. [21, 26] *If f is bounded above, i.e. $f^* = \sup_{x \in V} f(x)$ is finite, then $f^* \in K(f, V)$.*

3. DUALITY IN NON-COMPACT CASE

For the algebraic variety $\mathcal{V}$ defined in (3), let

$$C_{\mathbf{h}} = \mathbf{cl}\left(\mathbf{co}\left(\mathcal{V} \cap \mathbb{R}^n\right)\right),$$

i.e. the closure of the convex hull of $\mathcal{V} \cap \mathbb{R}^n$. Then, the problem (1) is equivalent to

$$c_0^* = \sup \; \mathbf{c}^T x \quad \text{ s.t. } x \in C_{\mathbf{h}}. \tag{4}$$

Let $\mathcal{V}^*$ be the dual variety to the projective closure of $\mathcal{V}$. When $C_{\mathbf{h}}$ contains no lines, i.e. $0^+ C_{\mathbf{h}}$ is pointed, the relation between the optimal value function of (1) and the defining polynomial of $\mathcal{V}^*$ is investigated in [19]. Now we prove the correctness of some results therein without the assumption of pointedness. It will yield an algorithm for solving parametric optimization problem (1) with generic parameters.

3.1 Smooth Case

In this subsection, we assume the algebraic variety $\mathcal{V}$ in (3) is smooth. Recall that $\mathbf{dom}(c_0^*(\mathbf{c} \mid C_{\mathbf{h}}))$ denotes the collection of the parameters' values $\gamma \in \mathbb{R}^n$ such that the supremum of $\gamma^T x$ on $C_{\mathbf{h}}$ is finite. We generalize Rostalski and Sturmfels' result [30, Theorem 5.23] to the non-compact case as follows.

THEOREM 3.1. *Suppose that $\mathcal{V}$ in (3) is smooth, then*

$$(-c_0^* : \gamma_1 : \cdots : \gamma_n) \in \mathcal{V}^* \tag{5}$$

for every $\gamma \in \mathbf{dom}(c_0^*(\mathbf{c} \mid C_{\mathbf{h}}))$.

PROOF. Fix a $\gamma \in \mathbf{dom}(c_0^*(\mathbf{c} \mid C_{\mathbf{h}}))$. By the definition, the supremum c_0^* of $f(X) = \gamma^T X$ on $\mathcal{V} \cap \mathbb{R}^n$ is finite. For the case when c_0^* is a critical value which can be attained, see the proof of [30, Theorem 5.23]. Now by Theorem 2.2, we suppose that c_0^* is an asymptotic critical value of f over $\mathcal{V} \cap \mathbb{R}^n$. Then, there exists a sequence $\{x^{(k)}\} \subseteq \mathcal{V} \cap \mathbb{R}^n$ such that $\|x^{(k)}\| \to \infty$, $f(x^{(k)}) \to c_0^*$ and $\|x^{(k)}\| \nu(d_{x^{(k)}} f) \to 0$. By [36, Lemma 2.1], for each $x^{(k)}$, we can find a vector $\gamma^{(k)}$ in the normal space of $\mathcal{V}$ at $\{x^{(k)}\}$ such that $\|\gamma^{(k)} - \gamma\| = \nu(d_{x^{(k)}} f)$. Then, $\|x^{(k)}\| \|\gamma^{(k)} - \gamma\| \to 0$, which implies $\|\gamma^{(k)} - \gamma\| \to 0$ and $(\gamma^{(k)})^T x^{(k)} \to c_0^*$. It can be checked that

$$(-(\gamma^{(k)})^T x^{(k)} : \gamma_1^{(k)} : \cdots : \gamma_n^{(k)}) \in \mathcal{V}^*.$$

Since $\mathcal{V}^*$ is closed, we have $(-c_0^* : \gamma_1 : \cdots : \gamma_n) \in \mathcal{V}^*$. $\square$

COROLLARY 3.2. *If $\mathcal{V}$ is irreducible, smooth and $C_{\mathbf{h}}$ contains no lines, then $\mathcal{V}^*$ is an irreducible hypersurface and its defining polynomial represents the optimal value function of (1).*

PROOF. Since $C_{\mathbf{h}}$ contains no lines, $0^+(C_{\mathbf{h}})$ is pointed. According to Theorem 2.1, $(0^+ C_{\mathbf{h}})^o$ is a n-dimensional convex set and $\mathbf{int}\left((0^+ C_{\mathbf{h}})^o\right)$ is contained in $\mathbf{dom}(c_0^*(\mathbf{c} \mid C_{\mathbf{h}}))$. Therefore, the affine cone of the Zariski closure of

$$\left\{(-c_0^* : \gamma_1 : \cdots : \gamma_n) \in (\mathbb{P}^n)^* \mid \gamma \in \mathbf{int}\left((0^+ C_{\mathbf{h}})^o\right)\right\} \tag{6}$$

has dimension $\geq n$. By [9, Theorem 12 (i), §3, Chpt. 9], the Zariski closure of (6) is of dimension $\geq n - 1$. By Theorem 3.1 and [30, Proposition 5.10], we have $\dim(\mathcal{V}^*) = n - 1$. As $\mathcal{V}$ is irreducible, $\mathcal{V}^*$ is an irreducible hypersurface [13, Proposition 1.3], and coincides with the Zariski closure of (6) according to [9, Proposition 10 (ii), §4, Chpt. 9]. Then, the conclusion follows. $\square$

In the sequel, we say that a property depending on some indeterminates is *generic* if there exists a non-empty Zariski open subset of the space endowed by these indeterminates over which the property holds (we will also say that the property holds for generic values of these indeterminates).

An algorithm can be derived from Theorem 3.1 for solving the parametric optimization (1) for generic parameter as described below. Denote $\mu = (\mu_1, \ldots, \mu_p)$. Let $\mathsf{J} \subseteq \mathbb{Q}[\mathbf{c}_0, \mathbf{c}, \mu, X]$ be the ideal generated by

$$\mathbf{c}^T X - \mathbf{c}_0, \; h_1, \ldots, h_p, \; \mathbf{c}_i - \sum_{j=1}^p \mu_j \frac{\partial h_j}{\partial X_i}, \; i = 1, \ldots, n.$$

Since $\mathbf{h}$ generates a radical ideal, for any $(c_0^*, \gamma, \bar{\mu}, \bar{x}) \in \mathbf{V}(\mathsf{J})$, c_0^* is a critical value of the function $\gamma^T X$ on $\mathcal{V}$ at a critical point $\bar{x}$.

ALGORITHM 3.1. $\mathsf{GenericParametricOptimization}(\mathbf{h})$
Input: *$h_1, \ldots, h_p \in \mathbb{Q}[X]$ which generate a radical ideal*
Output: *(Φ, Z) such that*

- *$\Phi \in \mathbb{Q}[\mathbf{c}_0, \mathbf{c}]$ and $Z \in \mathbb{Q}[\mathbf{c}]$*
- *For any $\gamma \in \mathbf{dom}(c_0^*(\mathbf{c} \mid C_{\mathbf{h}}))$ such that $Z(\gamma) \neq 0$, $\Phi(\mathbf{c}_0, \gamma)$ is not zero and its set of roots contains the optimum c_0^* of (1).*

Step 1 *Compute the reduced Gröbner basis G of $\mathsf{J} \cap \mathbb{Q}[\mathbf{c}_0, \mathbf{c}]$ with block lex order $X \succ \mu \succ \mathbf{c} \succ \mathbf{c}_0$.*
Step 2 *Set Γ to be the set of polynomials in G containing the variable $\mathbf{c}_0$.*
Step 3 *Set Φ to be the polynomial in Γ with the lowest degree in $\mathbf{c}_0$.*
Step 4 *Set Z to be the sum of squares of all coefficients of Φ in view of $\mathbb{Q}[\mathbf{c_1}, \ldots, \mathbf{c_n}][\mathbf{c}_0]$.*

THEOREM 3.3. *In Algorithm 3.1, we have $\Gamma \neq \emptyset$, $\mathcal{V}^* = \mathbf{V}(\Gamma)$ and the algorithm is correct.*

PROOF. By the definition of dual varieties and Theorem 3.1, it suffices to show that $\Gamma \neq \emptyset$. Let $\pi_{n+1}(\mathbf{V}(\mathsf{J}))$ be the projection of points in $\mathbf{V}(\mathsf{J})$ on their first $n + 1$ coordinates, then G is the corresponding elimination ideal. By the Closure Theorem [9], $\pi_{n+1}(\mathbf{V}(\mathsf{J})) \subseteq \mathbf{V}(G)$ and there exists a subvariety $W \subsetneq \mathbf{V}(G)$ such that $\mathbf{V}(G) \backslash W \subseteq \pi_{n+1}(\mathbf{V}(\mathsf{J}))$. Fix a point $(c_0^*, \gamma) \in \mathbf{V}(G)$. Suppose to the contrary that $\Gamma = \emptyset$, then $\mathbb{C} \times \gamma \subseteq \mathbf{V}(G)$. By Sard's Theorem, $\mathbb{C} \times \gamma \cap \pi_{n+1}(\mathbf{V}(\mathsf{J}))$ is an empty set or a finite set. Therefore, $\mathbb{C} \times \gamma \subseteq \mathbf{V}(G) \backslash \pi_{n+1}(\mathbf{V}(\mathsf{J})) \subseteq W$ except for at most finitely many points in $\mathbb{C} \times \gamma$. Since W is closed, $\mathbb{C} \times \gamma \subseteq W$. In particular, $(c_0^*, \gamma) \in W$ which means $\mathbf{V}(G) = W$, a contradiction. $\square$

REMARK 3.1. *As proved in Corollary 3.2, when $\mathcal{V}$ is irreducible, smooth and $C_{\mathbf{h}}$ contains no lines, there is only one polynomial in the set Γ in Algorithm 3.1. If $C_{\mathbf{h}}$ contains lines, Γ might consist of more than one polynomial and $\mathcal{V}^*$ may not be the Zariski closure of the set (6), see Example 3.1.*

Similar to [31, Theorem 6], with Algorithm 3.1 and procedures of deciding the emptiness of real algebraic varieties, we can determine whether a generic γ ($\gamma \notin \mathbf{V}(Z)$) belongs to $\mathbf{dom}(c_0^*(\mathbf{c} \mid C_{\mathbf{h}}))$ and the associated optimum c_0^* if it does.

EXAMPLE 3.1. *Consider the algebraic variety $\mathcal{V}$ defined by*

$$h(X_1, X_2) = X_1^2 X_2 - 1$$

which is irreducible, smooth, non-compact in $\mathbb{R}^2$ and $C_{\mathbf{h}}$ contains lines. Let $\gamma = (0, -1)$, then clearly $c_0^* = 0$. Running Algorithm 3.1, we get $\Gamma = \{4\mathbf{c}_0^3 + 27\mathbf{c}_1^2\mathbf{c}_2\}$ and hence

$$\Phi = 4\mathbf{c}_0^3 + 27\mathbf{c}_1^2\mathbf{c}_2, \qquad \mathbf{V}(Z) = \emptyset.$$

Hence, $\mathcal{V}^* = \mathbf{V}(\Gamma) \subseteq \mathbb{P}^2$ and $(0:0:-1) \in \mathcal{V}^*$.

Since $\mathbf{dom}(c_0^*(\mathbf{c} \mid C_{\mathbf{h}})) = \{\gamma \in \mathbb{R}^2 \mid \gamma_1 = 0, \gamma_2 < 0\}$ and $c_0^* = 0$ for any $\gamma \in \mathbf{dom}(c_0^*(\mathbf{c} \mid C_{\mathbf{h}}))$, the Zariski closure of

$$\{(-c_0^* : \gamma_1 : \gamma_2) \in (\mathbb{P}^2)^* \mid \gamma \in \mathbf{dom}(c_0^*(\mathbf{c} \mid C_{\mathbf{h}}))\} \qquad (7)$$

is $\{(0:0:\gamma_2) \in (\mathbb{P}^2)^* \mid \gamma_2 \in \mathbb{C}\}$ which is of dimension 0. Since we have $\dim \mathcal{V}^* = 1$, $\mathcal{V}^*$ is not the Zariski closure of the set (7).

3.2 Singular case

Now we suppose that $\mathcal{V}$ is irreducible, compact in $\mathbb{R}^n$ but is *not* smooth. We point out that the inclusion (5) might not hold in this case.

EXAMPLE 3.2. *Consider the astroid which is a real locus of a plane algebraic curve $\mathcal{V}$ defined by*

$$h(X_1, X_2) = \left(X_1^2 + X_2^2 - 1\right)^3 + 27X_1^2X_2^2.$$

It is obvious that for any linear function on $\mathcal{V} \cap \mathbb{R}^2$, its optimizer is one of the four singular points $\{(\pm 1, 0), (0, \pm 1)\}$. We have $\mathcal{V}^* = \mathbf{V}(\Gamma)$ where

$$\Gamma = \{-\mathbf{c}_1^2\mathbf{c}_2^2 + \mathbf{c}_0^2\mathbf{c}_1^2 + \mathbf{c}_2^2\mathbf{c}_0^2\}.$$

For a given $\gamma \in \mathbb{R}^2$, we have $c_0^* = \max\{|\gamma_1|, |\gamma_2|\} > 0$. It is easy to check $(-c_0^* : \gamma_1 : \gamma_2) \notin \mathcal{V}^*$ when $\gamma_1 \neq 0$ or $\gamma_2 \neq 0$, i.e. (5) does not hold.

Next we recursively construct a finite number of dual varieties such that (5) holds for the union of these varieties. For similar treatment, see [35]. The following algorithm has the same input as Algorithm 3.1 and returns a finite sequence of (Φ_k, Z_k) with the property that for any $\gamma \in \mathbf{dom}(c_0^*(\mathbf{c} \mid C_{\mathbf{h}}))$, there exists a k, such that if $Z_k(\gamma) \neq 0$, then c_0^* is contained in the roots of $\Phi_k(\mathbf{c}_0, \gamma)$.

ALGORITHM 3.2. SingularParametricOptimization($\mathbf{h}$)

Step 1 *Let $k = 1$ and $V_k = \mathcal{V}$.*

Step 2 *Compute an equidimensional decomposition $V_k = \cup_i V_{k,i}$ with $V_{k,i} = \mathbf{V}(I_{k,i})$ and each $I_{k,i}$ is a radical ideal.*

Step 3 *Run* GenericParametricOptimization($I_{k,i}$) *for each i and set $(\mathcal{V}^{(k)})^* = \cup_i V_{k,i}^*$.*

Step 4 *Compute the set $\Gamma_k \subseteq \mathbb{Q}[\mathbf{c}_0, \mathbf{c}]$ such that $\mathbf{V}(\Gamma_k) = (\mathcal{V}^{(k)})^*$.*

Step 5 *Set Φ_k to be the polynomial in Γ_k with the lowest degree in $\mathbf{c}_0$.*

Step 6 *Set Z_k to be the sum of squares of all coefficients in Φ_k in view of $\mathbb{Q}[\mathbf{c}_1, \ldots, \mathbf{c}_n][\mathbf{c}_0]$.*

Step 7 *Compute the singular locus $\widetilde{V}_{k,i}$ of each $V_{k,i}$ and set $V_{k+1} = \cup_i \widetilde{V}_{k,i}$. If $V_{k+1} \neq \emptyset$, then let $k = k+1$ and go to Step 2.*

The next theorem shows the correctness of Algorithm 3.2.

THEOREM 3.4. *The algorithm terminates in a finite number of steps and for every $\gamma \in \mathbb{R}^n$, we have*

$$(-c_0^* : \gamma_1 : \cdots : \gamma_n) \subseteq \cup_{k=1}^{l} (\mathcal{V}^{(k)})^*.$$

PROOF. Since $\widetilde{V}_{k,i}$ is the singular locus of $V_{k,i}$, $\dim(\widetilde{V}_{k,i}) < \dim(V_{k,i})$ and then the algorithm terminates in a finite number of steps. Since $\mathcal{V}$ is compact in $\mathbb{R}^n$, for every parameter γ, the optimum c_0^* is finite and attainable. If the optimizer x^* is a smooth point, by Theorem 3.1, $(-c_0^* : \gamma_1 : \ldots : \gamma_n) \in \mathcal{V}^*$. If x^* is a singular point of $\mathcal{V}$. Then there exist k and i such that x^* is regular in $V_{k,i}$ and $(-c_0 : \gamma_1 : \ldots : \gamma_n) \in V_{k,i}^*$. $\square$

EXAMPLE 3.2 (CONTINUED) The singular locus of $\mathcal{V}$ is defined by

$$\left\{h, X_1^5 - X_1, X_1^3X_2 + X_1X_2, 3X_1^4 - X_1^2 + 2X_2^2 - 2\right\},$$

and has four real points $\{(\pm 1, 0), (0, \pm 1)\}$. Running Algorithm 3.2, we get Γ_1 consisting of

$$(\mathbf{c}_0 - \mathbf{c}_2)(\mathbf{c}_0 + \mathbf{c}_2)(\mathbf{c}_0 - \mathbf{c}_1)(\mathbf{c}_0 + \mathbf{c}_1)(\mathbf{c}_0^2 + \mathbf{c}_1^2 - 2\mathbf{c}_1\mathbf{c}_2 + \mathbf{c}_2^2)$$
$$(\mathbf{c}_0^2 + \mathbf{c}_1^2 + 2\mathbf{c}_1\mathbf{c}_2 + \mathbf{c}_2^2).$$

By the discussion in Example 3.2, it is easy to check that $(-c_0^* : \gamma_1 : \gamma_2) \in \mathbf{V}(\Gamma_1) = (\mathcal{V}^{(1)})^*$ for every $\gamma \in \mathbb{R}^2$.

3.3 "Bad" parameters

It is clear that the polynomial Φ in Algorithm 3.1 gives no information about the optimal value of (1) with parameters belonging to $\mathbf{V}(Z)$. In particular, it might happen for the problem (1) reformulated from a general polynomial optimization problem by introducing a new variable.

Consider the polynomial optimization problem

$$f^* := \max_{x \in \mathbb{R}^n} f(x) \quad \text{s.t. } h_1(x) = \cdots = h_p(x) = 0,$$

where $f \in \mathbb{R}[X]$. If f is bounded from above on $\mathcal{V} \cap \mathbb{R}^n$, then we have $f^* \in K(f, \mathcal{V} \cap \mathbb{R}^n)$. Let

$$\mathcal{V}_{\mathbf{h},f} = \{(x, x_{n+1}) \in \mathbb{C}^{n+1} \mid x \in \mathcal{V}, x_{n+1} - f(x) = 0\}.$$

By Theorem 3.1, we have $(-f^* : 0 : \cdots : 0 : 1) \in \mathcal{V}_{\mathbf{h},f}^*$.

EXAMPLE 3.3. *[22, Example 2.1] Let $f = (X_1 + X_1^2X_2 + X_1^4X_2X_3)^2$. Running Algorithm 3.1 for $\mathcal{V}_{\mathbf{h},f}$ with $p = 0$, we get $\Gamma = \{\Phi\}$ where*

$$\Phi = 1073741824\mathbf{c}_0^{12}\mathbf{c}_2^4\mathbf{c}_4^2 + 268435456\mathbf{c}_0^{11}\mathbf{c}_1^2\mathbf{c}_2^4\mathbf{c}_4 - $$
$$+ 9865003008\mathbf{c}_0^{10}\mathbf{c}_2^4\mathbf{c}_3\mathbf{c}_4^3 + \cdots + 520093696\mathbf{c}_0^9\mathbf{c}_1\mathbf{c}_2^3\mathbf{c}_3^2\mathbf{c}_4^3.$$

We have $\Phi(\mathbf{c}_0, 0, 0, 0, 1) \equiv 0$ which gives no information about f^*. In fact, we have $\mathbf{V}(Z) = \mathbf{V}(\mathbf{c}_2\mathbf{c}_4, \mathbf{c}_3\mathbf{c}_4, \mathbf{c}_3\mathbf{c}_2\mathbf{c}_1)$. Hence, for $\mathbf{c}_2 = 0, \mathbf{c}_3 = 0$, we always have $\Phi(\mathbf{c}_0, \mathbf{c}_1, 0, 0, \mathbf{c}_4) \equiv 0$, i.e. $\mathbf{c}_0$ can be arbitrary values.

In next section, we aim to design a complete algorithm to solve this problem.

4. COMPLETE ALGORITHM

4.1 Overview

As above, let $\mathbf{h} = (h_1, \ldots, h_p) \subset \mathbb{Q}[X_1, \ldots, X_n]$ that generates a radical and equidimensional ideal and $\mathcal{V}$ be the algebraic set defined by $h_1 = \cdots = h_p = 0$; we assume that $\mathcal{V}$ is smooth and denote by r its codimension.

It might happen that for some "bad" parameters' values $\gamma = (\gamma_1, \ldots, \gamma_n)$, the defining polynomials of the dual variety $\mathcal{V}^*$ become identically zero. For such values, this gives no information about the optimum c_0^* of the map $x \to \gamma^T x$ on $\mathcal{V} \cap \mathbb{R}^n$. For instance, in Example 3.3, the polynomial Φ is a zero polynomial for any $\gamma = (\gamma_1, 0, 0, \gamma_4)$, $\gamma_1, \gamma_4 \in \mathbb{R}$. In this section, we describe an algorithm that allows to avoid this problem. It can be seen as a parametric version of [18] that provides a complete algorithm for polynomial optimization.

Our algorithm starts by computing a couple of the form $(\Phi, Z, \mathbf{P})$ where $\Phi \in \mathbb{Q}[\mathbf{c}_0, \mathbf{c}]$, $\mathbf{P} \subset \mathbb{Q}[\mathbf{c}]$ and $Z \in \mathbb{Q}[\mathbf{c}] - \langle \mathbf{P} \rangle$ such that for any $(\gamma_1, \ldots, \gamma_n) \in \mathbf{V}(\mathbf{P}) - \mathbf{V}(Z)$, $\Phi(\mathbf{c}_0, \gamma_1, \ldots, \gamma_n)$ is not identically 0 and the optimum of the restriction of the map $x \to \gamma^T x$ to $\mathcal{V} \cap \mathbb{R}^n$ is a root of $\Phi(\mathbf{c}_0, \gamma_1, \ldots, \gamma_n)$. Next, the algorithm is called recursively to study the parametric optimization problem under each constraint $\mathbf{P}_i$ which is a prime component of $\sqrt{\langle \mathbf{P} \rangle + \langle Z \rangle}$. Hence, we are led to run our algorithm over an integral domain $\mathbb{Q}[\mathbf{c}_1, \ldots, \mathbf{c}_n]/\mathcal{P}$ where $\mathcal{P} \subset \mathbb{Q}[\mathbf{c}_1, \ldots, \mathbf{c}_n]$ is a prime ideal. Since the domain on which the computations are performed is integral, all operations that we need to manipulate polynomial ideals are available; the only difference is that we need to compute pseudo-inverse of polynomials modulo $\mathcal{P}$, hence simulating computations over the fraction field of $\mathbb{Q}[\mathbf{c}_1, \ldots, \mathbf{c}_n]/\mathcal{P}$.

The routine that handles these computations over these integral rings is called $\mathsf{BasicParametricOptimization}$. It is a parametric variant of Algorithm $\mathsf{SetContainingLocalExtrema}$ in [18, Section 3]. One of its advantages is that in the fraction field of the integral domains $\mathbb{Q}[\mathbf{c}_1, \ldots, \mathbf{c}_n]/\mathcal{P}$, it performs a number of operations that is singly exponential in n (see [18, Section 6, Lemma 6.8]) to be compared with the doubly exponential complexity in n that is needed by Cylindrical Algebraic Decomposition.

Note that one can also use lazy representations of ideals and dynamic evaluation techniques (see e.g. [10, 23]) to work with these parameters as well as comprehensive Gröbner bases or comprehensive triangular sets (see e.g. [7, 37] and references therein). The description below is done assuming that our domain is integral for simplicity; this allows us to focus more on objects and properties related to polynomial optimization and introduced in [18].

Before describing in detail the recursive procedure that is sketched above, let us describe the objects and subroutines that we need and which are extracted from [18].

4.2 Basic objects and properties

We start with polar varieties (see e.g. [1, 3] and references therein) and their Noether position properties (see [32]). Let $\mathbf{P} \subset \mathbb{Q}[\mathbf{c}]$ be a finite polynomial sequence generating a prime ideal $\mathcal{P}$ and let $\mathbb{A} = \mathbb{Q}[\mathbf{c}]/\mathcal{P}$. Hence $\mathbb{A}$ is an integral ring.

For $\gamma \in \mathbf{V}(\mathbf{P})$, we consider the canonical projections $\pi_i : (x_1, \ldots, x_n) \to (x_1, \ldots, x_i)$ for $1 \le i \le n$ and the following projections

$$\pi_\gamma : x = (x_1, \ldots, x_n) \to \gamma^T x = \gamma_1 x_1 + \cdots + \gamma_n x_n$$

and for $1 \le i \le \dim(\mathcal{V}) = n - r$,

$$\pi_{\gamma,i} : x = (x_1, \ldots, x_n) \to (\gamma^T x, x_1, x_2, \ldots, x_i).$$

For $\vartheta \in \mathbb{C}$, we denote by $\mathcal{V}_{\gamma,\vartheta}$ the algebraic set defined by $\mathcal{V} \cap \pi_\gamma^{-1}(\vartheta)$. We consider

- the set of all $(r+1)$-minors of the truncated Jacobian matrix $\mathsf{Jac}([h_{i_1}, \ldots, h_{i_r}, \gamma^T X], \mathbf{X}_{>i})$ (columns corresponding to partial derivatives w.r.t $X_1, \ldots, X_i$ are omitted) for all subsets $\{i_1, \ldots, i_r\} \subset \{1, \ldots, p\}$ for $1 \le i \le n - r - 1$; we denote it by $\mathsf{M}(\mathbf{h}, \gamma, i)$. For convenience, let $\mathsf{M}(\mathbf{h}, \gamma, n-r) = \emptyset$.

  When the entries of γ are parameters $\mathbf{c} = (\mathbf{c}_1, \ldots, \mathbf{c}_n)$, the set of minors is denoted by $\mathsf{M}(\mathbf{h}, \mathbf{c}, i)$.

- the set of all $(r+1)$-minors of the Jacobian matrix $\mathsf{Jac}([h_{i_1}, \ldots, h_{i_r}, \gamma^T X])$ for all subsets $\{i_1, \ldots, i_r\} \subset \{1, \ldots, p\}$; we denote it by $S(\mathbf{h}, \gamma)$.

  When the entries of γ are parameters $\mathbf{c} = (\mathbf{c}_1, \ldots, \mathbf{c}_n)$, the set of minors is denoted by $\mathsf{S}(\mathbf{h}, \mathbf{c})$.

Let $\gamma \in \mathbb{C}^n - \{\mathbf{0}\}$ and $\vartheta \in \mathbb{C}$. Assume that $\mathcal{V}_{\gamma,\vartheta}$ is smooth and that the ideal $\langle \mathbf{h}, \gamma^T X - \vartheta \rangle$ is radical and equidimensional. The polar variety $W(\mathbf{h}, \gamma, \vartheta, i)$ associated to $\mathcal{V}_{\gamma,\vartheta}$ and $\pi_{\gamma,i}$ is the critical locus of the restriction to $\mathcal{V}_{\gamma,\vartheta}$ of π_i. It is defined by the vanishing of the polynomials in $\mathbf{h}$ and $\mathsf{M}(\mathbf{h}, \gamma, i)$ and the polynomial $\gamma^T X - \vartheta$.

We will denote by $W(\mathbf{h}, \gamma, i)$ the algebraic set defined by the vanishing of the polynomials in $\mathbf{h}$ and $\mathsf{M}(\mathbf{h}, \gamma, i)$, for $1 \le i \le n - r$.

The polar variety $C(\mathbf{h}, \gamma)$ associated to $\mathcal{V}$ and π_γ is the critical locus of the restriction to $\mathcal{V}$ of π_γ. It is defined by the vanishing of the polynomials in $\mathbf{h}$ and $\mathsf{S}(\mathbf{h}, \gamma)$.

In the sequel, we will use some properties of polar varieties that hold under generic changes of coordinates.

For $\mathbf{A} \in \mathrm{GL}_n(\mathbb{C})$ and $S \subset \mathbb{C}^n$, we denote by $S^{\mathbf{A}}$ the image of S by the map $x \to \mathbf{A}^{-1} x$.

Let $\mathbf{A} \in \mathrm{GL}_n(\mathbb{Q})$, we are interested in the parameters' values $\gamma \in \mathbf{V}(\mathbf{P})$ such that there exists a non-empty Zariski open set $\mathcal{O} \subset \mathbb{C}$ such that for any $\vartheta \in \mathcal{O}$, the following holds:

> $\mathfrak{P}_1$: $(\mathbf{h}, \gamma^T X - \vartheta)$ is radical and equidimensional and $\mathcal{V}_{\gamma,\vartheta}$ is smooth; note that by the Jacobian criterion, this is equivalent to saying that at any point of $\mathcal{V}_{\gamma,\vartheta}$, the rank of $(\mathbf{h}, \gamma^T X - \vartheta)$ is $r + 1$.
>
> $\mathfrak{P}_2(\mathbf{A})$: for $1 \le i \le n - r$, the polar variety $W(\mathbf{h}^{\mathbf{A}}, \gamma, \vartheta, i)$ is in Noether position with respect to the projection π_{i-1}.

For those parameters' values for which $\mathfrak{P}_1$ and $\mathfrak{P}_2(\mathbf{A})$ hold, we simply say that $\mathfrak{P}(\mathbf{A})$ holds.

We recall now the statement of [18, Proposition 4.2]. It emphasizes the interest of these properties for polynomial optimization.

PROPOSITION 4.1. *[18, Proposition 4.2] Let $\mathbf{A} \in \mathrm{GL}_n(\mathbb{C})$ and let $\gamma \in \mathbb{C}^n - \{\mathbf{0}\}$ such that $\mathfrak{P}(\mathbf{A})$ holds. Then the following holds:*

- *the algebraic set $\mathcal{C}_\gamma^{\mathbf{A}}$ defined as the Zarsiki closure of*

  $$\cup_{i=1}^{n-r} \left(\left(W(\mathbf{h}^{\mathbf{A}}, \gamma, i) - C(\mathbf{h}^{\mathbf{A}}, \gamma) \right) \cap \pi_{i-1}^{-1}(\mathbf{0}) \right)$$

  has dimension at most 1;

- *the union of $\pi_\gamma(C(\mathbf{h}^{\mathbf{A}}, \gamma))$ and the set of non-properness of the restriction of π_γ to $\mathcal{C}_\gamma^{\mathbf{A}}$ is finite and contains the extremum of the restriction of the map π_γ to $\mathcal{V} \cap \mathbb{R}^n$.*

From [18, Proposition 4.3], for any $\gamma \in \mathbf{V}(\mathbf{P})$ there exists a non-empty Zariski open set $\mathscr{A} \subset \mathrm{GL}_n(\mathbb{C})$ such that for any $\mathbf{A} \in \mathscr{A} \cap \mathrm{GL}_n(\mathbb{Q})$, $\mathfrak{P}(\mathbf{A})$ holds.

However, note that in order to use Proposition 4.1 for parametric optimization, we need to prove a stronger statement: there exists a non-empty Zariski open subset $\mathscr{A} \subset \mathrm{GL}_n(\mathbb{C})$ such that the following holds. For any $\mathbf{A} \in \mathscr{A} \cap \mathrm{GL}_n(\mathbb{Q})$, there exists a Zariski dense subset $U \subset \mathbf{V}(\mathbf{P})$ such that for $\gamma \in U$, $\mathfrak{P}(\mathbf{A})$ holds.

Basically, our algorithm $\mathsf{BasicParametricOptimization}$ identifies a polynomial Z such that $\mathbf{V}(\mathbf{P}) - \mathbf{V}(Z)$ is non-empty and computes a polynomial $\Phi(\mathbf{c}_0, \mathbf{c})$ such that for any $\gamma \in \mathbf{V}(\mathbf{P}) - \mathbf{V}(Z)$, $\Phi(\mathbf{c}_0, \gamma)$ defines the union of the finite algebraic sets in the second item of Proposition 4.1.

This is what we prove below but before doing that we introduce the data-structure and subroutines used by our algorithm.

4.3 Data-structures and subroutines

From now on, $\mathbf{P} \subset \mathbb{Q}[\mathbf{c}]$ is a polynomial sequence that generates a prime ideal $\mathcal{P}$. We denote by $\mathbb{A}$ the integral ring $\mathbb{Q}[\mathbf{c}]/\mathcal{P}$, by $\mathbb{K}$ the fraction field of $\mathbb{A}$ and by $\bar{\mathbb{K}}$ the algebraic closure of $\mathbb{K}$.

Data-structures. Let $\mathbf{F} \subset \mathbb{Q}[X_1, \ldots, X_n]$ that defines a finite algebraic set V in $\mathbb{C}^n$. Then, V can be encoded with a zero-dimensional rational parametrization which is a sequence of polynomials $\mathbf{Q} = (q, q_0, q_1, \ldots, q_n) \subset \mathbb{Q}[U]$, i.e. V is defined by

$$q(U) = 0,\ X_i = q_i(U)/q_0(U),\ q_0(U) \neq 0 \quad \text{for } 1 \le i \le n,$$

with $\mathsf{gcd}(q, q_0) = 1$ and q is unitary and its degree is the cardinality of V.

When $\mathbf{F}$ defines an algebraic curve $V \subset \mathbb{C}^n$, then a rational parametrization for V is a sequence of polynomials $\mathbf{Q} = (q, q_0, q_1, \ldots, q_n) \subset \mathbb{Q}[U, T]$ such that V is the algebraic closure of the set defined by

$$q(U,T) = 0,\ X_i = \frac{q_i(U,T)}{q_0(U,T)},\ q_0(U,T) \neq 0 \quad \text{for } 1 \le i \le n$$

with q unitary in U and T, its degree is the degree of the algebraic curve V and $\gcd(q, q_0) = 1$.

Below, we will also consider polynomial systems in the ring $\mathbb{A}[X_1, \ldots, X_n]$ where $\mathbb{A}$ is an integral ring. We denote by $\mathbb{K}$ the fraction field of $\mathbb{A}$ and by $\bar{\mathbb{K}}$ the algebraic closure of $\mathbb{K}$. Hence, the algebraic sets defined by these polynomial systems lie in $\bar{\mathbb{K}}^n$; whenever they define finite algebraic sets or algebraic curves, they can be encoded with rational parametrizations with coefficients in $\mathbb{K}$, which up to normalization can be turned into rational parameterizations with coefficients in $\mathbb{A}$.

Basic routines. We need to introduce the following routines.

The routine $\mathsf{SingularMinors}$ takes as input $\mathbf{h}$ and $\mathbf{P}$ and it returns $\tilde{\mathbf{G}} = (\mathbf{h}, \mathsf{S}(\mathbf{h}, \mathbf{c}))$.

The routine $\mathsf{SpecialCurve}$ takes as input $\mathbf{h}$ and $\mathbf{P}$ and returns $\tilde{\mathbf{F}} = (\tilde{\mathbf{F}}_1, \ldots, \tilde{\mathbf{F}}_{n-r})$ such that for $1 \leq i \leq n-r$, $\tilde{\mathbf{F}}_i$ is $\mathbf{h}, \mathsf{M}(\mathbf{h}, \mathbf{c}, i)$, $X_1, \ldots, X_{i-1}$.

These systems will allow us to compute parametrized representations of the sets $\mathcal{C}_\gamma$ for γ lying in a Zariski dense subset.

The routine $\mathsf{PointsPerComponents}$ takes as input $\mathbf{h} \in \mathbb{Q}[X_1, \ldots, X_n]$ and it returns a zero-dimensional rational parametrization that encodes a finite set of points contained in $\mathcal{V} = \mathbf{V}(\mathbf{h})$ and meeting all the connected components of $\mathcal{V} \cap \mathbb{R}^n$.

The routine $\mathsf{ValuesTakenByPoly}$ takes as input a zero-dimensional rational parametrization $\mathbf{Q} \subset \mathbb{Q}[U]$ that encodes a finite set of points V in $\mathbb{C}^n$, the sequence of polynomials $\mathbf{P} \subset \mathbb{Q}[\mathbf{c}]$. It returns $\Phi \subset \mathbb{Q}[\mathbf{c}_0, \mathbf{c}]$ and a polynomial $Z \in \mathbb{Q}[\mathbf{c}] - \langle \mathbf{P} \rangle$ such that for $\gamma \in \mathbf{V}(\mathbf{P}) - \mathbf{V}(Z)$, $\Phi(\mathbf{c}_0, \gamma)$ defines the set $\{\gamma^T x \mid x \in V\}$. It essentially consists of substituting the parametrization in the polynomial $\mathbf{c}^T X - \mathbf{c}_0$, clearing the denominators and eliminating the variable U with a resultant computation to get Φ. Note that these computations are done modulo $\mathbf{P}$ (hence in $\mathbb{A}$). Keeping track of exact divisions performed during the resultant computation needed to do this computation (or using specialization theorems, see e.g. [12]) yields the polynomial Z. Note that Z does not belong to $\mathbf{P}$ (else we wouldn't use its factors for performing divisions).

As above, the routine $\mathsf{ParametricValuesTakenByPoly}$ takes as input a zero-dimensional rational parametrization $\mathbf{Q}$ but with coefficients in $\mathbb{A}$, the sequence of polynomials $\mathbf{P} \subset \mathbb{Q}[\mathbf{c}]$. The parametrization $\mathbf{Q}$ encodes a finite set of points V in $\bar{\mathbb{K}}^n$. It returns $\Phi \subset \mathbb{Q}[\mathbf{c}_0, \mathbf{c}]$ and a polynomial $Z \in \mathbb{Q}[\mathbf{c}] - \langle \mathbf{P} \rangle$ such that for $\gamma \in \mathbf{V}(\mathbf{P}) - \mathbf{V}(Z)$, $\Phi(\mathbf{c}_0, \gamma)$ defines the set $\{\gamma^T x \mid x \in V\}$. As $\mathsf{ValuesTakenByPoly}$ does, this routine works using substitutions and resultant computations.

The following lemma is immediate.

LEMMA 4.2. *Let $\mathbf{Q}$ and $\mathbf{P}$ be as above and (Φ, Z) be the output of* $\mathsf{ParametricValuesTakenByPoly}(\mathbf{Q}, \mathbf{P})$. *Then, $Z \notin \langle \mathbf{P} \rangle$.*

Properness. We describe now a routine $\mathsf{CheckProperness}$ that takes as input $\mathbf{h}$, a matrix $\mathbf{A} \in \mathrm{GL}_n(\mathbb{Q})$ and $\mathbf{P} \subset \mathbb{Q}[\mathbf{c}]$ as above.

When there are no generic parameters' values in $\mathbf{V}(\mathbf{P})$ for which $\mathfrak{P}(\mathbf{A})$ holds, the routine $\mathsf{CheckProperness}$ simply returns (0). Else it returns $Z \in \mathbb{Q}[\mathbf{c}] - \langle \mathbf{P} \rangle$ such that for any $\gamma \in \mathbf{V}(\mathbf{P}) - \mathbf{V}(Z)$, property $\mathfrak{P}(\mathbf{A})$ holds.

Roughly speaking, the above routine identifies those parameters' values γ for which $\mathfrak{P}(\mathbf{A})$ holds.

LEMMA 4.3. *We use the above notation and assumptions. Then, there exists a non-empty Zariski open set $\mathscr{A} \subset \mathrm{GL}_n(\mathbb{C})$ such that for any $\mathbf{A} \in \mathscr{A} \cap \mathrm{GL}_n(\mathbb{Q})$ the following holds.*

Let Z be the output of $\mathsf{CheckProperness}(\mathbf{h}, \mathbf{A}, \mathbf{P})$. *Then, $\mathbf{V}(\mathbf{P}) - \mathbf{V}(Z)$ is Zariski dense in $\mathbf{V}(\mathbf{P})$ and for $\gamma \in \mathbf{V}(\mathbf{P}) - \mathbf{V}(Z)$, property $\mathfrak{P}(\mathbf{A})$ holds.*

PROOF. Note that by construction, under our assumptions, $Z \notin \langle \mathbf{P} \rangle$. Hence, since $\langle \mathbf{P} \rangle$ is prime, $\mathbf{V}(\mathbf{P}) - \mathbf{V}(Z)$ is Zariski dense in $\mathbf{V}(\mathbf{P})$.

It remains to prove that there exists a non-empty Zariski open set $\mathscr{A} \subset \mathrm{GL}_n(\mathbb{C})$ such that for any $\mathbf{A} \in \mathscr{A} \cap \mathrm{GL}_n(\mathbb{Q})$ and $\gamma \in \mathbf{V}(\mathbf{P}) - \mathbf{V}(Z)$, property $\mathfrak{P}(\mathbf{A})$ holds.

By [18, Proposition 4.3], for any $\gamma \in \mathbf{V}(\mathbf{P})$, there exists a non-empty Zariski open set $\mathscr{A} \in \mathrm{GL}_n(\mathbb{C})$ such that $\mathfrak{P}(\mathbf{A})$ holds for γ. We prove below that there exists $Z \notin \langle \mathbf{P} \rangle$ such that for any $\mathbf{A} \in \mathscr{A}$ and $\gamma' \in \mathbf{V}(\mathbf{P}) - \mathbf{V}(Z)$, $\mathfrak{P}(\mathbf{A})$ holds for γ'.

Consider a minimal Gröbner basis G of the ideal generated by $\langle \mathbf{h}, \mathbf{c}^T X - \mathbf{c}_0 \rangle$ and all $(r+1)$-minors of $\mathsf{Jac}(\mathbf{h}, \mathbf{c}^T X - \mathbf{c}_0)$ with $\mathbb{K}(\mathbf{c}_0)$ as a ground field. We claim that G is $\langle 1 \rangle$. Indeed, if it was not the case, this would imply that for any $\gamma \in \mathbf{V}(\mathbf{P})$ which does not cancel the finitely many denominators that appear in a computation of G, $\mathfrak{P}_1$ does not hold; hence a contradiction. We deduce that G is $\langle 1 \rangle$ as claimed and let Z' be the product of all denominators appearing during the computation of G.

Now let $\mathfrak{A}$ be an $n \times n$ matrix with entries $\mathfrak{A}_{i,j}$ as indeterminates.

By [33], one can ensure Noether position properties by setting the non-vanishing of some denominators of a minimal reduced Gröbner basis of the ideals generated by

$$\mathbf{h}^{\mathfrak{A}}, \mathsf{M}(\mathbf{h}^{\mathfrak{A}}, \mathbf{c}, i), \mathbf{c}^T \mathfrak{A} X - \mathbf{c}_0$$

with $\mathbb{K}(\mathfrak{A}_{i,j})$ as a ground field. The coefficients of these denominators lie in $\mathbb{K}$ which contains $\mathbb{Q}$. Now, we define the Zariski open set $\mathscr{A} \subset \mathrm{GL}_n(\mathbb{C})$ by the non-vanishing of the coefficients of the monomials in $\mathbf{c}, \mathbf{c}_0$. This set is non-empty because, as above, it would contradict that for any $\gamma \in \mathbf{V}(\mathbf{P})$, $\mathfrak{P}(\mathbf{A})$ holds for $\mathbf{A}$ generic.

Now, remark that for $\mathbf{A} \in \mathscr{A}$, one can define Z'' as the denominators that appear in the computation of the minimal reduced Gröbner basis of the ideals generated by

$$\mathbf{h}^{\mathbf{A}}, \mathsf{M}(\mathbf{h}^{\mathbf{A}}, \mathbf{c}, i), \mathbf{c}^T \mathbf{A} X - \mathbf{c}_0$$

with $\mathbb{K}$ as a ground field.

Taking $Z = Z'Z''$ ends the proof. $\square$

Rational parametrizations. Consider a sequence of polynomials polynomials $\mathbf{F} = (f_1, \ldots, f_s)$ and $\mathbf{G} = (g_1, \ldots, g_k)$ in $\mathbb{A}[X_1, \ldots, X_n]$ and $I \subset \mathbb{K}[X_1, \ldots, X_n]$ be the saturation of $\langle \mathbf{F} \rangle$ by $\langle \mathbf{G} \rangle$; we denote it by $\langle \mathbf{F} \rangle : \langle \mathbf{G} \rangle^\infty$. We assume that I has dimension 1 and is equidimensional.

We are interested in studying the complex solutions of $\mathbf{F}$ that are not solutions of $\mathbf{G}$ where the admissible values for the parameters $\mathbf{c}$ lie in the irreducible algebraic set associated to $\mathcal{P}$. We consider a routine $\mathsf{ParametricCurveRepresentation}$ that takes as input $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{P}$ and which returns a finite sequence of polynomials $\mathbf{Q} = (q, q_0, q_1, \ldots, q_n) \subset \mathbb{K}[U, T]$ and a polynomial Z in $\mathbb{Q}[\mathbf{c}] - \mathcal{P}$ such that for any $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbf{V}(\mathbf{P}) - \mathbf{V}(Z)$ the curve associated to $\langle \mathbf{F}_\gamma \rangle : \langle \mathbf{G} \rangle^\infty$ is the Zariski closure of the set defined by

$$q_\gamma(U, T) = 0, \quad X_i = q_{i,\gamma}(U, T)/q_{0,\gamma}(U, T), \quad q_{0,\gamma}(U, T) \neq 0$$

for $1 \leq i \leq n$ (where q_γ and $q_{i,\gamma}$ denote the polynomials of $\mathbf{Q}$ obtained by instantiating $\mathbf{c}$ to γ in q and $q_{i,\gamma}$ for $i \in \{0, \ldots, n\}$).

LEMMA 4.4. *Let $\mathbf{F}$ and $\mathbf{G}$ be as above and let $(\mathbf{Q}, Z)$ be the output of* $\mathsf{ParametricCurveRepresentation}(\mathbf{F}, \mathbf{G}, \mathbf{P})$. *Then the Krull dimension of $\langle \mathbf{P} \rangle + \langle Z \rangle$ is less than the Krull dimension of $\langle \mathbf{P} \rangle$.*

PROOF. Without loss of generality, one can assume that we are in generic coordinates. Using Gröbner bases with $\mathbb{K}$ as ground field and linear algebra in $\mathbb{K}(X_1)[X_2, \ldots, X_n]$ one can compute a rational parametrization of I (see e.g. [11, 4]). During this computation, some polynomials (which are not 0 modulo $\mathcal{P}$ by construction) are used to perform divisions. Taking Z as the product of these polynomials is a valid output and since these polynomials are not 0 modulo $\mathcal{P}$, Z is not. Since $\mathcal{P}$ is prime, we deduce that $\mathcal{P} + \langle Z \rangle$ has dimension less than the dimension of $\mathcal{P}$. $\square$

Reusing the above notations and $\mathsf{ParametricCurveRepresentation}$, it is straightforward to obtain a routine $\mathsf{UnionParametricCurve}$ that takes as input a sequence of sequences of polynomials $\mathbf{F} = (\mathbf{F}_1, \ldots, \mathbf{F}_l)$ a sequence of polynomials $\mathbf{G}$ and $\mathbf{P}$ such that the ideal $\langle \mathbf{F}_k \rangle : \langle \mathbf{G} \rangle^\infty \subset \mathbb{K}[X_1, \ldots, X_n]$ has dimension 1 and is equidimensional for $1 \leq k \leq l$. It returns a finite sequence of polynomials $\mathbf{Q} = (q, q_0, q_1, \ldots, q_n) \subset \mathbb{K}[U, T]$ and a polynomial Z in $\mathbb{Q}[\mathbf{c}] - \mathcal{P}$ such that for any $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbf{V}(\mathbf{P}) - \mathbf{V}(Z)$ the curve associated to $\bigcap_{k=1}^{l} \langle \mathbf{F}_{k\gamma} \rangle : \langle \mathbf{G} \rangle^\infty$ is the Zariski closure of the set defined by

$$q_\gamma(U, T) = 0, \quad X_i = q_{i,\gamma}(U, T)/q_{0,\gamma}(U, T) \quad q_{0,\gamma}(U, T) \neq 0$$

for $1 \leq i \leq n$ (where q_γ and $q_{i,\gamma}$ denotes the polynomials of $\mathbf{Q}$ obtained by instantiating $\mathbf{c}$ to γ in q and $q_{i,\gamma}$ for $i \in \{0, \ldots, n\}$). The following lemma is an immediate consequence of Lemma 4.4.

LEMMA 4.5. *Let* $\mathbf{F}$ *and* $\mathbf{G}$ *be as above and let* $(\mathbf{Q}, Z)$ *be the output of* $\mathsf{UnionCurveParametric}(\mathbf{F}, \mathbf{G}, \mathbf{P})$. *Then the Krull dimension of* $\langle \mathbf{P} \rangle + \langle Z \rangle$ *is less than the Krull dimension of* $\langle \mathbf{P} \rangle$.

Intersection of a curve with a variety. We describe now the routine ParametricIntersection which takes as input a one-dimensional rational parametrization $\mathbf{Q}$ of a curve $C \subset \bar{\mathbb{K}}^n$, a polynomial sequence $\mathbf{G} \in \mathbb{A}[X_1, \ldots, X_n]$ and $\mathbf{P}$. The sequence $\mathbf{G}$ defines an algebraic set H in $\bar{\mathbb{K}}^n$. Assume that the intersection of C and H is finite. Then, following [14] one can compute a parametric zero-dimensional rational parametrization $\mathbf{Q}'$ that encodes $C \cap H$. This is done by substituting in $\mathbf{G}$ the parametrizations of the X_i's hence reducing the computation to computing the intersection defined by the vanishing of two bivariate polynomials with coefficients in $\mathbb{K}$ (using resultant computations). Again, keeping track of the denominators appearing during the computation or using specialization theorems, one can finally return a parametric zero-dimensional parametrization $\mathbf{Q}$ and a polynomial $Z \notin \langle \mathbf{P} \rangle$ such that for any $\gamma \in \mathbf{V}(\mathbf{P}) - \mathbf{V}(Z)$, $\mathbf{Q}'_\gamma$ encodes $C_\gamma \cap H_\gamma$.

LEMMA 4.6. *Let* $\mathbf{Q}$, $\mathbf{G}$ *and* $\mathbf{P}$ *as above and* (Φ, Z) *be the output of* $\mathsf{ParametricIntersection}(\mathbf{Q}, f, \mathbf{P})$. *Then,* $\dim(\langle \mathbf{P} \rangle + \langle Z \rangle) < \dim(\mathbf{P})$.

Computing a set of non-properness. Let $\mathbf{Q}$ be a one dimensional rational parametrization with coefficients in $\mathbb{A}$; it defines an algebraic curve $C_1 \subset \bar{\mathbb{K}}^n$. Then, there exists a Zariski dense subset U of $\mathbf{V}(\mathbf{P})$ such that for $\gamma \in U$, $V(\mathbf{Q}_\gamma)$ defines an algebraic curve C_γ.

The routine ParametricSetOfNonProperness computes (Φ, Z) such that $\mathbf{V}(\mathbf{P}) - \mathbf{V}(Z) \subset U$ and is non-empty and such that for $\gamma \in \mathbf{V}(\mathbf{P}) - \mathbf{V}(Z)$, $\Phi(\mathbf{c}_0, \gamma)$ is not 0 and its set of roots contains the set of non-properness of the restriction of the map $x \to \gamma^T x$ to C_γ.

We denote by C_2 the algebraic curve $\{(\gamma_0, x) \mid x \in C_1$ and $\gamma_0 = \mathbf{c}^T x\}$. We also denote by $\mathfrak{C}_1$ the projective closure of C_1 in $\mathbb{P}^n(\bar{\mathbb{K}})$. For $x = (x_0 : x_1 : \cdots : x_n) \in \mathfrak{C}_1$ with $x_0 \neq 0$, we denote by $\tilde{x}$ the point $\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right)$ and by $\mathfrak{C}_2$ the quasi-projective set $\{(\gamma_0, x) \mid x \in \mathfrak{C}_1$ and $\mathbf{c}^T \tilde{x} = \gamma_0\}$.

Following algorithm given in [9] our routine reduces to the following steps:

- compute a representation of the projective closure $\mathfrak{C}_2$; this can be done using Gröbner bases with $\mathbb{K}$ as a ground field;
- compute the intersection of $\mathfrak{C}_2$ with the hyperplane at infinity defined by $X_0 = 0$; this is a finite set of points and again it can be done using Gröbner bases with $\mathbb{K}$ as a ground field.

Keeping track of all denominators that appear during the computations yields the polynomial $Z \notin \langle \mathbf{P} \rangle$ as above.

The following lemma is immediate.

LEMMA 4.7. *Let* $\mathbf{Q}$ *and* $\mathbf{P}$ *be as above and let* (Φ, Z) *be the output of* $\mathsf{ParametricSetOfNonProperness}(\mathbf{Q}, \mathbf{P})$. *Then* $\dim(\langle \mathbf{P} \rangle + \langle Z \rangle) < \dim(\mathbf{P})$.

4.4 Basic routine for parametric optimization

We describe now our basic subroutine BasicParametricOptimization. It can be seen as a parametrized version of Algorithm SetContainingLocalExtrema in [18].

This latter algorithm consists in reducing the problem of computing the optimum of a polynomial function restricted to a real algebraic set $\mathcal{V} \cap \mathbb{R}^n$ to the problem of computing the optimum of the same polynomial function restricted to a curve. Obviously, this is done in such a way that both optimization problems share the same optimum.

We describe the main steps and refer to the steps of algorithm BasicParametricOptimization corresponding to their parametric variants. The algorithm starts by computing sample points in $\mathcal{V} \cap \mathbb{R}^n$ and gets *(i)* the values attained by the polynomial function to optimize at those points (this corresponds to Steps 1-2). Next, it computes representations of linear sections of polar varieties that define algebraic curves (Step 5-6). Finally, it computes *(ii)* the set of non-properness of the restriction of the considered function to the curve (Step 7) and gets *(iii)* the critical values of this function restricted to the curve (Step 8-9). All this is done in such a way that the optimum lies in the set of values *(i)*, *(ii)* and *(iii)*.

Input: $\mathbf{h} = (h_1, \ldots, h_p) \subset \mathbb{Q}[X_1, \ldots, X_n]$ and $\mathbf{P} \subset \mathbb{Q}[\mathbf{c}]$
Properties: $\mathbf{P}$ generates a prime ideal and $\langle h_1, \ldots, h_p \rangle$ generates a radical equidimensional ideal defining a smooth algebraic set.
Output: (Φ, Z) such that

- $\Phi \in \mathbb{Q}[\mathbf{c}_0, \mathbf{c}]$ and $Z \in \mathbb{Q}[\mathbf{c}]$;
- Z is not 0 modulo $\langle \mathbf{P} \rangle$;
- For any $\gamma \in \mathbf{V}(\mathbf{P}) - \mathbf{V}(Z)$ such that $Z(\gamma) \neq 0$, $\Phi(\mathbf{c}_0, \gamma)$ is not zero and its set of roots contains the optimum of the restriction of π_γ to $\mathcal{V} \cap \mathbb{R}^n$.

$\mathsf{BasicParametricOptimization}(\mathbf{h}, \mathbf{P})$

1. $R = \mathsf{PointsPerComponents}(\mathbf{h})$
2. $(\Phi_0, Z_0) = \mathsf{ValuesTakenByPoly}(R, \mathbf{c}^T X, \mathbf{P})$
3. Choose randomly $\mathbf{A} \in \mathrm{GL}_n(\mathbb{C})$
4. $Z'_0 = \mathsf{CheckProperness}(\mathbf{h}, \mathbf{A}, \mathbf{P})$
5. $\tilde{\mathbf{F}} = \mathsf{SpecialCurve}(\mathbf{h}^{\mathbf{A}}, \mathbf{P})$, $\tilde{\mathbf{G}} = \mathsf{SingularMinors}(\mathbf{h}^{\mathbf{A}}, \mathbf{P})$
6. $(R_1, Z_1) = \mathsf{UnionParametricCurve}(\tilde{\mathbf{F}}, \tilde{\mathbf{G}}, \mathbf{P})$
7. $(\Phi_1, Z'_1) = \mathsf{ParametricSetofNonProperness}(R_1, \mathbf{P}, \mathbf{c}^T X)$
8. $R_2 = \mathsf{ParametricIntersection}(R_1, \tilde{\mathbf{G}}, \mathbf{P})$
9. $(\Phi_2, Z_2) = \mathsf{ParametricValuesTakesByPoly}(R_2, \mathbf{c}^T X, \mathbf{P})$
10. Take $Z = Z_0 Z'_0 Z_1 Z'_1 Z_2$ and $\Phi = \Phi_0 \Phi_1 \Phi_2$
11. return (Φ, Z)

THEOREM 4.8. *Let* $\mathbf{h}$ *and* $\mathbf{P}$ *be as above and* (Φ, Z) *be the output of* $\mathsf{BasicParametricOptimization}(\mathbf{h}, \mathbf{P})$. *Then,* $Z \notin \langle \mathbf{P} \rangle$ *and its output is correct.*

PROOF. The fact that $Z \notin \langle \mathbf{P} \rangle$ is an immediate consequence of the fact that $\langle \mathbf{P} \rangle$ is prime and that its factors Z_0, Z'_0, Z_1, Z'_1 and Z_2 do not belong to $\langle \mathbf{P} \rangle$ from Lemmata 4.2, 4.3, 4.5, 4.6 and 4.7.

It remains to prove the correctness of the output. Since $\mathbf{A}$ is chosen at random at Step 3, one can assume that $\mathbf{A}$ belongs to the non-empty Zariski open set $\mathscr{A}$ defined in Lemma 4.3.

Now, remark that for any $\gamma \in \mathbf{V}(\mathbf{P}) - \mathbf{V}(Z)$. By Lemma 4.3, property $\mathfrak{P}(\mathbf{A})$ holds.

Hence, without loss of generality one can assume that algorithm SetContainingLocalExtrema in [18] runs by choosing the matrix $\mathbf{A}$ selected at Step 3 of BasicParametricOptimization. On input $\gamma^T X$ and $\mathbf{h}$ the output of SetContainingLocalExtrema is a polynomial $\varphi \in \mathbb{Q}[\mathbf{c}_0]$ whose set of roots contains the optimum of the restriction of the map $x \to \gamma^T x$ to $\mathcal{V} \cap \mathbb{R}^n$. From Lemmata 4.2, 4.5, 4.6 and 4.7 this polynomial φ is exactly $\Phi(\mathbf{c}_0, \gamma)$. Hence, correctness of algorithm SetContainingLocalExtrema [18, Proposition 4.2] implies the one of BasicParametricOptimization. $\square$

4.5 Recursive procedure

We present now our recursive procedure. It uses the routine BasicParametricOptimization presented above. It also uses a routine PrimeDecomposition which takes as input a polynomial family $\mathbf{P} \subset \mathbb{Q}[\mathbf{c}]$ and a polynomial $Z \in \mathbb{Q}[\mathbf{c}]$. It returns polynomial families $\mathbf{P}_1, \ldots, \mathbf{P}_k$ such that

$$\sqrt{\langle \mathbf{P} \rangle + \langle Z \rangle} = \cap_{i=1}^k \langle \mathbf{P}_i \rangle$$

and $\langle \mathbf{P}_i \rangle$ is prime for $1 \leq i \leq k$.

Input: $\mathbf{h} = (h_1, \dots, h_p) \subset \mathbb{Q}[X_1, \dots, X_n]$ and $\mathbf{P}_0 \subset \mathbb{Q}[\mathbf{c}]$
Properties: $\mathbf{P}_0$ generates a prime ideal and $\langle h_1, \dots, h_p \rangle$ generates a radical equidimensional ideal defining a smooth algebraic set.
Output: a list of triples $(\Phi, Z, \mathbf{P})$ such that

- $\mathbf{P} \subset \mathbb{Q}[\mathbf{c}]$ generates a prime ideal; $Z \in \mathbb{Q}[\mathbf{c}] - \langle \mathbf{P} \rangle$ and $\Phi \in \mathbb{Q}[\mathbf{c}_0, \mathbf{c}]$;
- for any $\gamma \in \mathbf{V}(\mathbf{P}) - \mathbf{V}(Z)$, $\Phi(\mathbf{c}_0, \gamma)$ is not identically 0 and its set of roots contains the optimum of the restriction of the map $x \to \gamma^T x$ to $\mathcal{V} \cap \mathbb{R}^n$

and the union of the algebraic sets defined by the families $\mathbf{P}$ in the output is $\mathbf{V}(\mathbf{P}_0)$.

When calling this recusive algorithm with input $\mathbf{h}$ and $(0) \in \mathbb{Q}[\mathbf{c}]$ we get a list of triples $(\Phi_i, Z_i, \mathbf{P}_i)$ for $1 \leq i \leq k$ such that $\cup_{i=1}^{k} \mathbf{V}(\mathbf{P_i})$ is the whole parameters' space. Remark that with the above properties of the output, given $\gamma \in \mathbb{C}^n$, the optimum of the restrition of the map $x \to \gamma^T x$ is a root of the non-zero polynomial $\Phi_i(\mathbf{c}_0, \gamma)$ if $\gamma \in \mathbf{V}(\mathbf{P}_i) - \mathbf{V}(Z_i)$.

$\mathsf{ParametricOptimizationRec}(\mathbf{h}, \mathbf{P}_0)$

1. if $\langle \mathbf{P}_0 \rangle = \langle 1 \rangle$ then return []
2. $(\Phi, Z) = \mathsf{BasicParametricOptimization}(\mathbf{h}, \mathbf{P}_0)$
3. $(\mathbf{P}_1, \dots, \mathbf{P}_k) = \mathsf{PrimeDecomposition}(\mathbf{P}, Z)$
4. Let $\mathbf{L}_i = \mathsf{ParametricOptimizationRec}(\mathbf{h}, \mathbf{P}_i)$ for $1 \leq i \leq k$
5. return the union of $(\Phi, Z, \mathbf{P}_0)$ with $\mathbf{L}_1, \dots, \mathbf{L}_i$

THEOREM 4.9. *Algorithm* ParametricOptimizationRec *terminates and is correct.*

PROOF. Correctness follows straightforwardly from an induction on the depth of the recursion and the correctness of BasicParametricOptimization (see Theorem 4.8).

We prove now termination. Using again Theorem 4.8, note that the polynomial Z obtained at Step 2 is such that $\dim(\langle \mathbf{P}_0 \rangle + \langle Z \rangle) < \dim(\mathbf{P}_0)$ since $\mathbf{P}_0$ is prime and $Z \notin \langle \mathbf{P}_0 \rangle$. We deduce that at each recursive call, the dimension decreases which ends the proof. ☐

EXAMPLE 3.3 (CONTINUED) Since $\langle \mathbf{c}_2\mathbf{c}_4, \mathbf{c}_3\mathbf{c}_4, \mathbf{c}_3\mathbf{c}_2\mathbf{c}_1 \rangle$ represents the bad parameters' values for Algorithm 3.1, we need to consider its prime components $\langle \mathbf{c}_1, \mathbf{c}_4 \rangle$, $\langle \mathbf{c}_2, \mathbf{c}_4 \rangle$, $\langle \mathbf{c}_3, \mathbf{c}_4 \rangle$ and $\langle \mathbf{c}_2, \mathbf{c}_3 \rangle$ in our recursive procedure described above. Due to the limit of space, we do not provide all details. We only present the results obtained with $\mathbf{P} = \langle \mathbf{c}_2, \mathbf{c}_3 \rangle$, especially those of Step 7 in the subroutine BasicParametricOptimization but all the computations take a few minutes using Macaulay2 while the best implementations of CAD don't tackle this example. Following the paragraph on computing sets of non-properness, we obtain that the square-free parts of Φ_1 and $Z_1 Z_1'$ are respectively $\mathbf{c}_0\mathbf{c}_1\mathbf{c}_4$ and $\mathbf{c}_1\mathbf{c}_4$. Thus, we need recursive routines with $\mathbf{P}_1 = \langle \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3 \rangle$ and $\mathbf{P}_2 = \langle \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4 \rangle$, repectively. In both cases, all parameters are instantiated; the objective function is X_4 in the case of $\mathbf{P}_1$ and X_1 in the case of $\mathbf{P}_2$. In the case of $\mathbf{P}_1$, the problem is reduced to the non-parametric optimization problem with the objective X_4. Running the algorithm in [18], we get $\Phi_1 = \mathbf{c}_0$ which represents the asymptotic optimum that we are concerned about.

5. REFERENCES

[1] B. Bank, M. Giusti, J. Heintz, and L.-M. Pardo. Generalized polar varieties: Geometry and algorithms. *Journal of complexity*, 21(4):377–412, 2005.

[2] B. Bank, M. Giusti, J. Heintz, and M. Safey El Din. Intrinsic complexity estimates in polynomial optimization. *J. Complexity*, 30(4):430–443, 2014.

[3] B. Bank, M. Giusti, J. Heintz, M. Safey El Din, and E. Schost. On the geometry of polar varieties. *Applicable Algebra in Engineering, Communication and Computing*, 21(1):33–83, 2010.

[4] E. Becker, T. Mora, M. G. Marinari, and C. Traverso. The shape of the shape lemma. In *Proceedings of the international symposium on Symbolic and algebraic computation*, pages 129–133. ACM, 1994.

[5] D. P. C. Garcia and M. Morari. Model predictive control: theory and practice - a survey. *Automatica*, 25:335 – 348, 1989.

[6] F. Catanese, S. Hosten, A. Khetan, and B. Sturmfels. The maximum likelihood degree. *American Journal of Mathematics*, 128(3):671–677, 2006.

[7] C. Chen, O. Golubitsky, F. Lemaire, M. M. Maza, and W. Pan. Comprehensive triangular decomposition. In *Computer Algebra in Scientific Computing, 10th International Workshop, CASC 2007, Bonn, Germany, September 16-20, 2007, Proceedings*, pages 73–101, 2007.

[8] G. Collins. Quantifier elimination for real closed fields by cylindrical algebraic decompostion. volume 33 of *Lecture Notes in Computer Science*, pages 134–183. 1975.

[9] D. Cox, J. Little, and D. O'Shea. *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*. Undergraduate Texts in Mathematics. Springer, Springer Science+Business Media, LLC, 233 Spring Street, New York, NY 10013, USA, 3rd edition, 2007.

[10] J. Della Dora, C. Discrescenzo, and D. Duval. About a new method method for computing in algebraic number fields. In *EUROCAL 85 Vol. 2*, volume 204 of *LNCS*, pages 289–290. Springer, 1985.

[11] J.-C. Faugère, P. Gianni, D. Lazard, and T. Mora. Efficient computation of zero-dimensional Gröbner bases by change of ordering. *Journal of Symbolic Computation*, 16(4):329–344, 1993.

[12] J. v. Gathen and J. Gerhard. *Modern computer algebra*. Cambridge University Press, 1999.

[13] I. Gelfand, I. Gelfand, M. Kapranov, and A. Zelevinsky. *Discriminants, Resultants, and Multidimensional Determinants*. Mathematics (Birkhäuser). Birkhäuser Boston, 2008.

[14] M. Giusti, G. Lecerf, and B. Salvy. A Gröbner-free alternative for polynomial system solving. *Journal of Complexity*, 17(1):154–211, 2001.

[15] H.-C. Graf von Bothmer and K. Ranestad. A general formula for the algebraic degree in semidefinite programming. *Bulletin of the London Mathematical Society*, 41(2):193–197, 2009.

[16] A. Greuet, F. Guo, M. Safey El Din and L. Zhi. Global optimization of polynomials restricted to a smooth variety using sums of squares. *Journal of Symbolic Computation*, 47(5), 503-518, 2012.

[17] A. Greuet and M. Safey El Din. Deciding reachability of the infimum of a multivariate polynomial. In Proceedings of the 36th international symposium on Symbolic and algebraic computation, ISSAC '11, pages 131–138, New York, NY, USA, 2011. ACM.

[18] A. Greuet and M. Safey El Din. Probabilistic algorithm for polynomial optimization over a real algebraic set. *SIAM Journal on Optimization*, 24(3):1313–1343, 2014.

[19] F. Guo, C. Wang, and L. Zhi. Optimizing a linear function over a noncompact real algebraic variety. In *Proceedings of the 2014 Symposium on Symbolic-Numeric Computation*, SNC '14, pages 39–40, New York, NY, USA, 2014. ACM.

[20] S. Hosten, A. Khetan, and B. Sturmfels. Solving the likelihood equations. *Foundations of Computational Mathematics*, 5(4):389–407, 2005.

[21] Z. Jelonek and K. Kurdyka. Quantitative generalized bertini-sard theorem for smooth affine varieties. *Discrete & Computational Geometry*, 34(4):659–678, 2005.

[22] K. Kurdyka, P. Orro, and S. Simon. Semialgebraic sard theorem for generalized critical values. *Journal Differential Geom*, 56(1):67–92, 2000.

[23] X. Li, M. M. Maza, and W. Pan. Computations modulo regular chains. In J. R. Johnson, H. Park, and E. Kaltofen, editors, *Symbolic and Algebraic Computation, International Symposium, ISSAC 2009, Seoul, Republic of Korea, July 29-31, 2009, Proceedings*, pages 239–246. ACM, 2009.

[24] J. Nie and K. Ranestad. Algebraic degree of polynomial optimization. *SIAM Journal on Optimization*, 20(1):485–502, 2009.

[25] J. Nie, K. Ranestad, and B. Sturmfels. The algebraic degree of semidefinite programming. *Mathematical Programming*, 122(2):379–405, 2010.

[26] P. J. Rabier. Ehresmann fibrations and Palais-Smale conditions for morphisms of Finsler manifolds. *Annals of Mathematics*, 146(3):647–691, 1997.

[27] K. Ranestad. Algebraic degree in semidefinite and polynomial optimization. In M. F. Anjos and J. B. Lasserre, editors, *Handbook on Semidefinite, Conic and Polynomial Optimization*, volume 166 of *International Series in Operations Research & Management Science*, pages 61–75. Springer US, 2012.

[28] R. Rockafellar. *Convex Analysis*. Convex Analysis. Princeton University Press, 1970.

[29] P. Rostalski and B. Sturmfels. Dualities in convex algebraic geometry. *Rendiconti di Matematica, Serie VII*, 30:285–327, 2010.

[30] P. Rostalski and B. Sturmfels. Dualities. In G. Blekherman, P. A. Parrilo, and R. R. Thomas, editors, *Semidefinite Optimization and Convex Algebraic Geometry*, MOS-SIAM Series on Optimization, chapter 5, pages 203–250. Society for Industrial and Applied Mathematics, Philadelphia, PA, 2012.

[31] M. Safey El Din. Computing the global optimum of a multivariate polynomial over the reals. In *Proceedings of ISSAC 2008*, pages 71–78, 2008.

[32] M. Safey El Din and E. Schost. Polar varieties and computation of one point in each connected component of a smooth real algebraic set. In J. Sendra, editor, *Proceedings of ISSAC 2003*, pages 224–231. ACM Press, aug 2003.

[33] M. Safey El Din and E. Schost. Properness defects of projections and computation of at least one point in each connected component of a real algebraic set. *Discrete & Computational Geometry*, 32(3):417–430, 2004.

[34] M. Safey El Din and L. Zhi. Computing rational points in convex semialgebraic sets and sum of squares decompositions. *SIAM Journal on Optimization*, 20(6):2876–2889, 2010.

[35] R. Sinn. Algebraic boundaries of convex semi-algebraic sets. 2014. URL http://arxiv.org/pdf/1405.7822v2.pdf.

[36] H. Vui and P. Sòn. Representations of positive polynomials and optimization on noncompact semialgebraic sets. *SIAM Journal on Optimization*, 20(6):3082–3103, 2010.

[37] V. Weispfenning. Comprehensive Gröbner bases. *Journal of Symbolic Computation*, 14(1):1–29, 1992.