# Practical and theoretical issues for the computation of generalized critical values of a polynomial mapping


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**Abstract.** Let $f \in \mathbb{Q}[X_1, \ldots, X_n]$ be a polynomial of degree $D$. Computing the set of *generalized critical values* of the mapping $\widetilde{f} : x \in \mathbb{C}^n \rightarrow f(x) \in \mathbb{C}$ (i.e. $\{c \in \mathbb{C} \mid \exists (x_k)_{k\in\mathbb{N}} \quad f(x_k) \rightarrow c$ and $||x_k||.||d_{x_k} f|| \rightarrow 0$ when $k \rightarrow \infty\}$) is an important step in algorithms computing sampling points in semi-algebraic sets defined by a single inequality.

A previous algorithm allows us to compute the set of generalized critical values of $\widetilde{f}$. This one is based on the computation of the critical locus of a projection on a plane $P$. This plane $P$ must be chosen such that some global properness properties of some projections are satisfied. These properties, which are generically satisfied, are difficult to check in practice. Moreover, choosing randomly the plane $P$ induces a growth of the coefficients appearing in the computations.

We provide here a new certified algorithm computing the set of generalized critical values of $\widetilde{f}$. This one is still based on the computation of the critical locus on a plane $P$. The certification process consists here in checking that this critical locus has dimension 1 (which is easy to check in practice), without any assumption of global properness. Moreover, this allows us to limit the growth of coefficients appearing in the computations by choosing a plane $P$ defined by sparse equations. Additionally, we prove that the degree of this critical curve is bounded by $(D-1)^{n-1} - \mathfrak{d}$ where $\mathfrak{d}$ is the sum of the degrees of the positive dimensional components of the ideal $\langle \frac{\partial f}{\partial X_1}, \ldots, \frac{\partial f}{\partial X_n} \rangle$.

We also provide complexity estimates on the number of arithmetic operations performed by a probabilistic version of our algorithm.

Practical experiments at the end of the paper show the relevance and the importance of these results which improve significantly in practice previous contributions.


## 1 Introduction

Consider $f \in \mathbb{Q}[X_1, \ldots, X_n]$ of degree $D$ and the mapping $\widetilde{f} : x \in \mathbb{C}^n \rightarrow f(x)$. The set of generalized critical values of $\widetilde{f}$ is defined as the set of points $c \in \mathbb{C}$

such that there exists a sequence of points $(x_k)_{k\in\mathbb{N}}$ such that $f(x_k) \to c$ and $||x_k|||d_{x_k}f||| \to 0$ when $k$ tends to $\infty$ (see [20]). From [14, 20], this set of points contains:

- the classical set of critical values, i.e. the set of roots of the polynomial generating the principal ideal: $\langle f - T, \frac{\partial f}{\partial X_1}, \ldots, \frac{\partial f}{\partial X_n}\rangle \cap \mathbb{Q}[T]$;
- the set of asymptotic critical values which is the set of complex numbers for which there exists a sequence of points $(x_k)_{k\in\mathbb{N}} \subset \mathbb{C}^n$ such that $||x_k||$ tends to $\infty$ and $\left|\left|\left(X_i \frac{\partial f}{\partial X_j}\right)(x_k)\right|\right|$ tends to 0 when $k$ tends to $\infty$ for all $(i, j) \in \{1, \ldots, n\} \times \{1, \ldots, n\}$.

In this paper, we provide an efficient algorithm allowing us to compute the set of *generalized critical values* of the polynomial mapping $\widetilde{f}$.

*Motivation and description of the problem.* The interest of computing asymptotic critical values of a polynomial mapping comes from the following result which is proved in [28]: *Let $f \in \mathbb{Q}[X_1, \ldots, X_n]$, and $e \in ]0, e_0[$ where $e_0$ is less than the smallest positive generalized critical value of the mapping $x \to f(x)$. If there exists $x \in \mathbb{R}^n$ such that $f(x) = 0$ then each connected component of the semi-algebraic set defined by $f > 0$ contains a connected component of the real algebraic set defined by $f - e = 0$.* Thus, computing generalized critical values is a preliminary step of efficient algorithms computing sampling points in a semi-algebraic set defined by a single inequality, testing the positivity of a given polynomial, etc. In [28], the computation of generalized critical values is also used to decide if a given hypersurface contains real regular points. Once generalized critical values are computed, it remains to compute at least one point in each connected component in a real hypersurface which can be tackled using algorithms relying on the critical point method introduced in [13] (see also [26] and [25] for recent developments leading to practical efficiency).

Given $\mathbf{A} \in GL_n(\mathbb{C})$, we denote by $f^{\mathbf{A}}$ the polynomial $f(\mathbf{AX})$. In [28], the following result is proved (see [28, Theorem 3.6]): *There exists a Zariski-closed subset $\mathcal{A} \subsetneq GL_n(\mathbb{C})$ such that for all $\mathbf{A} \in GL_n(\mathbb{Q}) \setminus \mathcal{A}$, the set of asymptotic critical values of $x \to f(x)$ is contained in the set of non-properness of the projection on $T$ restricted to the Zariski-closure of the constructible set defined by $f^{\mathbf{A}} - T = \frac{\partial f^{\mathbf{A}}}{\partial X_1} = \cdots = \frac{\partial f^{\mathbf{A}}}{\partial X_{n-1}} = 0, \frac{\partial f^{\mathbf{A}}}{\partial X_n} \neq 0$.*

This result induces a probabilistic algorithm which consists in:

1. choosing randomly a matrix $\mathbf{A} \in GL_n(\mathbb{Q})$ and compute an algebraic representation of the Zariski-closure $\mathfrak{C}^{\mathbf{A}}$ of the constructible set defined by:
$$f^{\mathbf{A}} - T = \frac{\partial f^{\mathbf{A}}}{\partial X_1} = \cdots = \frac{\partial f^{\mathbf{A}}}{\partial X_{n-1}} = 0, \quad \frac{\partial f^{\mathbf{A}}}{\partial X_n} \neq 0$$
2. Compute the set of non-properness of the projection on $T$ restricted to $\mathfrak{C}^{\mathbf{A}}$.

Certifying this algorithm is done by checking that for $i = 1, \ldots, n-1$ the projection $\pi_i : (x_1, \ldots, x_n, t) \in \mathbb{C}^{n+1} \to (x_{n-i+1}, \ldots, x_n, t) \in \mathbb{C}^{i+1}$ restricted to the

Zariski-closure of the constructible set defined by

$$f^{\mathbf{A}} - T = \frac{\partial f^{\mathbf{A}}}{\partial X_1} = \cdots = \frac{\partial f^{\mathbf{A}}}{\partial X_{n-i}} = 0, \quad \frac{\partial f^{\mathbf{A}}}{\partial X_{n-i+1}} \neq 0$$

is proper.

The above algorithm allows us to deal with non-trivial examples and has been used to compute sampling points in a semi-algebraic set defined by a single inequality (see [7] for an application in computational geometry). Nevertheless, some improvements and theoretical issues could be expected:

1. how to limit the growth of coefficients appearing in the computations which are induced by the change of variables $\mathbf{A}$ ?
2. the certification of the above algorithm can be expensive on some examples; can we find a way to obtain a certified algorithm whose practical efficiency is better than the one of [28]?
3. can we improve the degree bounds on the geometric objects considered during the computations?

*Main contributions.* The main result of this paper is the following (see Theorem 3 below): *Let $f$ be a polynomial in $\mathbb{Q}[X_1, \ldots, X_n]$. Suppose that for all $i \in \{1, \ldots, n-1\}$, the Zariski-closure denoted by $W_i$ of the constructible set defined by $f - T = \frac{\partial f}{\partial X_1} = \cdots = \frac{\partial f}{\partial X_{n-i}} = 0$, $\frac{\partial f}{\partial X_{n-i+1}} \neq 0$ has dimension $i$. Then, the set of asymptotic critical values of $f$ is contained in the set of non-properness of the projection $(x_1, \ldots, x_n, t) \in \mathbb{C}^n \to t$ restricted to $W_1$.*

Note that this strongly simplifies the certification process of the algorithm designed in [28] since it is now reduced to compute the dimension of a Zariski-closed algebraic set. This also allows us to use simpler matrices $\mathbf{A}$ (for which the aforementioned projections $\pi_i$ may be not proper) to avoid a growth of the coefficients. This result is obtained by using local properness of these projections $\pi_i$ instead of global properness which is used in the proof of [28, Theorem 3.6].

Additionally, we prove that, *There exists a Zariski-closed subset $\mathcal{A} \subsetneq \mathbb{C}^n$ such that for all $(a_1, \ldots, a_n) \in \mathbb{C}^n \setminus \mathcal{A}$, the ideal*

$$\left( \langle L\frac{\partial f}{\partial X_1} - a_1, L\frac{\partial f}{\partial X_2} - a_2, \ldots, L\frac{\partial f}{\partial X_n} - a_n \rangle \cap \mathbb{Q}[X_1, \ldots, X_n] \right) + \langle f - T \rangle$$

*has either dimension $1$ in $\mathbb{Q}[X_1, \ldots, X_n, T]$ or it equals $\langle 1 \rangle$. Moreover, if the determinant of the Hessian matrix associated to $f$ is not identically null, there exists a Zariski-closed subset $\mathcal{A} \subsetneq \mathbb{C}^n$ such that the above ideal has dimension $1$* (see Proposition 1).

Thus, if the determinant of the Hessian matrix of $f$ is not null, we are able to apply the aforementioned result by performing linear change of variables to compute asymptotic critical values by computing a set of non-properness of a projection restricted to a curve. The degree of this curve is crucial to estimate the complexity of our algorithm. We prove in Theorem 4 (see below) that it is bounded by $(D-1)^{n-1} - \mathfrak{d}$ where $\mathfrak{d}$ is the sum of the degrees of the positive dimensional irreducible components of the variety associated to $\langle \frac{\partial f}{\partial X_1}, \ldots, \frac{\partial f}{\partial X_n} \rangle$. Note that $\mathfrak{d}$ is an intrinsic quantity. This last result improves the degree bounds provided in [28].

We describe two versions of this algorithm. The first one is certified and uses Gröbner bases to perform algebraic elimination. The second one is probabilistic and uses the geometric resolution of algorithm of [19].

We have implemented the certified version of the algorithm we have obtained using Gröbner bases. We did experiments comparing

- the algorithm we obtained,
- the one which is designed in [28]
- the one designed in [20]
- an algorithm based on CAD computing the asymptotic critical values of a polynomial.

It appears that the algorithm we design in this paper is significantly faster than the previous ones. Compared to the one given [28] which is based on similar geometric techniques, the gain comes from the fact the growth of the coefficients appearing in our algorithm is indeed better controlled.

*Organization of the paper.* Section 2 is devoted to recall basic definitions and properties about generalized critical values of a polynomial mapping. Section 3 is devoted to the proof of the results presented above. Section 4 is devoted to present practical experiments showing the relevance of our approach.

## 2 Preliminaries

In this section, we recall the definitions and basic properties of generalized critical values which can be found in [20].

**Definition 1.** *A complex number $c \in \mathbb{C}$ is a critical value of the mapping $\widetilde{f} : y \in \mathbb{C}^n \to f(y)$ if and only if there exists $z \in \mathbb{C}^n$ such that $f(z) = c$ and $\frac{\partial f}{\partial X_1}(z) = \cdots = \frac{\partial f}{\partial X_n}(z) = 0$.*

*A complex number $c \in \mathbb{C}$ is an asymptotic critical value of the mapping $\widetilde{f} : y \in \mathbb{C}^n \to f(y)$ if and only if there exists a sequence of points $(z_\ell)_{\ell \in \mathbb{N}} \subset \mathbb{C}^n$ such that:*

- *$f(z_\ell)$ tends to $c$ when $\ell$ tends to $\infty$.*
- *$||z_\ell||$ tends to $+\infty$ when $\ell$ tends to $\infty$.*
- *for all $(i,j) \in \{1, \ldots, n\}^2$ $||X_i(z_\ell)||.||\frac{\partial f}{\partial X_j}(z_\ell)||$ tends to $0$ when $\ell$ tends to $\infty$.*

*The set of generalized critical values is the union of the sets of critical values and asymptotic critical values of $\widetilde{f}$.*

In the sequel, we denote by $K_0(f)$ the set of critical values of $\widetilde{f}$, by $K_\infty(f)$ the set of asymptotic critical values of $\widetilde{f}$, and by $K(f)$ the set of generalized critical values of $\widetilde{f}$ (i.e. $K(f) = K_0(f) \cup K_\infty(f)$).

In [20], the authors prove the following result which can be seen as a generalized Sard's theorem for generalized critical values.

**Theorem 1.** *Let $f$ be a polynomial in $\mathbb{Q}[X_1, \ldots, X_n]$ of degree $D$. The set of generalized critical values $K(f)$ of the mapping $\widetilde{f} : x \in \mathbb{C}^n \to f(x) \in \mathbb{C}$ is Zariski-closed in $\mathbb{C}$. Moreover, $D\sharp K_\infty(f) + \sharp K_0(f) \le D^n - 1$*

The main interest of the set of *generalized critical values* relies on its topological properties which are summarized below and proved in [20].

**Theorem 2.** *The mapping $f_{\mathbb{C}} : x \in \mathbb{C}^n \to f(x) \in \mathbb{C}$ realizes a locally trivial fibration in $\mathbb{C}^n \setminus f_{\mathbb{C}}^{-1}(K(f_{\mathbb{C}}))$.*
*The mapping $f_{\mathbb{R}} : x \in \mathbb{R}^n \to f(x) \in \mathbb{R}$ realizes a locally trivial fibration in $\mathbb{R}^n \setminus f_{\mathbb{R}}^{-1}(K(f_{\mathbb{R}}))$.*

Thus, $K(f)$ is Zariski-closed, degree bounds on $K(f)$ are Bézout-like degree bounds and its topological properties ensure that there is no topological change of the fibers of $f$ taken above any interval of $\mathbb{R}$ which has an empty intersection with $K(f)$.

In the sequel, for the sake of simplicity, we identify a polynomial $f \in \mathbb{Q}[X_1, \ldots, X_n]$ with the mapping $f_{\mathbb{C}} : x \in \mathbb{C}^n \to f(x) \in \mathbb{C}$.

# 3 Main Results and Algorithms

## 3.1 Geometric results

In the sequel, we consider maps between complex or real algebraic varieties. The notion of properness of such maps will be relative to the topologies induced by the metric topologies of $\mathbb{C}$ or $\mathbb{R}$. A map $\phi : V \to W$ of topological spaces is said to be proper at $w \in W$ if there exists a neighborhood $B$ of $w$ such that $f^{-1}(\overline{B})$ is compact (where $\overline{B}$ denotes the closure of $B$). The map $\phi$ is said to be proper if it is proper at all $w \in W$.

The following lemma is used in the proof of the main result of this section.

**Lemma 1.** *Let $\Delta_{n-j}$ be the Zariski-closure of the constructible set defined by*

$$\frac{\partial f}{\partial X_1} = \cdots = \frac{\partial f}{\partial X_{n-j}} = 0, \quad \frac{\partial f}{\partial X_{n-j+1}} \neq 0.$$

*Suppose that for $j = 1, \ldots, n-1$, $\Delta_{n-j}$ has dimension $j$ and that its intersection with the hypersurface defined by $\frac{\partial f}{\partial X_{n-j+1}} = 0$ is regular and non-empty.*

*Consider the projection $\pi_{n-j+2} : (x_1, \ldots, x_n) \in \mathbb{C}^n \to (x_{n-j+2}, \ldots, x_n) \in \mathbb{C}^{j-1}$ and suppose its restriction to $\Delta_{n-j}$ to be dominant. There exists a Zariski-closed subset $\mathcal{Z} \subsetneq \mathbb{C}^{j-1}$ such that if $\alpha \notin \mathcal{Z}$ and if there exists a sequence of points $(x_k)_{k\in\mathbb{N}} \in \pi_{n-j+2}^{-1}(\alpha) \cap \Delta_{n-j}$, such that $\frac{\partial f}{\partial X_{n-j+1}}(x_k) \to 0$ when $k \to \infty$, then there exists a point in $x \in \Delta_{n-j}$ such that $\pi_{n-j+2}(x) = \alpha$ and $\frac{\partial f}{\partial X_{n-j+1}}(x) = 0$.*

*Proof.* Let $x$ be a point of $\Delta_{n-j+1}$, which has, by assumption, dimension $j-1$. Then $x$ belongs to an irreducible component of dimension $j-1$ of the intersection of a component $C'$ of the variety $V$ defined by $\frac{\partial f}{\partial X_1} = \cdots = \frac{\partial f}{\partial X_{n-j}} = 0$ with the hyperurface $H$ defined by $\frac{\partial f}{\partial X_{n-j+1}} = 0$. The component $C'$ has thus a dimension which is less than $j+1$. Remark now that each component of $V$ has dimension greater than $j-1$ (since it is defined by the vanishing of $n-j$ polynomials). Thus, $C'$ has dimension $j$ and its intersection with the hypersurface $H$ is regular. Then, $C'$ is an irreducible component of $\Delta_{n-j}$. Consider $\mathfrak{C}$ the union of such irreducible components containing points of $\Delta_{n-j+1}$.

Finally, each point in $\Delta_{n-j+1}$ lies in $\mathfrak{C}$. Thus, it is sufficient to prove that for a generic choice of $\alpha \in \mathbb{C}^{j-1}$, $\pi_{n-j+2}^{-1}(\alpha) \cap \Delta_{n-j+1}$ is zero-dimensional and not isolated in the variety $\mathfrak{C}$.

Consider the ideal $I = \langle \frac{\partial f}{\partial X_1}, \ldots, \frac{\partial f}{\partial X_{n-j}} \rangle : \langle \frac{\partial f}{\partial X_{n-j+1}} \rangle^\infty \subset \mathbb{Q}[X_1, \ldots, X_n]$ and the ideal $J = I + \langle \frac{\partial f}{\partial X_{n-j+1}} - U \rangle$. Remark that $I$ is equi-dimensional since it has dimension $j$ and contains $\langle \frac{\partial f}{\partial X_1}, \ldots, \frac{\partial f}{\partial X_{n-j}} \rangle$ which has dimension at least $j$. Moreover, by assumption, $\dim(J + \langle U \rangle) = \dim(I) - 1$ and $\pi_{n-j+2}$ is dominant. Then, for all $k \in \{1, \ldots, n-j+1\}$ $J \cap \mathbb{Q}[X_k, X_{n-j+2}, \ldots, X_n, U] + \langle U \rangle$ is generated by a non-constant polynomial $P_k$. If for all $k \in \{1, \ldots, n-j+1\}$, $\alpha$ does not belong to the leading coefficient of $P_k$ seen as a univariate polynomial in $X_k$, $\pi_{n-j+2}^{-1}(\alpha)$ has a zero-dimensional intersection $\mathfrak{A}$ with the variety defined by $\Delta_{n-j+1}$. This intersection lies in $\mathfrak{C}$.

Remark now that $\mathfrak{C}$ is equi-dimensional since $I$ is equi-dimensional, so that the points in $\mathfrak{A}$ are not isolated in $\mathfrak{C}$. □

**Theorem 3.** *Let $f$ be a polynomial in $\mathbb{Q}[X_1, \ldots, X_n]$. Suppose that for all $i \in \{1, \ldots, n-1\}$, the Zariski-closure denoted by $W_i$ of the constructible set defined by $f - T = \frac{\partial f}{\partial X_1} = \cdots = \frac{\partial f}{\partial X_{n-i}} = 0$, $\frac{\partial f}{\partial X_{n-i+1}} \neq 0$ has dimension $i$. Then, the set of asymptotic critical values of $f$ is contained in the set of non-properness of the projection $(x_1, \ldots, x_n, t) \in \mathbb{C}^n \to t$ restricted to $W_1$.*

The proof is based on similar arguments than the one of [28, Theorem 3.6]. We consider below the projections: $\Pi_i : (x_1, \ldots, x_n, t) \mapsto (x_{n-i+2}, \ldots, x_n, t)$ (for $i = n, \ldots, 2$).

*Proof.* Given an integer $j$ in $\{n+1, \ldots, 2\}$, we say that property $\mathfrak{P}_j$ is satisfied if and only if the following assertion is true: *let $c \in K_\infty(f)$, there exists a sequence of points $(z_\ell)_{\ell \in \mathbb{N}}$ such that for all $\ell \in \mathbb{N}$, $z_\ell \in W_{j-1}$; $f(z_\ell) \to c$ when $\ell \to \infty$; $||z_\ell||$ tends to $\infty$ when $\ell$ tends to $\infty$; and $||z_\ell||.||d_{z_\ell} f|| \to 0$ when $\ell \to \infty$.*

Suppose now $\mathfrak{P}_{j+1}$ is true. We show below that this implies $\mathfrak{P}_j$. Since $\mathfrak{P}_{j+1}$ is supposed to be true, then there exists a sequence of points $(z_\ell)_{\ell \in \mathbb{N}}$ such that for all $\ell \in \mathbb{N}$, $z_\ell \in W_j$, $f(z_\ell) \to c$ when $\ell \to \infty$, $||z_\ell||$ tends to $\infty$ when $\ell$ tends to $\infty$ and $||z_\ell||.||d_{z_\ell} f|| \to 0$ when $\ell \to \infty$.

We prove below that one can choose such a sequence $(z_\ell)_{\ell \in \mathbb{N}}$ in $W_{j-1}$.

Consider the mapping $\phi : W_j \subset \mathbb{C}^{n+1} \to \mathbb{C}^{2j+1}$ which associates to a point $x = (x_1, \ldots, x_n, t) \in W_j$ the point:

$$\left( x_{n-j+2}, \ldots, x_n, t, \frac{\partial f}{\partial X_{n-j+1}}(x), \left( x_{n-j+r} \sum_{k=n-j+1}^{n} ||\frac{\partial f}{\partial X_k}(x)|| \right)_{r=1,\ldots,j} \right)$$

Remark that using the isomorphism between $\mathbb{C}^n$ and $\mathbb{R}^{2n}$, it is easy to prove that $\phi$ is a semi-algebraic map. Denote by

$$(a_{n-j+2},\dots,a_n,a_{n+1},a_{0,n-j+1},a_{n-j+1,n-j+1},\dots,a_{n,n-j+1})$$

the coordinates of the target space of $\phi$.

By assumption, the restriction of $\Pi_j$ to $W_j$ has finite fibers. Then, there exists a semi-algebraic subset $\mathcal{Z} \subsetneq \mathbb{C}^{2j+1} \simeq \mathbb{R}^{4j+2}$ such that specializing the coordinates $(a_{n-j+2},\dots,a_n,a_{0,n-j+1},a_{n-j+1,n-j+1})$ of the target space of $\phi$ to a point

$$\alpha_{n-j+2},\dots,\alpha_n,\alpha_{0,n-j+1},\alpha_{n-j+1,n-j+1}$$

outside $\mathcal{Z}$ defines a finite set of points in the image of $\phi$. Indeed, these points are the images of the points in $W_j$ such that their $X_i$ coordinate (for $i=n-j+2,\dots,n$) equals $\alpha_i$ and $X_{n-j+1}\sum_{k=n-j+1}^{n}||\frac{\partial f}{\partial X_k}||$ equals $\alpha_{n-j+1,n-j+1}$.

Given a point $\underline{\alpha}=(\alpha_{n-j+2},\dots,\alpha_n)\in\mathbb{C}^{j-1}$ and a complex number $\theta=(\eta_1)\in\mathbb{C}$, such that $(\alpha_{n-j+2},\dots,\alpha_n,\eta_1)\notin\mathcal{Z}$, we denote by $y(\underline{\alpha},\beta)$ a point in the image of $\phi$ obtained by specializing the first $(j-1)$ coordinates (corresponding to $x_{n-j+2},\dots,x_n$) to $\underline{\alpha}$ and the $j+2$-th coordinate (corresponding to $x_{n-j+1}\sum_{k=n-j+1}^{n}||\frac{\partial f}{\partial X_k}||$). We also denote by $x(\underline{\alpha},\theta)$ a point in the pre-image of $y(\underline{\alpha},\theta)$ by $\phi$.

Consider $c\in K_\infty(f)$. Then, since $\mathfrak{P}_{j+1}$ is supposed to be true, there exists a sequence of points $(z_\ell)_{\ell\in\mathbb{N}}\subset\mathbb{C}^n$ in the Zariski-closure of the constructible set defined by: $\frac{\partial f}{\partial X_1}=\cdots=\frac{\partial f}{\partial X_{n-j}}=0, \quad \frac{\partial f}{\partial X_{n-j+1}}\neq 0$ such that $f(z_\ell)$ tends to $c$ when $\ell$ tends to $\infty$, $||z_\ell||$ tends to $\infty$ when $\ell$ tends to $\infty$, and $||z_\ell||.||d_{z_\ell}f||$ tends to 0 when $\ell$ tends to $\infty$.

Consider the images by $\phi$ of the points $(z_\ell,f(z_\ell))$ and their first $j-1$ coordinates $\underline{\alpha_\ell}$ and $\theta_\ell$ of their $j+2$-th coordinate. We consider now the double sequence $(\underline{\alpha}_i,\theta_\ell)_{(i,\ell)\in\mathbb{N}\times\mathbb{N}}$.

Note that, by construction, $\theta_\ell$ tends to 0 when $\ell$ tends to $\infty$ and that the last $j+1$ coordinates of $y(\underline{\alpha}_{i_0},\theta_\ell)$ tend to zero when $i_0$ is fixed and $\ell$ tends to $\infty$ if $X_{n-j+1}(x(\alpha_{i_0},\theta_\ell))$ does not tend to 0 when $\ell$ tends to $\infty$.

If for all $\ell\in\mathbb{N}$, $\frac{\partial f}{\partial X_{n-j+1}}(z_\ell)=0$ the result is obtained. Else, one can suppose that for all $\ell\in\mathbb{N}$, $\frac{\partial f}{\partial X_{n-j+1}}(z_\ell)\neq 0$.

Remark that without loss of generality, we can do the assumption: for all $(i,j)\in\mathbb{N}\times\mathbb{N}$, $x(\underline{\alpha}_i,\theta_\ell)$ is not a root of $\frac{\partial f}{\partial X_{n-j+2}}$ and $(\underline{\alpha}_i,\theta_\ell)\notin\mathcal{Z}$.

Moreover, if $j=n$ remark that the set of non-properness of $\Pi_n$ restricted to the hypersurface defined by $f-T=0$ is defined as the set of complex solutions of the leading coefficient of $f$ seen as a univariate polynomial in $X_1$. Thus, without loss of generality, one can suppose that for all $i$ and for all $t\in\mathbb{C}$, $(\underline{\alpha}_i,t)$ does not belong to this set of non-properness. Else, up to a linear change of variables on the variables $X_{n-j+2},\dots,X_n$, one can suppose that the assumptions of Lemma 1 are satisfied and then we choose $\underline{\alpha}_i$ outside the Zariski-closed subset $\mathcal{Z}$ exhibited in Lemma 1.

Remark that, since $\phi$ is semi-algebraic, $X_{n-j+1}(x(\underline{\alpha},\theta))$ is a root of a univariate polynomial with coefficients depending on $\underline{\alpha}$ and $\theta$. Then, for a fixed integer

$i_0$, since $\theta_\ell$ tends to (0) when $\ell$ tends to $\infty$, $X_{n-j+1}(x(\underline{\alpha}_{i_0}, \theta_\ell))$ has either a finite limit or tends to $\infty$ when $\ell$ tends to $\infty$.

*In the sequel, we prove that for $i_0 \in \mathbb{N}$, $y(\underline{\alpha}_{i_0}, \theta_\ell)$ has a finite limit in $\mathbb{C}^{2n+1}$ when $\ell$ tends to $\infty$.* Suppose first that $X_{n-j+1}(x(\underline{\alpha}_{i_0}, \theta_\ell))$ has a finite limit when $\ell$ tends to $\infty$. Then, $f(x(\underline{\alpha}_{i_0}, \theta_\ell))$ remains bounded (since $X_{n-j+1}(x(\underline{\alpha}_{i_0}, \theta_\ell))$ has a finite limit and since $\frac{\partial f}{\partial X_1}, \ldots, \frac{\partial f}{\partial X_{n-j}}$ vanish at $x(\underline{\alpha}_{i_0}, \theta_\ell)$). Thus, it has consequently a finite limit. Moreover, without loss of generality, one can suppose that $X_{n-j+1}(x(\underline{\alpha}_{i_0}, \theta_\ell))$ does not tend to 0 which implies that $\frac{\partial f}{\partial X_{n-j+1}}(x(\underline{\alpha}_{i_0}, \theta_\ell))$ tends to 0 when $\ell$ tends to $\infty$.

Suppose now that $X_{n-j+1}(x(\underline{\alpha}_{i_0}, \theta_\ell))$ tends to $\infty$ when $\ell$ tends to $\infty$. This immediately implies that $\frac{\partial f}{\partial X_{n-j+1}}(x(\underline{\alpha}_{i_0}, \theta_\ell))$ tends to 0 when $\ell$ tends to $\infty$. Since $X_{n-j+1}(x(\underline{\alpha}_{i_0}, \theta_\ell))$ tends to $\infty$ when $\ell$ tends to $\infty$, and $\left(X_k \frac{\partial f}{\partial X_{n-j+1}}\right)(x(\underline{\alpha}_{i_0}, \theta_\ell))$ tends to 0 when $\ell$ (for $k \in \{n-j+1, \ldots, n\}$) tends to $\infty$, using [28, Remark 2.2] and the curve selection Lemma at infinity (see [20, Lemma 3.3, page 9], this implies there exists a semi-algebraic arc $\gamma_{i_0} : [0,1[\rightarrow \mathbb{R}^n$ such that:

- $\gamma_{i_0}([0,1[)$ is included in the intersection of $W_j$ and of the linear subspace defined by $X_k = X_k(\underline{\alpha}_{i_0})$ for $k = n-j+2, \ldots, n$, which implies that

$$\sum_{p=1}^{n} \left(X_p \frac{\partial f}{\partial X_p}\right)(\gamma_{i_0}(\rho)) = \left(X_{n-j+1} \frac{\partial f}{\partial X_{n-j+1}}\right)(\gamma_{i_0}(\rho))$$

- $||\gamma_{i_0}(\rho)|| \rightarrow \infty$ and $||X_{n-j+1}(\gamma_{i_0}(\rho))||.||\frac{\partial f}{\partial X_{n-j+1}}(\gamma_{i_0}(\rho))|| \rightarrow 0$ when $\rho$ tends to 1.

From Lojasiewicz's inequality at infinity [4, 2.3.11, p. 63], this implies that there exists an integer $N \geq 1$ such that: $\forall \rho \in [0,1[$, $||\frac{\partial f}{\partial X_{n-j+1}}(\gamma_{i_0}(\rho)))|| \leq ||X_{n-j+1}(\gamma_{i_0}(\rho))||^{-1-\frac{1}{N}}$. Following the same reasoning as in [20, Lemma 3.4, page 9], one can re-parametrize $\gamma_{i_0}$ such that $\gamma_{i_0}$ becomes a semi-algebraic function from $[0, +\infty[$ to $\mathbb{R}^n$ and $\lim_{\rho \rightarrow 1} ||\dot{\gamma}_{i_0}(\rho)|| = 1$. Thus, the following yields: $\forall p \in [0, +\infty[$, $||\frac{\partial f}{\partial X_{n-j+1}}(\gamma_{i_0}(\rho))||.||\dot{\gamma}_{i_0}(\rho)|| \leq ||X_{n-j+1}(\gamma_{i_0}(\rho))||^{-1-\frac{1}{N}}.||\dot{\gamma}_{i_0}(\rho)||$ and there exists $B \in \mathbb{R}$ such that $\int_0^\infty ||\gamma_{i_0}(\rho)||^{-1-\frac{1}{N}}.||\dot{\gamma}_{i_0}(\rho)||d\rho \leq B$.. Since

$$\int_0^\infty ||\gamma_{i_0}(\rho)||^{-1-\frac{1}{N}}.||\dot{\gamma}_{i_0}(\rho)||d\rho \geq \int_0^\infty ||X_{n-j+1}(\gamma_{i_0}(\rho))||^{-1-\frac{1}{N}}.||\dot{\gamma}_{i_0}(\rho)||d\rho$$

and $\int_0^\infty ||\frac{\partial f}{\partial X_{n-j+1}}(\gamma_{i_0}(\rho))||.||\dot{\gamma}_{i_0}(\rho)||d\rho \geq ||\int_0^\infty \frac{\partial f}{\partial X_{n-j+1}}(\gamma_{i_0}(\rho)).\dot{\gamma}_{i_0}(\rho)d\rho||$, one has finally $||\int_0^\infty \frac{\partial f}{\partial X_{n-j+1}}(\gamma_{i_0}(\rho)).\dot{\gamma}_{i_0}(\rho)d\rho|| \leq B$. Thus, the restriction of $f$ is bounded along $\gamma_{i_0}$.

Finally, we have proved that $y(\underline{\alpha}_{i_0}, \theta_\ell)$ tends to a point whose $j+1$-th coordinates is null.

Let $y_{i_0}$ be the limit of $y(\underline{\alpha}_{i_0}, \theta_\ell)$ and let $p_{i_0} \in \mathbb{C}^n$ be $(\underline{\alpha}_{i_0}, c_{i_0})$ and $p_\ell \in \mathbb{C}^n$ be the point whose coordinates are the $j$-first coordinates of $y(\underline{\alpha}_{i_0}, \theta_\ell)$.

We prove now that $y_{i_0}$ belongs to the image of $\phi$. If $j = n$ this is a consequence of the fact that $(\alpha_{i_0}, c_{i_0})$ does not belong to the set of non-properness of $\Pi_n$ restricted to the vanishing set of $f - T = 0$. If $j < n$ this is an immediate consequence of Lemma 1.

Thus, $\Pi_{j+1}^{-1}(p_{i_0}) \cap W_{j-1} \neq \emptyset$ and one can extract a converging subsequence from $(x(\underline{\alpha}_{i_0}, \theta_\ell))_{\ell\in\mathbb{N}}$ and let $x_{i_0}$ be the limit of the chosen converging subsequence.

It remains to prove that:

- $(f(x_{i_0}))_{i_0\in\mathbb{N}}$ tends to $c$ when $i_0$ tends to $\infty$
- $\left(X_i \frac{\partial f}{\partial X_j}\right)(x_{i_0})$ for $(i,j) \in \{1, \ldots, n\}$ tends to $0$ when $i_0$ tends to $\infty$.

which is a consequence of the continuity of the polynomials $f$ and $X_i \frac{\partial f}{\partial X_j}$ for $i = 2, \ldots, n$, and the definition of the sequence of points $x(\alpha_i, \theta_\ell)$.

$\square$

**Proposition 1.** *Let $f \in \mathbb{Q}[X_1, \ldots, X_n]$ be a polynomial of degree $D \geq 2$. There exists a Zariski-closed subset $\mathcal{A} \subsetneq \mathbb{C}^n$ such that for all $(a_1, \ldots, a_n) \in \mathbb{C}^n \setminus \mathcal{A}$, the ideal*

$$\left(\langle L\frac{\partial f}{\partial X_1} - a_1, L\frac{\partial f}{\partial X_2} - a_2, \ldots, L\frac{\partial f}{\partial X_n} - a_n\rangle \cap \mathbb{Q}[X_1, \ldots, X_n]\right) + \langle f - T\rangle$$

*has either dimension 1 in $\mathbb{Q}[X_1, \ldots, X_n, T]$ or it equals $\langle 1\rangle$. Moreover, if the determinant of the Hessian matrix associated to $f$ is not identically null, there exists a Zariski-closed subset $\mathcal{A} \subsetneq \mathbb{C}^n$ such that the above ideal has dimension 1.*

*Proof.* This is an immediate consequence of Sard's theorem (see [4, Theorem 2.5.11 and 2.5.12]) applied to the mapping $(x, \ell) \in \mathbb{C}^n \times \mathbb{C} \to \left(\ell\frac{\partial f}{\partial X_1}, \ldots, \ell\frac{\partial f}{\partial X_n}\right)$ $\square$

**Theorem 4.** *Let $\mathfrak{d}$ be the sum of the degrees of the positive-dimensional primes associated to the ideal $\langle\frac{\partial f}{\partial X_1}, \ldots, \frac{\partial f}{\partial X_n}\rangle$. The degree of the curve associated to the ideal $\left(\langle L\frac{\partial f}{\partial X_n} - 1, \frac{\partial f}{\partial X_1}, \ldots, \frac{\partial f}{\partial X_{n-1}}\rangle \cap \mathbb{Q}[X_1, \ldots, X_n]\right) + \langle f - T\rangle$ is dominated by $(D-1)^{n-1} - \mathfrak{d}$.*

*Proof.* From [9], the sum of the degrees of the prime ideals associated to the radical of the ideal $I = \langle\frac{\partial f}{\partial X_2}, \ldots, \frac{\partial f}{\partial X_n}\rangle$ is dominated by $(D-1)^{n-1}$. Consider the intersection $\mathcal{P}$ of these primes which contain $\frac{\partial f}{\partial X_1}$. Remark now that $\mathcal{P} + \langle\frac{\partial f}{\partial X_n}\rangle = \mathcal{P}$ and then, that the variety associated to $\mathcal{P} + \langle\frac{\partial f}{\partial X_n}\rangle$ is the union of the irreducible components of positive dimension associated to the radical of the ideal $\langle\frac{\partial f}{\partial X_1}, \ldots, \frac{\partial f}{\partial X_n}\rangle$. Note now that the degree of the curve defined by the ideal $J = \langle L\frac{\partial f}{\partial X_n} - 1, \frac{\partial f}{\partial X_1}, \ldots, \frac{\partial f}{\partial X_{n-1}}\rangle \cap \mathbb{Q}[X_1, \ldots, X_n]$ is bounded by the one of $I : \mathcal{P}^\infty$ and then is bounded by $(D-1)^{n-1} - \mathfrak{d}$ since $\mathfrak{d}$ is the sum of the degrees of the primes associated to $\mathcal{P}$. At last, note that $J + \langle f - T\rangle$ has the same degree than the one of $J$.

### 3.2 Algorithms and complexity

Our algorithm takes as input a polynomial $f \in \mathbb{Q}[X_1, \ldots, X_n]$ and outputs a non-zero univariate polynomial in $\mathbb{Q}[T]$ whose set of roots contains the set of generalized critical values of the mapping $x \in \mathbb{C}^n \to f(x)$. We focus on the computation of the *asymptotic critical values*, the case of the critical values being already investigated in [28]. Our procedure makes use of algebraic elimination algorithms to represent algebraic varieties defined as the Zariski-closure of the constructible sets defined by $f^{\mathbf{A}} - T = \frac{\partial f^{\mathbf{A}}}{\partial X_1} = \cdots = \frac{\partial f^{\mathbf{A}}}{\partial X_{n-1}} = 0, \quad \frac{\partial f}{\partial X_n} \neq 0$. Below, we show how to use Gröbner bases or the geometric resolution algorithm in our procedures computing the set of asymptotic critical values of $f$.

*Using Gröbner bases.* From Proposition 1, if the determinant of the Hessian matrix of $f$ is not zero, the set of matrices $\mathbf{A}$ such that the Zariski-closure $C_{\mathbf{A}}$ of the complex solution set of $f^{\mathbf{A}} - T = \frac{\partial f^{\mathbf{A}}}{\partial X_1} = \cdots = \frac{\partial f^{\mathbf{A}}}{\partial X_{n-1}} = 0, \quad \frac{\partial f^{\mathbf{A}}}{\partial X_n} \neq 0$ has dimension 1 is Zariski-opened in $GL_n(\mathbb{C})$. From Theorem 3, it suffices to find $\mathbf{A} \in GL_n(\mathbb{Q})$ such that $C_{\mathbf{A}}$ has dimension 1 and to compute the set of non-properness of the restriction to $C_{\mathbf{A}}$ of the projection $\pi : (x_1, \ldots, x_n, t) \to t$. The computation of the set of non-properness requires as input a Gröbner basis encoding the variety to which the considered projection is restricted. Such a routine is shortly described in [28] (see also [26] or [16] for a complete description); it is named `SetOfNonProperness` in the sequel.

**Algorithm computing $K_\infty(f)$ using Gröbner bases**

- **Input:** a polynomial $f$ in $\mathbb{Q}[X_1, \ldots, X_n]$.
- **Output:** a univariate polynomial $P \in \mathbb{Q}[T]$ such that its zero-set contains $K_\infty(f)$.

- Let $D = \det(\texttt{Hessian}(f))$. If $D = 0$ then return 1
- Choose $\mathbf{A} \in GL_n(\mathbb{C})$.
- Compute a Gröbner basis $G$ the ideal generated by

$$f^{\mathbf{A}} - T, \frac{\partial f^{\mathbf{A}}}{\partial X_1}, \ldots, \frac{\partial f^{\mathbf{A}}}{\partial X_{n-1}}, L\frac{\partial f^{\mathbf{A}}}{\partial X_n} - 1$$

  and its dimension $d$. If $d \neq 1$ then return to the previous step.
- Return `SetOfNonProperness`$(G, \{T\})$

In the above algorithm, one can first choose matrices $\mathbf{A}$ performing a sparse linear change of variables in order to reduce the height of the integers appearing in the computations. Nevertheless, the use of Gröbner bases as a routine of algebraic elimination does not allow us to obtain a complexity which is polynomial in the quantity bounding the degree of the curve defined as the Zariski-closure of the complex solution set of $f^{\mathbf{A}} - T = \frac{\partial f^{\mathbf{A}}}{\partial X_1} = \cdots = \frac{\partial f^{\mathbf{A}}}{\partial X_{n-1}} = 0 \frac{\partial f^{\mathbf{A}}}{\partial X_n} - 1 \neq 0$.

*Using the geometric resolution algorithm.* To this end, we consider the geometric resolution algorithm (see [11], [12, 19] and references therein). This algorithm is probabilistic and returns a rational parametrization of the complex solution set of the input (see [29] for situations where the input contains a parameter). Here is how it can be used to compute the set of asymptotic critical values of the mapping $x \in \mathbb{C}^n \to f(x) \in \mathbb{C}$.

| **Probabilistic Algorithm computing $K_\infty(f)$ using the Geometric Resolution Algorithm** |
|---|
| – **Input:** a polynomial $f$ in $\mathbb{Q}[X_1, \ldots, X_n]$.<br>– **Output:** a univariate polynomial $P \in \mathbb{Q}[T]$ such that its zero-set contains $K_\infty(f)$.<br><br>– Consider $T$ as a parameter in the polynomial system $f^{\mathbf{A}} - T = \frac{\partial f^{\mathbf{A}}}{\partial X_1} = \cdots = \frac{\partial f^{\mathbf{A}}}{\partial X_{n-1}} = 0, \frac{\partial f^{\mathbf{A}}}{\partial X_n} \neq 0$ and compute a geometric resolution.<br>– Return the least common multiple of the denominators in the coefficients of the polynomial $q$. |

Using Theorem 4 and Proposition 1, one obtains the following complexity result as a by-product of the complexity estimates given in [19].

**Theorem 5.** *The above probabilistic algorithm computing $K_\infty(f)$ performs at most $\mathcal{O}(n^7\delta^{4n})$ arithmetic operations in $\mathbb{Q}$ where $\delta$ is bounded by $(D-1)^{n-1} - \mathfrak{d}$, where $\mathfrak{d}$ denotes the sum of the degrees of the positive dimensional primes associated to the radical of the ideal generated by $\langle \frac{\partial f}{\partial X_1}, \ldots, \frac{\partial f}{\partial X_n} \rangle$.*

The above complexity estimate improves the one of [28, Theorem 4.3]. Remark that $\mathfrak{d}$ is intrinsic.

## 4 Practical Results

We have implemented the algorithm presented in the preceding section using Gröbner bases. The Gröbner engine which is used is FGb, release 1.26, [8] which is implemented in `C` by J.-C. Faugère. Computing rational parametrization of the complex roots of a zero-dimensional ideal from a Gröbner basis is done by RS, release 2.0.37, [21] which is implemented in `C` by F. Rouillier. Isolation of real roots of univariate polynomials with rational coefficients is done by RS using the algorithm provided in [23].

The resulting implementation is a part of the RAGLib Maple library (release 2.24) [24].

All the computations have been performed on a PC Intel Pentium Centrino Processor 1.86 GHz with 2048 Kbytes of Cache and 1024 MB of RAM.

### 4.1 Description of the test-suite.

Our test-suite is based on polynomials coming from applications. Most of the time, the user-question is to decide if the considered polynomial has constant sign on $\mathbb{R}^n$ or to compute at least one point in each connected component outside its vanishing set. As explained in the introduction, the computation of generalized critical values is a preliminary step of efficient algorithms dealing with these problems.

The following polynomial appears in a problem of algorithmic geometry studying the Voronoi Diagram of three lines in $\mathbb{R}^3$. In [7], the authors focus on determining topology changes of the Voronoi diagram of three lines in $\mathbb{R}^3$. The question was first reduced to determining if the zero-set of the discriminant of the following polynomial with respect of the variable $u$ contains real regular points.

This discriminant has degree 30. This discriminant is the product of a polynomial of degree 18 and several polynomials up to an odd power whom zero-set could not contain a real regular point since they are sums of squares. The polynomial of degree 18 is **Lazard II**. D. Lazard and S. Lazard have also asked to determine if the following polynomial which is denoted by **Lazard I** in the sequel is always positive.

$$
\begin{aligned}
&16\,a^2\left(\alpha^2+1+\beta^2\right)u^4+16\,a\left(-\alpha\,\beta\,a^2+ax\alpha+2\,a\alpha^2+2\,a+2\,a\beta^2+ay\beta-\alpha\,\beta\right)u^3+\\
&\left(\left(24\,a^2+4\,a^4\right)\alpha^2+\left(-24\,\beta\,a^3-24\,a\beta-8\,ya^3+24\,xa^2-8\,ay\right)\alpha+24\,a^2\beta^2+4\,\beta^2-\right.\\
&\left.8\,\beta\,xa^3+4\,y^2a^2+24\,y\beta\,a^2-8\,ax\beta+16\,a^2+4\,x^2a^2\right)u^2+\left(-4\,\alpha\,a^3+4\,ya^2-\right.\\
&\left.4\,ax-8\,a\alpha+8\,\beta\,a^2+4\,\beta\right)(\beta-a\alpha+y-ax)\,u+\left(a^2+1\right)(\beta-a\alpha+y-ax)^2
\end{aligned}
$$

The following polynomial appears in [15]. The problem consists in determining the conditions on $a$, $b$, $c$ and $d$ such that the ellipse defined by $\frac{(x-c)^2}{a^2}+\frac{(y-d)^2}{b^2}=1$ is inside the circle defined by $x^2+y^2-1=0$. The problem is reduced to compute at least one point in each connected component of the semi-algebraic set defined as the set of points at which the polynomial below (which is denoted by **Ellipse-Circle** in the sequel) does not vanish.

$$
\begin{aligned}
&4\,a^6c^2d^2+2\,a^2b^2d^6-6\,a^2b^2d^4+a^4c^4+2\,a^4c^2d^6-6\,a^2b^2c^4-6\,a^4b^2c^4+4\,a^6b^2d^2+\\
&a^8b^4+6\,b^4c^2d^2-2\,b^6c^4d^2+a^8d^4+6\,a^2b^6d^2-8\,a^4b^4d^2-4\,a^4b^2d^6-6\,b^4c^4d^2-8\,a^4b^4c^2+\\
&6\,a^6b^2c^2-8\,a^2b^4c^2+6\,a^4b^4d^4-2\,b^4c^2d^4-4\,a^2b^4c^6-4\,a^6b^4c^2-6\,a^2b^4d^4-2\,a^4c^4d^2+\\
&10\,a^4b^2d^4-2\,a^2b^8c^2-6\,a^2b^6c^4+a^4b^8+6\,a^2b^2d^2+6\,a^6b^4d^2-4\,a^4b^6d^2+b^4d^4+b^4c^8+\\
&10\,a^2b^4c^4+6\,a^2b^2c^2+4\,a^2b^6c^2+a^4d^8+4\,b^6c^2d^2+6\,a^4b^6c^2-8\,a^4b^2d^2+\\
&4\,a^4b^2c^2-2\,a^8b^2d^2+6\,a^4c^2d^2+4\,a^2b^4d^2-6\,a^6b^2d^4+6\,a^4b^4c^4-2\,a^6c^2d^4+\\
&2\,b^4c^6d^2+2\,a^2b^2c^6-6\,a^4c^2d^4+b^8c^4+2\,a^4b^2-4\,a^4d^2+a^4-2\,b^6-2\,a^6+a^8+\\
&b^8+b^4+2\,a^2b^4+2\,b^6c^6-2\,b^8c^2-6\,b^6c^4+2\,a^6b^4-2\,a^2b^2-2\,a^6b^6+2\,a^4b^6-\\
&2\,a^2b^8-6\,a^4b^2c^4d^2+2\,a^2b^4c^4d^2+2\,a^4b^2c^2d^4-6\,a^2b^4c^2d^4-6\,a^4b^2c^2d^2-6\,a^2b^4c^2d^2+\\
&4\,a^2b^2c^4d^4+2\,a^2b^2c^2d^6+2\,a^2b^2c^4d^2+2\,a^2b^2c^2d^4-10\,a^2b^2c^2d^2+6\,a^2b^6c^2d^2-\\
&6\,a^4b^4+2\,a^2b^6-2\,a^8b^2+2\,a^6b^2+6\,a^6b^2c^2d^2-10\,a^4b^4c^2d^2-4\,b^4c^6+6\,b^4c^4+6\,b^6c^2-\\
&2\,a^6c^2+2\,a^2b^2c^6d^2+a^4c^4d^4-2\,a^4c^2-2\,b^6d^2-4\,a^4d^6+2\,a^6d^6-\\
&2\,a^8d^2-6\,a^6d^4+6\,a^6d^2+b^4c^4d^4-4\,b^4c^2+6\,a^4d^4-2\,b^4d^2\left(a^2-b^2\right)
\end{aligned}
$$

The polynomials $\sum_{i=1}^{n}\prod_{j\neq i}(X_i - X_j)$ which are called in the sequel $\mathbf{LL}_n$ are studied in [18]. They are used as a benchmark for algorithms decomposing polynomials in sums of squares (see also [30]). In the sequel we consider **LL5** (which has degree 4 and contains 5 variables), **LL6** (which has degree 5 and contains 6 variables) and **LL7** (which has degree 6 and contains 7 variables).

We also consider polynomials coming from the Perspective-Three-Point Problem [10] which is an auto-calibration problem. Classifying the number of solutions on some instances of this problem leads to compute at least one point in each connected component outside a hypersurface. We consider two instances of this problem leading to study

- a polynomial denoted by **P3Piso** of degree 16 having 4 variables and 153 monomials.
- a polynomial denoted **P3P** of degree 16 having 5 variables and 617 monomials.

These polynomials are too big for being printed here.

### 4.2 Practical Results

We only report on timings for the computation of asymptotic critical values.

Below, in the column **JK** we give the timings for computing asymptotic critical values by using the algorithm of [20]. We obviously use the same Gröbner engine `FGb` for both algorithms.

Using similar arguments than the ones used in [1], one can prove that Cylindrical Algebraic Decomposition can compute a Zariski-closed set containing the generalized critical values of the mapping $f : x \to f(x)$ by computing a CAD adapted to $f - T$ (where $T$ is a new variable) and considering $T$ as the smallest variable. The column **CAD** contains the timings of an implementation of the open CAD algorithm in Maple which is due to G. Moroz and F. Rouillier.

The column **S07** contains the timings obtained using the *probabilistic* algorithm described in [28] to compute generalized critical values. In particular, we don't count the time required to certify the output of this algorithm.

The column **Algo** contains the timings obtained with the implementation of the algorithm described in this paper.

The symbol $\infty$ means that the computations have been stopped after 2 days of computations without getting a result.

It appears that on all the considered problems, the algorithm given in [20] does provide an answer in a reasonable amount of time.

On problems having at most 4 variables, the open CAD algorithm behaves well (except on polynomials having a big degree) and our implementation has comparable timings. On problems having more variables, our implementation ends with reasonable timings while open CAD either does not end after 2 days of computations or requires too much memory. This is mainly due to the highest degrees appearing in the projection step of CAD while the degrees of the polynomials appearing during the execution of our algorithms is better controlled. The same conclusions hold when we take into account the computation of classical critical values.

In comparison with the algorithm provided in [28], our algorithm performs better on harder problems. On some problems, we obtain a speed-up of 30. This is mainly due to the fact that the growth of coefficients appearing in our algorithm is better controlled than the ones appearing in the algorithm designed in [28]: we take here advantage of Theorem 3 to choose sparse matrices **A**. Note nevertheless that on smaller problems, our algorithm may be slower: this is mainly due to the search of an appropriate projection (preserving the sparsity of the initial problem) used for the computation of asymptotic critical values.

**Table 1.** Computation time obtained on a PC Intel Pentium Centrino Processor, 1.86 GHz with 2048 Kbytes of Cache and 1024 MB of RAM.

| BM | ♯vars | Degree | JK | Algo | S07 | CAD |
|---|---|---|---|---|---|---|
| **Lazard I** | 6 | 8 | $\infty$ | 14 sec. | 2 sec. | $\infty$ |
| **Lazard II** | 5 | 18 | $\infty$ | 192 sec. | 3 hours | $\infty$ |
| **Ellipse-Circle** | 4 | 12 | $\infty$ | 0.7 sec. | 90 sec. | 5 min. |
| **LL5** | 5 | 4 | $\infty$ | 0.2 sec. | 0.1 sec. | 20 sec. |
| **LL6** | 6 | 5 | $\infty$ | 9 sec. | 2 sec. | $\infty$ |
| **LL7** | 7 | 6 | $\infty$ | 28 sec. | 139 sec. | $\infty$ |
| **P3Piso** | 4 | 16 | $\infty$ | 1000 sec. | 2 hours | 20 min. |
| **P3P** | 5 | 16 | $\infty$ | 1100 sec. | 7 hours | $\infty$. |

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