Generalized critical values and testing sign conditions on a polynomial

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Abstract. Let f be a polynomial in $\mathbb{Q}[X_1, \ldots, X_n]$ of degree D. We focus on testing the emptiness of the semi-algebraic set defined by $f > 0$ (or $f < 0$ or $f \neq 0$). To this end, the problem is reduced to computing at least one point in each connected component of a hypersurface defined by $f - e = 0$ for $e \in \mathbb{Q}$ *positive and small enough*. We provide an algorithm allowing us to determine a positive rational number e which is small enough in this sense. This is based on the efficient computation of the set of *generalized critical values* of the mapping $f : y \in \mathbb{C}^n \to f(y) \in \mathbb{C}$ which is the union of the classical set $K_0(f)$ of critical values of the mapping f and $K_\infty(f)$ of *asymptotic critical values* of the mapping f. Our algorithm computes *separately* $K_0(f)$ (with a complexity within $\mathcal{O}\left(n^7 D^{4n}\right)$ arithmetic operations in $\mathbb{Q}$) and $K_\infty(f)$ (with a complexity within $\mathcal{O}(n^7 D^{4n)})$ arithmetic operations in $\mathbb{Q}$). This is asymptotically optimal since $\sharp K_\infty(f) \leq D^{n-1} - 1$ and $\sharp K_0(f) \leq D^{n-1}$. Then, we show how to use the computation of generalized critical values in order to obtain an efficient algorithm deciding the emptiness of a semi-algebraic set defined by a single inequality or a single inequation. At last, we show how to apply our contribution to determining if a hypersurface contains real regular points. We provide complexity estimates for probabilistic versions of the latter algorithms which are within $\mathcal{O}(n^7 D^{4n})$ arithmetic operations in $\mathbb{Q}$.

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1. Introduction

Let f be a polynomial in $\mathbb{Q}[X_1, \ldots, X_n]$ of degree D and $\mathcal{S}_+ \subset \mathbb{R}^n$ (resp. $\mathcal{S}_-$ and $\mathcal{S}$) be the semi-algebraic set defined by $f > 0$ (resp. $f < 0$ and $f \neq 0$). The aim of this paper is to provide an efficient algorithm *in practice* which computes at least one point in each connected component of $\mathcal{S}$ (resp. $\mathcal{S}_-$ and $\mathcal{S}$).

This question is of first importance since solving parametric polynomial systems of equations and inequalities is reduced to compute at least one point in each connected component of the complementary of a real hypersurface (see [25]). This question also appears as a black box used in algorithms solving quantifier elimination problems (see [7]).
Algorithms computing a Cylindrical Algebraic Decomposition (see [9]) allow us to produce one point in each connected component of $\mathcal{S}_+$, $\mathcal{S}_-$ or $\mathcal{S}$. Nevertheless the complexity of such algorithms is doubly exponential in the number of variables and their implementations are limited to problems having 3 or 4 variables.
Algorithms based on the critical point method are provided in [18, 19, 20, 29, 5, 6]. The classical strategy is to exhibit a hypersurface such that each connected component of $\mathcal{S}_+$ (resp. $\mathcal{S}_-$ or $\mathcal{S}$) contains a connected component of the real counterpart of the exhibited hypersurface. Indeed, given an infinitesimal ε, denote by $\mathcal{H}_\varepsilon \subset \mathbb{C}\langle\varepsilon\rangle^n$ the hypersurface defined by $f-\varepsilon$. By the mean value theorem, each connected component of the embedding of $\mathcal{S}_+$ in $\mathbb{R}\langle\varepsilon\rangle^n$ contains a connected component of $\mathcal{H}_\varepsilon \cap \mathbb{R}\langle\varepsilon\rangle^n$. Hence, the problem is reduced to compute at least one point in each connected component of the real counterpart of the hypersurface $\mathcal{H}_\varepsilon$.

Computing at least one point in each connected component of a real hypersurface. Consider a hypersurface $\mathcal{H} \subset \mathbb{C}^n$. We focus now on algorithms computing at least one point in each connected component of $\mathcal{H} \cap \mathbb{R}^n$. This problem is tackled by the critical point method. Its principle is the following: choose a polynomial mapping $\phi : \mathcal{H} \cap \mathbb{R}^n \to \mathbb{R}$ reaching its extrema in each connected component of $\mathcal{H} \cap \mathbb{R}^n$ and such that its critical locus is zero-dimensional. When $\mathcal{H}$ is smooth, ϕ can be the square of the euclidean distance to a generically chosen point of $\mathbb{Q}^n$. When, additionally, $\mathcal{H} \cap \mathbb{R}^n$ is known to be compact, ϕ can be the projection on a line.
In [5], computing sampling points in $\mathcal{H} \cap \mathbb{R}^n$ is reduced to computing sampling points of a smooth hypersurface whose real counterpart is compact by introducing several infinitesimals. Thus, projection functions are used. The algorithms are deterministic and their complexity is $(2D)^{\mathcal{O}(n)}$ arithmetic operations in $\mathbb{Q}$. Algebraic manipulations are performed to avoid a computation of Gröbner bases and lead to encode critical points as solutions of a zero-dimensional polynomial system generating an ideal having *always* a degree $2D(2D-1)^n$. Moreover, all the computations are performed over a Puiseux series field. Thus, there is no hope to obtain an efficient practical behaviour of these algorithms.
In [32, 1, 4, 3], the authors use the square of the euclidean distance to a generically chosen point A in $\mathbb{Q}^n$. Algorithms dealing with the case where $\mathcal{H}$ is not smooth are provided in [32, 1]. The one of [32] uses infinitesimal deformations. The one of [1] processes by performing a recursive study of the singular locus until it has dimension 0 or is empty. Because of the choice of A, the deterministic complexity of the algorithm of [32] is $D^{\mathcal{O}(n^2)}$. Nevertheless, in practice, the first choice is suitable to obtain zero-dimensional critical loci, so that under this assumption, *which is satisfied in practice*, the complexity of [32] is $D^{\mathcal{O}(n)}$. The complexity of [1] is not well-controlled even if in singular situations it behaves better than the

ones based on infinitesimal deformations. The algorithms of [4, 3] use the geometric resolution algorithm which is probabilistic. Their complexity is polynomial in n, the evaluation complexity of the input polynomial and an intrinsic geometric degree δ which is dominated by D^n.
In the smooth case, these contributions are improved in [35]: generic projection functions are used even in non-compact situations instead of distance functions to a generic point. The genericity of the choice of projection functions is necessary to ensure properness properties. As in the case of algorithms using distance functions, in practice, the first choices are suitable. Using elimination algorithms based on the geometric resolution, this leads to a probabilistic algorithm whose arithmetic complexity is polynomial in n, the evaluation complexity of the input polynomial, and an intrinsic geometric degree δ which is dominated by $D(D-1)^{n-1}$. One can also use Gröbner bases. Making the assumptions that the first choice of projections is suitable, the complexity becomes $D^{\mathcal{O}(n)}$. This work is generalized to the case of singular hypersurfaces in [37]. The algorithms relying on [35] are the most efficient in practice and are implemented in [37].
The output of all these algorithms are critical points encoded by a rational parameterization:

$$\begin{cases} X_n & = & \frac{q_n(T)}{q_0(T)} \\ & \vdots & \\ X_1 & = & \frac{q_1(T)}{q_0(T)} \\ q(T) & = & 0 \end{cases}$$

where T is a new variable, and $q, q_0, q_1, \ldots, q_n$ are univariate polynomials in $\mathbb{Q}[t]$. Such a rational parametrization can be obtained either by linear algebra computations in a quotient-algebra (see [31]) or directly by the geometric resolution algorithm (see [16, 14, 15, 17, 27]).

As recalled above, the classical strategy to compute at least one point in each connected component implies to apply the aforementioned algorithms in the case of a hypersurface defined by a polynomial with coefficients in $\mathbb{Q}\langle\varepsilon\rangle$. Thus, the output is a rational parameterization with coefficients in $\mathbb{Q}(\varepsilon)$. Once it is obtained, a small enough specialization for ε is obtained by computing the discriminant of q with respect to T and choosing a specialization less than the smallest absolute value of the real roots of this discriminant. Thus, the final output is smaller than the rational parameterization with coefficients in $\mathbb{Q}(\varepsilon)$. Moreover, computing rational parameterizations with coefficients in $\mathbb{Q}(\varepsilon)$ is hard in practice: infinitesimal arithmetics spoil the practical behaviour of elimination algorithms due to problems appearing in memory management and the overcost of arithmetic operations.

Substituting infinitesimal deformations by a pre-computation of generalized critical values. Remark that in order to obtain one point in each connected component in $\mathcal{S}_+$ (resp. $\mathcal{S}_-$ or $\mathcal{S}$), it is sufficient to substitute *a priori* the infinitesimal ε appearing in $f-\varepsilon$ by a small enough positive rational number $e \in \mathbb{Q}$. The problem is to ensure that the chosen rational number is small enough which means here that

for each connected component S of $\mathcal{S}_+$, there exists a connected component of the real counter part of the hypersurface defined by $f - e = 0$ which is contained in S. This can be done by determining $e_0 \in \mathbb{R}$ such that for all $e \in]0, e_0[$, there exists a diffeomorphism φ such that the following diagram commutes:

$$\begin{array}{ccc} f^{-1}(e)\times]0,e_0[& \xrightarrow{\varphi} & f^{-1}(]0,e_0[) \\ & \searrow^{\pi} & \downarrow f \\ & &]0,e_0[\end{array}$$

where π is the canonical projection on the second member of the cartesian product $f^{-1}(e)\times]0,e_0[$.
Such a topological property is obtained by ensuring that the interval $I =]0,e_0[$ has an empty intersection with the set of *generalized critical values* of the polynomial mapping $\widetilde{f} : x \in \mathbb{R}^n \to f(x) \in \mathbb{R}$ which is denoted by $K(f)$ in the sequel. This set is defined and studied in [28]. A real number $c \in \mathbb{R}$ is a generalized critical value of a mapping $\widetilde{f}$ if and only if it is either a critical value of $\widetilde{f}$ or there exists a sequence of points $(z_\ell)_{\ell\in\mathbb{N}}$ such that $f(z_\ell)$ tends to c when ℓ tends to ∞, $||z_\ell||$ tends to ∞ when ℓ tends to ∞ and $||z_\ell||.||d_{z_\ell} f||$ tends to 0 when ℓ tends to ∞. In the latter case, c is said to be an *asymptotic critical value*. Degree bounds are provided in [23]. An algorithm computing them is described in [28]. This algorithm works as follows: denoting by I the ideal

$$I = \langle f - T, \left(\frac{\partial f}{\partial X_i} - a_i\right)_{i\in\{1,\ldots,n\}}, \left(X_i\frac{\partial f}{\partial X_j} - a_{i,j}\right)_{(i,j)\in\{1,\ldots,n\}^2}\rangle$$

where $a_1, \ldots, a_{1,1}, \ldots, a_{n,n}$ and T are new variables, compute

$$J = I \cap \mathbb{Q}[T, a_1, \ldots, a_n, a_{1,1}, \ldots, a_{n,n}]$$

. Generalized critical values are solutions of

$$J + \langle a_1, \ldots, a_n, a_{11}, a_{n,n}\rangle.$$

Thus, the complexity of this algorithm is at least $D^{\mathcal{O}(n^2)}$ since it requires to deal with $n^2 + 2n + 1$ variables. Obviously, its practical behaviour is unefficient.
We provide here an algorithm computing efficiently the set of generalized critical values of a polynomial mapping from $\mathbb{R}^n$ to $\mathbb{R}$. A probabilistic version of this algorithm has a complexity within $D^{\mathcal{O}(n)}$ arithmetic operations in $\mathbb{Q}$ which is polynomial in the size of the output in worst-cases.
This allows us to substitute the use of infinitesimal deformations by a pre-computation of generalized critical values in order to compute at least one point in each connected component of a semi-algebraic set defined by a single inequality. The algorithm we obtain is efficient in practice and its probabilistic versions have a complexity within $\mathcal{O}(n^7 D^{4n})$ arithmetic operations in $\mathbb{Q}$. We also show how to apply our contribution to the problem of deciding if a hypersurface contains real regular points.

Plan of the paper. The paper is organized as follows. In Section 2, we recall the definition and basic properties of *generalized critical values* which can be found in [28]. In Section 3, we provide geometric results which, up to a generic linear change of the variables $X_1, \ldots, X_n$, characterize *generalized critical values* as the set of non-properness of a projection on a line restricted to a 1-dimensional polar variety. In Section 4, we show how to obtain a first algorithm computing generalized critical values which is directly based on the geometric results of Section 3. Then, we prove that these computations reduce to computing classical critical values on the one hand, and critical values at infinity on the other hand. In Section 5, we describe an algorithm computing at least one point in each connected component of a semi-algebraic set defined by a single inequality, which is based on the computation of *generalized critical values*. In Section 6, we show how to apply our contributions to determining if a hypersurface contains real regular points. Finally, Section 7 contains some benchmarks illustrating the efficiency of our algorithms.

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2. Definition and first properties of generalized critical values

In this section, we recall the definitions and basic properties of generalized critical values which can be found in [28].

Definition 2.1. A complex number $c \in \mathbb{C}$ is a critical value of the mapping $f : y \in \mathbb{C}^n \to f(y)$ if and only if there exists $z \in \mathbb{C}^n$ such that $f(z) = c$ and $\frac{\partial f}{\partial X_1}(z) = \cdots = \frac{\partial f}{\partial X_n}(z) = 0$.
A complex number $c \in \mathbb{C}$ is an asymptotic critical value of the mapping $f : y \in \mathbb{C}^n \to f(y)$ if and only if there exists a sequence of points $(z_\ell)_{\ell \in \mathbb{N}} \subset \mathbb{C}^n$ such that:

- $f(z_\ell)$ tends to c when ℓ tends to ∞.
- $||z_\ell||$ tends to $+\infty$ when ℓ tends to ∞.
- for all $(i,j) \in \{1, \ldots, n\}$ $||X_i(z_\ell)||.||\frac{\partial f}{\partial X_j}(z_\ell)||$ tends to 0 when ℓ tends to ∞.

Example. Consider the following polynomial in $Q[X_1, X_2]$

$$f = X_1(X_1X_2 - 1)$$

and the mapping $\widetilde{f} : (x_1, x_2) \to f(x_1, x_2)$. This mapping has obviously no critical value since $\langle f - T, \frac{\partial f}{\partial X_1}, \frac{\partial f}{\partial X_2} \rangle = \mathbb{Q}[X_1, X_2, T]$. Suppose now that there exists a sequence of points z_ℓ such that:

- $||z_\ell||$ tends to $+\infty$ when ℓ tends to ∞.
- for all $(i,j) \in \{1,2\}$ $||X_i(z_\ell)||.||\frac{\partial f}{\partial X_j}(z_\ell)||$ tends to 0 when ℓ tends to ∞.

This implies that $X_1^2(z_\ell)$ tends to 0 when ℓ tends to ∞, which implies that $X_1(z_\ell)$ tends to 0 when ℓ tends to ∞, and $X_2X_1^2(z_\ell)$ tends to 0 when ℓ tends to ∞. Finally, $f(z_\ell)$ tends to 0 when ℓ tends to ∞. Thus, 0 is an asymptotic critical value of the mapping $\widetilde{f}$. We will see further that it is the only one.

Consider now the following example in 3 variables:

$$f = X_1 + X_1^2 X_2 + X_1^4 X_2 X_3$$

In [28], the authors prove that the set of generalized critical values of the mapping sending $x \in \mathbb{C}^n$ to $f(x)$ is $\{0\}$ by using a similar reasoning as the above.

In [28], the authors prove the following result.

Theorem 2.2. *Let f be a polynomial in $\mathbb{Q}[X_1, \ldots, X_n]$ of degree D. The set of generalized critical values $K(f)$ of the mapping $\widetilde{f} : x \in \mathbb{C}^n \to f(x) \in \mathbb{C}$ is Zariski-closed in $\mathbb{C}$ and $D\sharp K_\infty(f) + \sharp K_0(f) \leq D^n - 1$*

Consider a mapping $f_\mathbb{C} : \mathbb{C}^n \to \mathbb{C}$ and an open subset $F_\mathbb{C}$ of $\mathbb{C}$. We say that $f_\mathbb{C}$ realizes a locally trivial fibration on $\mathbb{C}^n \setminus f_\mathbb{C}^{-1}(F_C)$ if for all connected open set (for the euclidean topology) $U_\mathbb{C} \subset \mathbb{C} \setminus F_\mathbb{C}$, for all $e \in U_\mathbb{C}$ denoting by $\pi_\mathbb{C}$ the projection on the second member of the cartesian product $f_\mathbb{C}^{-1}(e) \times U_\mathbb{C}$, the following diagram

$$\begin{array}{ccc} f_\mathbb{C}^{-1}(e) \times U_\mathbb{C} & \xrightarrow{\varphi} & f_\mathbb{C}^{-1}(U_\mathbb{C}) \\ & \searrow^{\pi} & \downarrow f \\ & & U_\mathbb{C} \end{array}$$

The above definition is also used for polynomial mappings from $\mathbb{R}^n$ to $\mathbb{R}$. Consider a mapping $f_\mathbb{R} : \mathbb{C}^n \to \mathbb{C}$ and an open subset $F_\mathbb{R}$ of $\mathbb{C}$. We say that $f_\mathbb{R}$ realizes a locally trivial fibration on $\mathbb{C}^n \setminus f_\mathbb{R}^{-1}(F_C)$ if for all connected open set (for the euclidean topology) $U_\mathbb{R} \subset \mathbb{C} \setminus F_\mathbb{R}$, for all $e \in U_\mathbb{R}$ denoting by $\pi_\mathbb{R}$ the projection on the second member of the cartesian product $f_\mathbb{R}^{-1}(e) \times U_\mathbb{R}$, the following diagram

$$\begin{array}{ccc} f_\mathbb{R}^{-1}(e) \times U_\mathbb{R} & \xrightarrow{\varphi} & f_\mathbb{R}^{-1}(U_\mathbb{R}) \\ & \searrow^{\pi} & \downarrow f \\ & & U_\mathbb{R} \end{array}$$

is commutative.
The main interest of the set *generalized critical values* relies on its topological properties which are summarized below and proved in [28].

Theorem 2.3. *The mapping $f_\mathbb{C}$ realizes a locally trivial fibration in $\mathbb{C}^n \setminus f_\mathbb{C}^{-1}(K(f_\mathbb{C}))$. The mapping $f_\mathbb{R}$ realizes a locally trivial fibration in $\mathbb{R}^n \setminus f_\mathbb{R}^{-1}(K(f_\mathbb{R}))$.*

Example. Consider the examples given above. We have proved that for both examples 0 is an asymptotic critical value. Remark that the fiber of both considered mappings above 0 is reducible while a generic fiber is irreducible. This is characteristic to a change of topology.

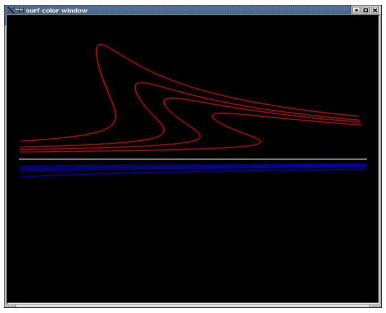

FIGURE 1. Existence of generalized critical values and no change in topology

Nevertheless, note that a mapping can realize a locally trivial fibration even if there exists a generalized critical value in I. To illustrate this fact, consider the following example:

$$f = -X_2(2X_1^2X_2^2 - 9X_1X_2 + 12)$$

which realizes a locally trivial fibration around 0 as shown in Figure 1 but is such that $K(f) = \{0\}$.

Thus, $K(f)$ is Zariski-closed, degree bounds on $K(f)$ are Bézout-like degree bounds and its topological properties ensure that there is no topological change in the fibers of f taken above any interval of $\mathbb{R}$ which has an empty intersection with $K(f)$.

Denote by $GL_n(\mathbb{C})$ the set of invertible matrices with coefficients in $\mathbb{C}$. Consider now $\mathbf{A} \in GL_n(\mathbb{C})$ and denote by $f^{\mathbf{A}}$ the polynomial $f(\mathbf{AX})$ where $\mathbf{X}$ denotes $(X_1, \ldots, X_n)$. Moreover, given $\{f_1, \ldots, f_s\}$ in $\mathbb{Q}[X_1, \ldots, X_n]$ and an algebraic variety $\mathcal{V} \subset \mathbb{C}^n$ defined by $f_1 = \cdots = f_s = 0$, we denote by $\mathcal{V}^{\mathbf{A}}$ the algebraic variety defined by $f_1^{\mathbf{A}} = \cdots = f_s^{\mathbf{A}} = 0$.

The following lemma is an immediate consequence of Definition 2.1 and will be used in the sequel.

Lemma 2.4. *For all $\mathbf{A} \in GL_n(\mathbb{Q})$, $K(f)$ equals $K(f^{\mathbf{A}})$, $K_0(f)$ equals $K_0(f^{\mathbf{A}})$ and $K_\infty(f)$ equals $K_\infty(f^{\mathbf{A}})$.*
If c is a critical value (resp. an asymptotic critical value) of f, then for all $e \in \mathbb{Q}$, $c - e$ is a critical value (resp. an asymptotic critical value) of $f + e$.

The following lemma is also immediate and is used further.

Lemma 2.5. *Let f be a polynomial in $\mathbb{Q}[X_1, \ldots, X_n]$. Consider $c \in \mathbb{C}$ and $(z_\ell)_{\ell \in \mathbb{N}} \subset \mathbb{C}^n$ be a sequence of points such that:*

- *$f(z_\ell)$ tends to c when ℓ tends to ∞;*
- *$||z_\ell||$ tends to ∞ when ℓ tends to ∞;*
- *$||z_\ell||.||d_{z_\ell} f||$ tends to 0 when ℓ tends to ∞.*

Denote by $\mathbf{X}$ *the vector* $X_1, \ldots, X_n$. *There exists a Zariski-closed subset* $\mathcal{A} \subsetneq GL_n(\mathbb{C})$ *such that for all* $\mathbf{A} \in GL_n(\mathbb{Q}) \setminus \mathcal{A}$, $||\mathbf{AX}(z_\ell)||$ *tends to* ∞ *when* ℓ *tends to* ∞.

In the sequel, for simplicity, we identify a polynomial $f \in \mathbb{Q}[X_1, \ldots, X_n]$ with the mapping $f_{\mathbb{C}} : x \in \mathbb{C}^n \to f(x) \in \mathbb{C}$.

3. Geometric results

Let f be a polynomial in $\mathbb{Q}[X_1, \ldots, X_n]$, $\mathcal{H} \subset \mathbb{C}^{n+1}$ be the hypersurface defined by $f - T = 0$ (where T is a new variable). For simplicity we denote by f the mapping sending a point $x \in \mathbb{C}^n$ to $f(x)$. Given $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$, we denote by $F_i : \mathbb{C}^n \to \mathbb{C}^{n+1}$ the polynomial mapping sending x to:

$$((\partial f/\partial X_i)(x), (X_1 \partial f/\partial X_i)(x), \ldots, (X_n \partial f/\partial X_i)(x))$$

and by $\widetilde{F_i} : \mathbb{C}^n \to \mathbb{C}^{in+i+1}$ the polynomial mapping sending x to:

$$(F_1(x), F_2(x), \ldots, F_i(x), f(x)).$$

We consider in the sequel the polynomial mapping $\phi : \mathbb{C}^n \to \mathbb{C}^{n^2+n+1}$ sending $x = (x_1, \ldots, x_n)$ to

$$(F_1(x), \ldots, F_n(x), f(x))$$

which coincides with $\widetilde{F_n}$. For any polynomial mapping ψ, we denote by Γ_ψ the image of ψ and by $\overline{\Gamma}_\psi$ its Zariski-closure. For $(i, j) \in \{1, \ldots, n\}^2$, we introduce new variables a_i, and $a_{i,j}$ such that $\overline{\Gamma}_\phi$ is defined by a set of generators of the ideal:

$$\langle f - T, (\partial f/\partial X_i - a_i)_{i \in \{1,\ldots,n\}}, (X_i.\partial f/\partial X_j - a_{i,j})_{(i,j) \in \{1,\ldots,n\}^2} \rangle$$

intersected with the polynomial ring $\mathbb{Q}[T, a_1, \ldots, a_n, a_{1,1}, \ldots, a_{n,n}]$.
Let $L_i \subset \mathbb{C}^{in+i+1}$ be the coordinate axis of T, i.e. the line defined by:

$$a_1 = \cdots = a_i = a_{1,1} = \cdots = a_{n,1} = \cdots = a_{1,i} = \cdots = a_{n,i} = 0.$$

The line L_n is denoted by L in the sequel.
Jelonek and Kurdyka prove that $\overline{\Gamma}_\phi \cap L$ equals the set of generalized critical values of f. The set of *asymptotic critical values* of f, denoted by $K_\infty(f)$, is characterized as the intersection of the set of non-properness of ϕ with L.

3.1. Geometric characterization of generalized critical values under properness assumptions

In the sequel, for $i = n, \ldots, 2$, we consider projections:

$$\begin{array}{rccc} \Pi_i : & \mathbb{C}^{n+1} & \to & \mathbb{C}^i \\ & (x_1, \ldots, x_n, t) & \mapsto & (x_{n-i+2}, \ldots, x_n, t) \end{array}$$

For $i = 1, \ldots, n-1$, let $W_{n-i} \subset \mathbb{C}^{n+1}$ denotes the Zariski-closure of the constructible set defined by:

$$f - T = \frac{\partial f}{\partial X_1} = \cdots = \frac{\partial f}{\partial X_i} = 0, \quad \frac{\partial f}{\partial X_{i+1}} \neq 0.$$

For simplicity, W_n denotes $\mathcal{H}$.

In the sequel, we consider maps between complex or real algebraic varieties. The notion of properness of such maps will be relative to the topologies induced by the metric topologies of $\mathbb{C}$ or $\mathbb{R}$. A map $\phi : V \to W$ of topological spaces is said to be proper at $w \in W$ if there exists a neighborhood B of w such that $f^{-1}(\overline{B})$ is compact (where $\overline{B}$ denotes the closure of B). The map ϕ is said to be proper if it is proper at all $w \in W$.

In the sequel, we suppose that there exists a Zariski-closed subset $\mathcal{A} \subsetneq GL_n(\mathbb{Q})$ such that for all $\mathbf{A} \in GL_n(\mathbb{Q}) \setminus \mathcal{A}$ given $j \in \{2, \ldots, n\}$, the property $\mathcal{P}_j(\mathbf{A})$ is satisfied if and only if for all $i \in \{j, \ldots, n\}$, the mapping Π_i restricted to $W_i^{\mathbf{A}}$ is proper and the restriction of the map Π_{i+1} to W_i is birational onto its image.

Remark 3.1. Remark that from the algebraic Bertini-Sard theorem [39], if $\mathcal{P}(\mathbf{A})$ is true, Π_i restricted to W_i is a finite map and then $W_i^{\mathbf{A}}$ has dimension i.

We prove below that if $\mathcal{P}_2(\mathbf{A})$ is satisfied, given $c \in K_\infty(f)$, there exists a sequence of points $(z_\ell)_{\ell \in \mathbb{N}}$ in $W_1^{\mathbf{A}}$ such that:

- $f(z_\ell)$ tends to c when ℓ tends to ∞
- $||z_\ell||$ tends to ∞ when ℓ tends to ∞
- $||z_\ell||.||d_{z_\ell} f||$ tends to 0 when ℓ tends to ∞

so that the existence of asymptotic critical values can be read off in W_1 which has dimension 1.

Proposition 3.2. *Consider $c \in K_\infty(f)$. There exists a Zariski-closed subset $\mathcal{A} \subsetneq GL_n(\mathbb{C})$ such that for all $A \in GL_n(\mathbb{Q}) \setminus \mathcal{A}$, there exists a sequence of points $(z_\ell)_{\ell \in \mathbb{N}}$ such that:*

- *for all $\ell \in \mathbb{N}$, $z_\ell \in W_{n-1}^{\mathbf{A}}$;*
- *$f^{\mathbf{A}}(z_\ell) \to c$ while $\ell \to \infty$;*
- *$||z_\ell||$ tends to ∞ when ℓ tends to ∞;*
- *$||z_\ell||.||d_{z_\ell} f^{\mathbf{A}}|| \to 0$ while $\ell \to \infty$.*

Proof. For the sake of simplicity, suppose that $\mathcal{P}_n(\mathbf{I}_n)$ is satisfied.
Consider the mapping $\phi : \mathcal{H} \subset \mathbb{C}^{n+1} \to \mathbb{C}^{2n+2}$ which associates to a point $x = (x_1, \ldots, x_n, t) \in \mathcal{H}$ the point:

$$\left(x_2, \ldots, x_n, t, \frac{\partial f}{\partial X_1}(x), x_1 \frac{\partial f}{\partial X_1}(x), \ldots, x_n \frac{\partial f}{\partial X_1}(x), t \frac{\partial f}{\partial X_1}(x) \right) \in \mathbb{C}^{2n+2}$$

Denote by $(a_2, \ldots, a_n, a_{n+1}, a_{0,1}, a_{1,1}, \ldots, a_{n+1,1})$ the coordinates of the target space of ϕ. Since for $i = 2, \ldots, n$, $X_i = a_i$, $T = a_{n+1}$ and $X_1 = \frac{a_{1,1}}{a_{0,1}}$, $X_1, \ldots, X_n$ and T can be expressed as rational functions of coordinates in the target space

of ϕ and then, the map ϕ is birational onto its image. Moreover, the graph of ϕ, denoted by Γ_ϕ is an irreducible algebraic variety of $\mathbb{C}^{3n+3}$ of dimension n. Then, specializing n coordinates outside a Zariski closed subset, of maximal dimension $n-1$, in the target space of ϕ determines a unique point in the pre-image of ϕ.
In the sequel, given a point $\underline{\alpha} = (\alpha_2, \ldots, \alpha_n) \in \mathbb{C}^{n-1}$ (resp. a complex number $\theta \in \mathbb{C}$), chosen outside a Zariski-closed subset of $\mathbb{C}^{n-1}$ (resp. $\mathbb{C}$), we denote by $y(\underline{\alpha}, \beta)$ the point in the image of ϕ obtained by specializing the first $(n-1)$ coordinates (corresponding to $x_2, \ldots, x_n$) to $\underline{\alpha}$ and the $n+2$-th coordinate (corresponding to $x_1 \frac{\partial f}{\partial X_1}$). Since ϕ is birational and since $\underline{\alpha}$ and β are chosen generically, one can define $x(\underline{\alpha}, \beta)$ as the unique pre-image of $y(\underline{\alpha}, \beta)$.
Consider $c \in K_\infty(f)$, then there exists a sequence of points $(z_\ell)_{\ell \in \mathbb{N}} \subset \mathbb{C}^n$ such that:

- $f(z_\ell)$ tends to c when ℓ tends to ∞
- $||z_\ell||$ tends to ∞ when ℓ tends to ∞.
- $||z_\ell||.||d_{z_\ell} f||$ tends to 0 when ℓ tends to ∞.

Consider the images by ϕ of the points $(z_\ell, f(z_\ell))$ and their first $n-1$ coordinates α_ℓ and their $n+2$-th coordinates θ_ℓ. Since $f(z_\ell)$, $1/||z_\ell||$ and $||z_\ell||.||d_{z_\ell} f||$ are Cauchy sequences, the doubly-indexed sequence $\underline{\alpha}_i, \theta_\ell$ is such that:

- (a) $f(x(\underline{\alpha}_i, \theta_\ell))$ tends to c when i and ℓ tend to ∞;
- (b) $||d_{x(\underline{\alpha}_i, \theta_\ell)} f||$ tends to 0 when i and ℓ tend to ∞.
- (c) $||x(\underline{\alpha}_i, \theta_\ell)||$ tends to ∞ when i and ℓ tend to ∞.
- (d) $||x(\underline{\alpha}_i, \theta_\ell)||.||d_{x(\underline{\alpha}_i, \theta_\ell)} f||$ tends to 0 when i and ℓ tend to ∞.

Note that such a choice implies that θ_ℓ tends to 0 when ℓ tends to ∞. Moreover, without loss of generality, by disturbing infinitesimally $\underline{\alpha}_i$ and θ_ℓ, one can suppose that:

- (e) for all $i \in \mathbb{N}$, $\underline{\alpha}_i$ is chosen outside the Zariski-closed subset defined as the Zariski-closure of the projection of W_{n-2} onto $X_2, \ldots, X_n$.
- (f) from Lemma 2.5, one can suppose that up to a generic linear change of coordinates, $X_1(\alpha_i, \theta_\ell)$ tends to ∞ when i and ℓ tend to ∞.

Since the map ϕ is birational, there exists an n-variate rational fraction Q such that $X_1(x(\underline{\alpha}, \theta))$ is obtained by evaluating this rational fraction at $\underline{\alpha}, \theta$. Then, for a fixed integer i_0, $X_1(x(\underline{\alpha}_{i_0}, \theta_\ell))$ has either a finite limit or tends to ∞ when ℓ tends to ∞. In the sequel, we prove that in both cases, $y(\underline{\alpha}_{i_0}, \theta_\ell)$ tends to a point whose last $n+2$ coordinates are null.
Suppose first that $X_1(x(\underline{\alpha}_{i_0}, \theta_\ell))$ has a finite limit when ℓ tends to ∞. Up to a generic linear change of variables, due to property (f), one can suppose that $||X_1(x(\underline{\alpha}_i, \theta_\ell))||$ tends to ∞ when i and ℓ tend to ∞. Thus, one can choose i_0 large enough to ensure that if $X_1(x(\underline{\alpha}_{i_0}, \theta_\ell))$ has a finite limit when ℓ tends to ∞, this limit is not 0. This implies that the $n+1$-th coordinate of $y(\underline{\alpha}_{i_0}, \theta_\ell)$ tends to zero when $\ell \to \infty$.
This also implies that for $j = n+3, \ldots, 2n+1$, the j-th coordinate of $y(\underline{\alpha}_{i_0}, \theta_\ell)$ tend to 0 when $\ell \to \infty$ since these coordinates can be rewritten as the product

of one coordinate of $\underline{\alpha}_{i_0}$ and the $(n+1)$-th of $y(\underline{\alpha}_{i_0}, \theta_\ell)$ which tends to 0 when ℓ tends to ∞.
Moreover, $f(x(\underline{\alpha}_{i_0}, \theta_\ell))$ remains bounded (since $X_1(x(\underline{\alpha}_{i_0}, \theta_\ell))$ has a finite limit), and has consequently a finite limit. Finally, this allows us to conclude that the last coordinate of $y(\underline{\alpha}_{i_0}, \theta_\ell)$ tends to 0 when ℓ tends to ∞. Thus, in this case, one has proved that $y(\underline{\alpha}_{i_0}, \theta_\ell)$ tends to a point whose last $n+2$ coordinates are null.

Suppose now that $X_1(x(\underline{\alpha}_{i_0}, \theta_\ell))$ tends to ∞ when ℓ tends to ∞. This immediately implies that the $n+1$-th coordinate and, for $j = n+3, \ldots, 2n+1$, the j-th coordinates of $y(\underline{\alpha}_{i_0}, \theta_\ell)$ tend to 0 when ℓ tend to ∞. It remains to prove that the last coordinate of $y(\underline{\alpha}_{i_0}, \theta_\ell)$ tends to 0 when ℓ tends to ∞.
Since $X_1(x(\underline{\alpha}_{i_0}, \theta_\ell))$ tends to ∞ when ℓ tends to ∞, from the curve selection Lemma at infinity (see [28, Lemma 3.3, page 9], this implies there exists a semi-algebraic arc $\gamma_{i_0} : [0,1[\to \mathbb{R}$ such that

$$||\gamma_{i_0}(\rho)|| \to \infty \quad \text{and} \quad ||\gamma_{i_0}(\rho)||.||\frac{\partial f}{\partial X_1}(\gamma_{i_0}(\rho))|| \to 0$$

when ρ tends to 1. From Lojasiewicz's inequality at infinity [8, 2.3.11, p. 63], this implies that there exists an integer $N \geq 1$ such that:

$$\forall \rho \in [0,1[, \quad ||\frac{\partial f}{\partial X_1}(\gamma_{i_0}(\rho)))|| \leq ||\gamma_{i_0}(\rho)||^{-1-\frac{1}{N}}$$

Following the same reasoning as in [28, Lemma 3.4, page 9], one can reparameterize γ_{i_0} such that γ_{i_0} becomes a semi-algebraic function from $[0,+\infty[$ and $\lim_{\rho \to 1} ||\dot{\gamma}_{i_0}(\rho)|| = 1$. Thus, the following yields:

$$\forall p \in [0,+\infty[, \quad ||\frac{\partial f}{\partial X_1}(\gamma_{i_0}(\rho))||.||\dot{\gamma}_{i_0}(\rho)|| \leq ||\gamma_{i_0}(\rho)||^{-1-\frac{1}{N}}.||\dot{\gamma}_{i_0}(\rho)||$$

and

$$\int_0^\infty ||\gamma_{i_0}(\rho)||^{-1-\frac{1}{N}}||.||\dot{\gamma}_{i_0}(\rho)||d\rho$$

is bounded. Since

$$\int_0^\infty ||\frac{\partial f}{\partial X_1}(\gamma_{i_0}(\rho))||.||\dot{\gamma}_{i_0}(\rho)||d\rho \geq ||\int_0^\infty \frac{\partial f}{\partial X_1}(\gamma_{i_0}(\rho)).\dot{\gamma}_{i_0}(\rho)d\rho||$$

This obviously implies that $f(X_1, \underline{\alpha}_{i_0})$ is bounded along γ_{i_0}. Hence we have proved that $y(\underline{\alpha}_{i_0}, \theta_\ell)$ tends to a point whose last $n+2$ coordinates are null.

Let $y_{i_0} = (\underline{\alpha}_{i_0}, c_{i_0}, 0, \ldots, 0)$ be the limit of $y(\underline{\alpha}_{i_0}, \theta_\ell)$ and let $p_{i_0} \in \mathbb{C}^n$ be $(\underline{\alpha}_{i_0}, c_{i_0})$ and $p_\ell \in \mathbb{C}^n$ be the point whose coordinates are the n-first coordinates of $y(\underline{\alpha}_{i_0}, \theta_\ell)$. We prove now that y_{i_0} belongs to the image of ϕ.
Since the restriction to $\mathcal{H}$ of Π_n is supposed to be proper, for all $\ell \in \mathbb{N}$, $\Pi_n^{-1}(p_\ell) \cap \mathcal{H} \neq \emptyset$ and there exists a ball centered at p_{i_0} such that $\Pi_n^{-1}(\mathcal{B})$ is compact. Moreover, remark that $x(\underline{\alpha}_{i_0}, \theta_\ell)$ belongs to $\Pi_n^{-1}(p_\ell)$.
Thus, one can extract a converging subsequence from $(x(\underline{\alpha}_{i_0}, \theta_\ell))_{\ell \in \mathbb{N}}$ and let x_{i_0} be the limit of the chosen converging subsequence. Note that we have proved above that the evaluation of $\frac{\partial f}{\partial X_1}$ at $x(\underline{\alpha}_{i_0}, \theta_\ell)$ tends to 0 when ℓ tends to ∞ which implies

that $\frac{\partial f}{\partial X_1}$ vanishes at x_{i_0}. Moreover, from property (e), $\frac{\partial f}{\partial X_2}$ does not vanish at x_{i_0}. Hence x_{i_0} belongs to W_{n-1} and $\phi(x_{i_0}) = y_{i_0}$ which implies that y_{i_0} belongs to the image of ϕ. To end the proof, note that, from properties (a), (c) and (d), $(f(x_{i_0}))_{i_0\in\mathbb{N}}$ (resp. $(||x_{i_0}||)_{i_0\in\mathbb{N}}$ and $\left(||d_{x_{i_0}}f||\right)_{i_0\in\mathbb{N}}$ and $\left(||x_{i_0}||.||d_{x_{i_0}}f||\right)_{i_0\in\mathbb{N}}$) has the same limit when i_0 tends to ∞ as $(f(x(\underline{\alpha}_i,\theta_\ell)))_{(i,\ell)\in\mathbb{N}\times\mathbb{N}}$ (resp. $||d_{x(\underline{\alpha}_i,\theta_\ell)}f||$ and $||x(\underline{\alpha}_i,\theta_\ell)||$ and $||x(\underline{\alpha}_i,\theta_\ell)||.||d_{x(\underline{\alpha}_i,\theta_\ell)}f||$) when i and ℓ tend to ∞. □

Finally, the above result tells that under some assumptions on the properness of some projections and the dimension of a polar variety, generalized critical values can be read off in the polar variety W_1.

Proposition 3.3. *Consider $c \in K_\infty(f)$. There exists a a Zariski-closed subset $\mathcal{A} \subsetneq GL_n(\mathbb{C})$ such that for all $A \in GL_n(\mathbb{Q}) \setminus \mathcal{A}$, there exists a sequence of points $(z_\ell)_{\ell\in\mathbb{N}}$ such that:*

- *for all $\ell \in \mathbb{N}$, $z_\ell \in W_1^{\mathbf{A}}$;*
- *$f^{\mathbf{A}}(z_\ell) \to c$ when $\ell \to \infty$;*
- *$||z_\ell||$ tends to ∞ when ℓ tends to ∞;*
- *$||z_\ell||.||d_{z_\ell}f^{\mathbf{A}}|| \to 0$ when $\ell \to \infty$.*

The proof of the above result uses the same techniques than the ones used in the proof of Proposition 3.2.

3.2. Ensuring properness properties

We prove now that up to a generic linear change of variables, the assumption $\mathcal{P}_1(\mathbf{A})$ which is summarized in the following proposition.

Proposition 3.4. *There exists a Zariski-closed subset $\mathcal{A} \subsetneq GL_n(\mathbb{C})$ such that for all $\mathbf{A} \in GL_n(\mathbb{Q}) \setminus \mathcal{A}$ and for all $j \in \{1,\ldots,n-1\}$:*

- *Π_j restricted to W_j is proper.*
- *the restriction of Π_{j+1} to W_j is bi-rational onto its image.*

In [35], the authors prove that given a hypersurface $\mathcal{H} \subset \mathbb{C}^{n+1}$, there exists a Zariski-closed subset $\mathcal{A} \subsetneq GL_{n+1}(\mathbb{C})$ such that for $j \in \{1,\ldots,n-1\}$ and for all $\mathbf{A} \in GL_{n+1}(\mathbb{Q}) \setminus \mathcal{A}$, Π_j restricted to $W_j^{\mathbf{A}}$ is proper and satisfies a Nœther normalization property.
This result can not be used as stated in [35], since we consider here the hypersurface defined by $f - T = 0$ and allow only change of variables on $X_1,\ldots,X_n$. Nevertheless, the incremental intersection process, originate from [16, 15, 14], which is used in the proof of [35] allows us to state:

Proposition 3.5. *For $i = 1,\ldots,n$, denote by $\Delta_i^{\mathbf{A}}$ the ideals associated to the Zariski-closure of the constructible set defined by:*

$$\frac{\partial f^{\mathbf{A}}}{\partial X_1} = \cdots = \frac{\partial f^{\mathbf{A}}}{\partial X_i} = 0, \quad \frac{\partial f^{\mathbf{A}}}{\partial X_{i+1}} \neq 0$$

There exists a Zariski-closed subset $\mathcal{A} \subsetneq GL_n(\mathbb{C})$ such that:

- *for all* $i \in \{1,\ldots,n\}$ *and for all prime* $P_i^{\mathbf{A}}$ *associated to* $\Delta_i^{\mathbf{A}}$*, the extension* $\mathbb{C}[\mathbf{X}_{\geq i+1}] \to \mathbb{C}[\mathbf{X}]/P_i^{\mathbf{A}}$ *is integral, where* $\mathbf{X}_{\geq i+1}$ *denotes* $X_{i+1},\ldots,X_n$ *and* $\mathbf{X}$ *denotes* $X_1,\ldots,X_n$*.*
- *for all* $i \in \{2,\ldots,n-1\}$*, the restriction of the projection* $\pi_i : (x_1,\ldots,x_n) \to (x_i,\ldots,x_n) \in \mathbb{C}^{n-i+1}$ *to the algebraic variety defined by* $\Delta_i^{\mathbf{A}}$ *is birational onto its image.*

Using *mutatis mutandis* the proof of [35, Proposition 3, Section 2.5], which is based on [21, Lemma 3.10] relating the properness of π_i to the fact that the above extensions are integral yields the following result:

Lemma 3.6. *Denote by* π_{i+1} *the projection* $(x_1,\ldots,x_n) \in \mathbb{C}^n \to (x_{i+1},\ldots,x_n) \in \mathbb{C}^{n-i}$*. There exists a Zariski-closed subset* $\mathcal{A} \subsetneq GL_n(\mathbb{C})$ *such that for all* $\mathbf{A} \in GL_n(\mathbb{Q}) \setminus \mathcal{A}$ *and for all* $i \in \{1,\ldots,n\}$*,* π_{i+1} *restricted to the algebraic variety defined by* $\Delta_i^{\mathbf{A}}$ *is proper.*

Now, we prove that if π_i restricted to the algebraic variety associated to $\Delta_i^{\mathbf{A}}$ is proper, then Π_i restricted to $W_i^{\mathbf{A}}$ is proper. Indeed, suppose there exists $(x,t) \in \mathbb{C}^{i-1} \times \mathbb{C}$ such that Π_i restricted to $W_i^{\mathbf{A}}$ is not proper at (x,t). This means that there exists a ball $B \times U \subset \mathbb{C}^{i-1} \times \mathbb{C}$ containing (x,t) such that $\Pi_i^{-1}(B) \cap W_i^{\mathbf{A}}$ is not compact. Remark that in that case, the only variables which can tend to infinity are $X_1,\ldots,X_{n-i}$ or X_{n-i+1}. This implies that the projection of $\Pi_i^{-1}(B) \cap W_i^{\mathbf{A}}$ onto $X_1,\ldots,X_n$ is not compact. Moreover, the projection of $\Pi_i^{-1}(B) \cap W_i^{\mathbf{A}}$ onto $X_1,\ldots,X_n$ is contained in the pre-image by π_i of B which contains x. Thus, the non-properness of Π_i restricted to $W_i^{\mathbf{A}}$ at (x,t) implies the non-properness of π_i restricted to $\Delta_i^{\mathbf{A}}$ at x.
The fact that the restriction of Π_i to $W_i^{\mathbf{A}}$ is birational comes immediately from the fact that the restriction of π_i to $\Delta_i^{\mathbf{A}}$ is birational.
This ends the proof of Proposition 3.4.

We are now ready to state our main geometric result which characterizes the set of *generalized critical values* of f.

3.3. Main geometric result

The combination of Proposition 3.4 and Lemma 2.5 leads to the following result.

Theorem 3.7 (Geometric characterization of generalized critical values). *There exists a Zariski-closed subset* $\mathcal{A} \subsetneq GL_n(\mathbb{C})$ *such that for all* $\mathbf{A} \in GL_n(\mathbb{Q}) \setminus \mathcal{A}$ *the set* $K_\infty(f)$ *of asymptotic critical values of* f *is contained in the set of non-properness of the projection* π_T *restricted to the Zariski-closure of the constructible set defined by:*

$$f^{\mathbf{A}} - T = \frac{\partial f^{\mathbf{A}}}{\partial X_2} = \cdots = \frac{\partial f^{\mathbf{A}}}{\partial X_n} = 0, \quad \frac{\partial f^{\mathbf{A}}}{\partial X_1} \neq 0.$$

Remark 3.8. Remark that the above result only states that $K_\infty(f)$ is contained in the set of non-properness $\mathcal{Z}$ of the projection $\Pi : (x_1,\ldots,x_n,t) \in \mathbb{C}^{n+1} \to t \in \mathbb{C}$

restricted to W_1. The latter set is zero-dimensional (see [21]). Nevertheless, this inclusion can be strict since some points in $\mathcal{Z}$ can depend on $\mathbf{A}$.

Example. In [36], the authors use [21, Lemma 3.10] to compute the set of non-properness of a projection restricted to an algebraic variety. Denoting by $I^{\mathbf{A}}$ the ideal associated to $W_1^{\mathbf{A}}$, this algorithm specializes in our case to computing the characteristic polynomial of the multiplication by X_1 in $\mathbb{Q}(T)[X_1,\ldots,X_n]/I^{\mathbf{A}}$. The set of non-properness of the projection on T is the reunion of the zero-sets of the denominators of this characteristic polynomial seen as univariate in X_1.
Consider the polynomial which is already studied in Section 2

$$f = X_1 + X_1^2X_2 + X_1^4X_2X_3$$

Performing the linear change of variables below

$$\begin{array}{rcl} X_1 & \leftarrow & X_1 + X_2 + X_3 \\ X_2 & \leftarrow & X_1 + 2X_2 + 3X_3 \\ X_3 & \leftarrow & X_1 + 4X_2 + 9X_3 \end{array}$$

one finds as a set of non-properness for the projection on T the zero-set of the univariate polynomial below

$$256\,T^2\,(20\,T+1)$$

Performing the linear change of variables below

$$\begin{array}{rcl} X_1 & \leftarrow & 10213\,X_1 + 41543\,X_2 + 51532\,X_3 \\ X_2 & \leftarrow & X_1 + 44904\,X_2 + 10334\,X_3 \\ X_3 & \leftarrow & X_1 + 58200\,X_2 + 1597\,X_3 \end{array}$$

one finds as a set of non-properness for the projection on T the zero-set of the univariate polynomial below

$$T^2\,(898540\,T+117941)\,.$$

Thus $K_\infty(f)$ is the gcd of these univariate polynomials and is $\{0\}$.

4. The algorithm and its complexity

Given $f \in \mathbb{Q}[X_1,\ldots,X_n]$, we show now how to compute the set of generalized critical values $K(f)$ of the mapping $x \in \mathbb{C}^n \to f(x) \in \mathbb{C}$. Since $K(f) = K_0(f) \cup K_\infty(f)$, we focus first on the computation of $K_0(f)$ and then we deal with the computation of $K_\infty(f)$.
Our algorithms rely on tools coming from polynomial system solving. We use Gröbner bases and the Geometric resolution algorithm. Gröbner bases are a standard tool in polynomial system solving since it allows to test the membership of a polynomial to an ideal, to compute elimination ideals, and to reduce the resolution of zero-dimensional polynomial systems to linear algebra computations. Gröbner bases have a complexity within $D^{\mathcal{O}(n)}$ arithmetic operations in $\mathbb{Q}$ when the input polynomial family generates a zero-dimensional ideal (see []).

The geometric resolution algorithm [17, 27] is more recent and goes back to [16, 14, 15]. The input is a polynomial system of equation and inequations encoded by a straight-line program and defining a constructible set. It returns generic points in each equi-dimensional component of the Zariski-closure of the constructible set defined by the input. These generic points are encoded by rational parameterizations

$$\begin{cases} X_n & = & \frac{q_n(T)}{q_0(T)} \\ & \vdots & \\ X_1 & = & \frac{q_1(T)}{q_0(T)} \\ q(T) & = & 0 \end{cases}$$

where T is a new variable. Thus the output of the geometric resolution algorithm is a list of $n+2$-tuples of univariate polynomials $(q, q_0, q_1, \ldots, q_n)$. This algorithm is probabilistic, but its complexity is well-controlled. We denote by $M(x)$ the cost of multipyling univariate polynomials of degree x and the notation $p \in \mathcal{O}_{\log}(x)$ means that $p \in \mathcal{O}(x \log x^a)$ for some constant a.

Theorem 4.1 (Complexity result for geometric resolution). [27] *Let $g_1, \ldots, g_S$ and g be polynomials of degree bounded by D in $\mathbb{Q}[X_1, \ldots, X_n]$, represented by a Straight-Line Program of length $\mathcal{L}$. There exists an algorithm computing a geometric resolution of the Zariski-closure $V(g_1, \ldots, g_S) \setminus V(g)$ whose arithmetic complexity is:*

$$\mathcal{O}_{\log}(Sn^4(n\mathcal{L} + n^4)M(D\mathfrak{d}))^3$$

where $\mathfrak{d}$ is the maximum of the sums of the algebraic degrees of the irreducible components of the intermediate varieties defined as the Zariski-closures of the constructible sets $g_1 = \cdots = g_i = 0, g \neq 0$ for i in $1, \ldots, S$.

In practice, Gröbner bases remains, in general, the fastest tool to solve polynomial systems, in particular when the algorithms [12, 13] are used. The geometric resolution algorithm is implemented as a `Magma` package by G. Lecerf (see [26]).
Hereafter, we describe how to compute $K_0(f)$ and $K_\infty(f)$ using Gröbner bases and the geometric resolution algorithm. When using Gröbner bases, one obtains a deterministic algorithm and an efficient behaviour in practice (see Section 7). When using the geometric resolution algorithm, we obtain a probabilistic algorithm whose complexity is well-controlled.

Computation of $K_0(f)$. The first step of an algorithm computing $K(f)$ is obviously the computation of the set of critical values $K_0(f)$ of f. This is encoded as the set of roots of a univariate polynomial. Denote by I the ideal

$$\langle f - T, \frac{\partial f}{\partial X_1}, \ldots, \frac{\partial f}{\partial X_n} \rangle.$$

Sard's Theorem ensures that there exists $P \in \mathbb{Q}[T]$ such that: $\langle P \rangle = I \cap \mathbb{Q}[T]$ and, by definition, the set of roots of P is $K_0(f)$.
Gröbner bases allow such computations of elimination ideals.

Algorithm computing $K_0(f)$ using Gröbner bases

- **Input:** a polynomial f in $\mathbb{Q}[X_1,\ldots,X_n]$.
- **Output:** a univariate polynomial $P \in \mathbb{Q}[T]$ such that its zero-set is $K_0(f)$.
- Compute a Gröbner basis G for an elimination ordering $[X_1,\ldots,X_n] > [T]$ of the ideal generated by:
$$\langle f - T, \frac{\partial f}{\partial X_1}, \ldots, \frac{\partial f}{\partial X_n} \rangle.$$
- Return the element of G belonging to $\mathbb{Q}[T]$.

Remark that $\sharp K_0(f) \leq (D-1)^n$ since it is defined as the values taken by a polynomial on each isolated primary component of an ideal defined by n polynomials of degree $D-1$. So, one could expect to obain an algorithm computing $K_0(f)$ having a complexity within $(D-1)^{\mathcal{O}(n)}$. This aim can be reached by using the geometric resolution Algorithm. The first step is the computation of rational parametrizations of generic points in each equi-dimensional component of the algebraic variety defined by:
$$\frac{\partial f}{\partial X_1} = \cdots = \frac{\partial f}{\partial X_n} = 0.$$
Once they are obtained, one can obtain the values taken by f at these points which are encoded by a univariate polynomial.

Probabilistic Algorithm computing $K_0(f)$ using the Geometic Resolution Algorithm

- **Input:** a polynomial f in $\mathbb{Q}[X_1,\ldots,X_n]$.
- **Output:** a univariate polynomial $P \in \mathbb{Q}[T]$ such that its zero-set is $K_0(f)$.
- Let G be the rational parametrizations returned by the geometric resolution algorithm taking as input $\frac{\partial f}{\partial X_1},\ldots,\frac{\partial f}{\partial X_n}$.
- For each element $g = (q, q_0, q_1, \ldots, q_n)$ of G, substitute for $i = 1,\ldots,n$ in $f - T$ the variables X_i by $\frac{q_i}{q_0}$. Put the result to the same denominator and compute the resultant of the obtained polynomial with respect to the variable T.
- Return the product of the computed polynomials.

The complexity of the above algorithm is dominated by the cost of computing a geometric resolution of the algebraic variety defined by:
$$\frac{\partial f}{\partial X_1} = \cdots = \frac{\partial f}{\partial X_n}$$

Computation of $K_\infty(f)$. It remains to show how to compute $K_\infty(f)$. Following Remark 3.8 and Example 3.3, this task can be achieved by linear algebra computations in the quotient ring $\mathbb{Q}(T)[X_1,\ldots,X_n]/I^{\mathbf{A}}$ where $I^{\mathbf{A}}$ is the ideal associated to $W_1^{\mathbf{A}}$.

Deterministic Algorithm. In order to obtain a deterministic algorithm, we must check that the chosen linear change of variables $\mathbf{A}$ is generic enough. This can be done by first testing that the projection on X_1 restricted the Zariski-closure of the constructible set defined

$$\frac{\partial f^{\mathbf{A}}}{\partial X_1} = \cdots = \frac{\partial f^{\mathbf{A}}}{\partial X_{n-1}} = 0, \quad \frac{\partial f^{\mathbf{A}}}{\partial X_n} \neq 0$$

is proper and that the projection on (X_1, X_2) is birational. The property of properness can be tested by using algorithms to compute sets of non-properness in [36, 25]. Once this is ensured, one can specialize X_1 to any rational value and test the property of birationality by computing a rational univariate representation. Once this check is done, it is sufficient to instantiate X_1 in $f^{\mathbf{A}}$ and repeat the above.

In the sequel we denote by `SetOfNonProperness` a subroutine taking as input a polynomial system of equations and inequations and a set of variables and computes the set of non-properness of the projection on the variables given as input restricted to the Zariski-closure of the constructible set defined by the input polynomial system.

Algorithm computing $K_\infty(f)$ using Gröbner bases

- **Input:** a polynomial f in $\mathbb{Q}[X_1,\ldots,X_n]$.
- **Output:** a univariate polynomial $P \in \mathbb{Q}[T]$ such that its zero-set is $K_0(f)$.
- Choose randomly $\mathbf{A} \in GL_n(\mathbb{C})$ and check it is generic enough until this test returns true.
- Return `SetOfNonProperness`$([f^{\mathbf{A}} - T = \frac{\partial f^{\mathbf{A}}}{\partial X_1} = \cdots = \frac{\partial f^{\mathbf{A}}}{\partial X_{n-1}} = 0, \frac{\partial f^{\mathbf{A}}}{\partial X_n} \neq 0], \{T\})$

Probabilistic Algorithm. As in the case of the computation of $K_0(f)$, Gröbner bases do not allow to obtain complexity results even if the first choice of $\mathbf{A}$ to be correct. To reach this aim, one also uses extensions of the geometric resolution algorithms allowing to lift the parameter. Here, in the input polynomial system

$$f^{\mathbf{A}} - T = \frac{\partial f^{\mathbf{A}}}{\partial X_1} = \cdots = \frac{\partial f^{\mathbf{A}}}{\partial X_{n-1}} = 0, \frac{\partial f^{\mathbf{A}}}{\partial X_n} \neq 0$$

T is considered as the parameter. From [2], if $\mathbf{A}$ is generic enough, this defines a zero-dimensional system generating a radical ideal. The output is a geometric

resolution

$$\begin{cases} X_n & = & \frac{q_n(X_1,T)}{q_0(X_1,T)} \\ & \vdots & \\ X_2 & = & \frac{q_2(X_1,T)}{q_0(X_1,T)} \\ q(X_1,T) & = & 0 \end{cases}$$

The set of non properness of the projection on T restricted to the Zariski-closure of the constructible set defined by the input polynomial system is contained the least commun multiple of the denominators of the coefficients of q.

Probabilistic Algorithm computing $K_0(f)$ using the Geometic Resolution Algorithm

- **Input:** a polynomial f in $\mathbb{Q}[X_1, \ldots, X_n]$.
- **Output:** a univariate polynomial $P \in \mathbb{Q}[T]$ such that its zero-set is $K_0(f)$.
- Consider T as a parameter in the polynomial system $f^{\mathbf{A}} - T = \frac{\partial f^{\mathbf{A}}}{\partial X_1} = \cdots = \frac{\partial f^{\mathbf{A}}}{\partial X_{n-1}} = 0, \frac{\partial f^{\mathbf{A}}}{\partial X_n} \neq 0$ and compute a geometric resolution.
- Lift the parameter.
- Return the least common multiple of the denominators in the coefficients of the polynomial q.

The probabilistic versions of the algorithms computing $K_0(f)$ and $K_\infty(f)$ lead to the following complexity result.

Theorem 4.2 (Complexity result). *The above probabilistic algorithm computing $K_0(f)$ performs at most $\mathcal{O}(n^7 D^{4n})$ arithmetic operations in $\mathbb{Q}$.*
The above probabilistic algorithm computing $K_\infty(f)$ performs at most $\mathcal{O}(n^7 D^{4n})$ arithmetic operations in $\mathbb{Q}$.

5. Application I: testing the emptiness of a semi-algebraic set defined by a single inequality

In this section, we show how to use the above algorithm to compute at least one point in each connected component of a semi-algebraic set defined by a single inequality.
This result can be proved using classical techniques of real algebraic geometry.

Theorem 5.1 (Semi-algebraic sets). *Let f be a polynomial in $\mathbb{Q}[X_1, \ldots, X_n]$ and $\mathcal{S}$ be the semi-algebraic set defined by $f > 0$. Let $e \in \mathbb{Q}$ be such that $0 < e < \min(|r|, r \in K(f) \cap \mathbb{R})$.*
Consider the hypersurface $\mathcal{H}_e$ defined by $f - e = 0$. Then, for each connected component S of $\mathcal{S}$, there exists a connected component C of $\mathcal{H}_e \cap \mathbb{R}^n$ such that $C \subset S$.

Remark 5.2. From Theorem 5.1, deciding the emptiness of the semi-algebraic set defined by $f > 0$ is reduced to decide if a hypersurface defined by a polynomial with coefficients in $\mathbb{Q}$ contains real points.
Substituting f by $-f$ one can deal with semi-algebraic sets defined by $f < 0$.
At last, computing at least one point in each connected component of the semi-algebraic set defined by $f \neq 0$ is done by computing at least one point in each connected component of the semi-algebraic sets defined by $f > 0$ and $f < 0$.

The Algorithm. The algorithm relies on Theorem 5.1. Given a polynomial f in $\mathbb{Q}[X_1, \ldots, X_n]$ of degree D, the algorithm computes at least one point in each connected component of the semi-algebraic set defined by $f > 0$. The first step is the computation of the set of generalized critical values of the mapping $f : x \in \mathbb{C}^n \to f(x) \in \mathbb{C}$. Using the probabilistic version of the algorithm provided in Section 4, this can be done within $\mathcal{O}(n^7 D^{4n})$ arithmetic operations in $\mathbb{Q}$.
Isolating the real solutions of the polynomial encoding the set of generalized critical values of f is done within $\mathcal{O}(D^{3n})$ arithmetic operations in $\mathbb{Q}$ using the variant of Uspensky's algorithm designed in [33]. Choosing a positive rational number e between 0 and the smallest positive real generalized critical value is immediate.
It remains to compute at least one point in each connected component of the real counterpart of the hypersurface defined by $f - e = 0$. This can be done using the algorithm designed in [35] within $\mathcal{O}(n^7 D^{3n})$ arithmetic operations in $\mathbb{Q}$. This leads to the following theorem.

Theorem 5.3 (Complexity result). *Let f be a polynomial in $\mathbb{Q}[X_1, \ldots, X_n]$ of degree D and $\mathcal{S}$ be the semi-algebraic set defined by $f > 0$. The probabilistic version of the above algorithm computes at least one point in each connected component of $\mathcal{S}$ with a complexity within $\mathcal{O}(n^7 D^{4n})$ arithmetic operations in $\mathbb{Q}$.*

6. Application II: determining the existence of real regular points in a hypersurface

In this section, we focus on the following problem: given a polynomial $f \in \mathbb{Q}[X_1, \ldots, X_n]$, decide if the hypersurface $\mathcal{H}$ defined by $f = 0$ contains real regular points. Hence, the problem consists in deciding if the real dimension of $\mathcal{H} \cap \mathbb{R}^n$ equals the complex dimension of $\mathcal{H}$. This problem appears in many applications (in particular in automated geometric reasoning or in algorithmic geometry) studying generic geometric situations.

This can be solved using the Cylindrical Algebraic Decomposition but the complexity of this method is doubly exponential in the number of variables and, in practice, this method is limited to problems having 3 or 4 variables.
Such a problem can also be tackled by computing the *real radical* of the ideal $\langle f \rangle \subset \mathbb{Q}[X_1, \ldots, X_n]$ (which is the radical ideal of $\mathbb{Q}[X_1, \ldots, X_n]$ whose associated algebraic variety is the smallest one – for the inclusion ordering – containing $\mathcal{H} \cap \mathbb{R}^n$). This can be done by using the algorithms designed in []. Nevertheless, the

complexity of such methods seems to be doubly exponential in the number of variables and no efficient implementations have been obtained from these works.
The real dimension of $\mathcal{H}$ can be computed using [7, Chapter 14]. The complexity of this algorithm is $D^{\mathcal{O}(n^2)}$. Nevertheless, this algorithm does not provide satisfactory results in practice due to the use of several infinitesimals and some growth of degree which lead to a high complexity constant (which is here as an exponent) and are difficult to manage in practical implementations.
All the methods above compute exactly the real dimension of $\mathcal{H} \cap \mathbb{R}^n$ which is stronger than the expected output. In the case where f is square-free, the problem in which we are interested can be tackled by deciding if all the semi-algebraic sets $\mathcal{S}_i \subset \mathbb{R}^n$ defined by $f = 0, \frac{\partial f}{\partial X_i} \neq 0$ (for $i = 1, \ldots, n$) are empty or not. Each semi-algebraic set $\mathcal{S}_i$ is studied by studying the real algebraic sets of $\mathbb{R}\langle\varepsilon\rangle^n$ defined by $f = \frac{\partial f}{\partial X_i} - \varepsilon = 0$ and $f = \frac{\partial f}{\partial X_i} + \varepsilon = 0$. The complexity of this method is $D^{\mathcal{O}(n)}$ but we are lead here to study n distinct semi-algebraic sets defined by an equation and an inequation.

In the sequel, we show how to determine the existence of real regular points in a hypersurface to the problem of deciding the emptiness of a single semi-algebraic set defined by a single inequality. The probabilistic version of our algorithm has a complexity within $\mathcal{O}(n^7 D^{4n})$ arithmetic operations in $\mathbb{Q}$.

Theorem 6.1 (Existence of regular real points). *Let f be a square-free polynomial in $\mathbb{Q}[X_1, \ldots, X_n]$ and $\mathcal{H} \subset \mathbb{C}^n$ be the hypersurface defined by $f = 0$. There exist regular real points in $\mathcal{H}$ if and only if there exist $(x, x') \in \mathbb{R}^n \times \mathbb{R}^n$ such that $f(x) > 0$ and $f(x') < 0$.*

Proof. Suppose first that $\mathcal{H}$ contains real regular points and let y be such a point. Since f is square-free, one has $\mathbf{grad}_y(f) \neq \mathbf{0}$. Now, considering the line passing through y and supported by the vector $\mathbf{grad}_y(f)$ and a Taylor development of f along this line near y, it is clear that f is positive and negative along this line.
Suppose now that $\mathcal{H}$ does not contain a real regular zero. Then, the real locus of $\mathcal{H}$ (which may be empty) is contained in the singular locus of $\mathcal{H}$. Since the co-dimension of the singular locus of $\mathcal{H}$ is greater than 1, the complementary of $\mathcal{H} \cap \mathbb{R}^n$ in $\mathbb{R}^n$ is connected. This implies that either the semi-algebraic set defined by $f > 0$ is empty or the semi-algebraic set defined by $f < 0$ is empty. □

The Algorithm. The algorithm based on Theorem 6.1 works as follows. The input of the algorithm is a polynomial f in $\mathbb{Q}[X_1, \ldots, X_n]$ of degree D. Compute the square-free part of f.
Determine the sign of f on a randomly chosen point at which f does not vanish. In practice, this step is immediate while in theory, on has to test each point in a grid of size D^{n^2} to be sure to find a point at which f does not vanish. Since our complexity estimates are based on probabilistic algorithms, we suppose that the cost of this step is the one of the evaluation of f, i.e. $\mathcal{O}_{\log}(D^n)$ arithmetic operations in $\mathbb{Q}$.

If f is found to be positive on the test-point, test the emptiness of the semi-algebraic defined by $f < 0$, else test the emptiness of the semi-algebraic set defined by $f > 0$. Using the algorithm designed in Section 5 using the computation of generalized critical values, this is done within $\mathcal{O}(n^7 D^{4n})$ arithmetic operations in $\mathbb{Q}$.

7. Practical results

We have implemented the algorithms presented in Sections 4, 5 and 6 using Gröbner bases.

The Gröbner engine which is used is FGb [11] which is implemented in `C` by J.-C. Faugère. Computing rational parametrizations of the complex roots of a zero-dimensional ideal from a Gröbner basis is done by RS which is implemented in `C` by F. Rouillier. Isolation of real roots of univariate polynomials with rational coefficients is done by RS using the algorithm provided in [30].

The resulting implementation is a part of the development version of the RAGLib Maple library [34]. We do not describe implementation details allowing us to avoid an *explicit* linear change of variables by using a choice of generic projections. We also don't describe modular tests which allow us to test if the chosen projections are good. However, observe that the first choices have always been correct.

All the computations have been performed on a PC Intel Pentium Centrino Processor 1.86 GHz with 2048 Kbytes of Cache and 512 MB of RAM.

7.1. Description of the test-suite.

The following polynomial has been submitted by D. Lazard and S. Lazard. It appears in a problem of algorithmic geometry studying the Voronoi Diagram of three lines. The question was first to determine if the zero-set of discriminant of the following polynomial with respect of the variable u contains real regular points. This discriminant has degree 30. This discriminant is the product of a polynomial of degree 18 and several polynomials up to an odd power whom zero-set could not contain a real regular point since they are sums of squares. The polynomial of degree 18 is **Lazard II**. D. Lazard and S. Lazard have also asked to determine if the following polynomial which is denoted by **Lazard I** in the sequel is always positive.

$$\begin{aligned}
&16\,a^2\left(\alpha^2+1+\beta^2\right)u^4+16\,a\left(-\alpha\,\beta\,a^2+ax\alpha+2\,a\alpha^2+2\,a+2\,a\beta^2+ay\beta-\alpha\,\beta\right)u^3+\\
&\left(\left(24\,a^2+4\,a^4\right)\alpha^2+\left(-24\,\beta\,a^3-24\,a\beta-8\,ya^3+24\,xa^2-8\,ay\right)\alpha+24\,a^2\beta^2+4\,\beta^2-\right.\\
&\left.8\,\beta\,xa^3+4\,y^2a^2+24\,y\beta\,a^2-8\,ax\beta+16\,a^2+4\,x^2a^2\right)u^2+\left(-4\,\alpha\,a^3+4\,ya^2-\right.\\
&\left.4\,ax-8\,a\alpha+8\,\beta\,a^2+4\,\beta\right)\left(\beta-a\alpha+y-ax\right)u+\left(a^2+1\right)\left(\beta-a\alpha+y-ax\right)^2
\end{aligned}$$

In the sequel, we denote by **Lazard I** the above polynomial and by **Lazard II** the discriminant of **Lazard I** with respect to the variable u.

The following polynomial appears in [24]. The problem consists in determining the conditions on a, b, c and d such that the ellipse defined by:

$$\frac{(x-c)^2}{a^2} + \frac{(y-d)^2}{b^2} = 1$$

is inside the circle defined by $x^2 + y^2 - 1 = 0$.

$$\begin{aligned}
&4a^6c^2d^2 + 2a^2b^2d^6 - 6a^2b^2d^4 + a^4c^4 + 2a^4c^2d^6 - 6a^2b^2c^4 - 6a^4b^2c^4 + 4a^6b^2d^2 + \\
&a^8b^4 + 6b^4c^2d^2 - 2b^6c^4d^2 + a^8d^4 + 6a^2b^6d^2 - 8a^4b^4d^2 - 4a^4b^2d^6 - 6b^4c^4d^2 - 8a^4b^4c^2 + \\
&6a^6b^2c^2 - 8a^2b^4c^2 + 6a^4b^4d^4 - 2b^4c^2d^4 - 4a^2b^4c^6 - 4a^6b^4c^2 - 6a^2b^4d^4 - 2a^4c^4d^2 + \\
&10a^4b^2d^4 - 2a^2b^8c^2 - 6a^2b^6c^4 + a^4b^8 + 6a^2b^2d^2 + 6a^6b^4d^2 - 4a^4b^6d^2 + b^4d^4 + b^4c^8 + \\
&10a^2b^4c^4 + 6a^2b^2c^2 + 4a^2b^6c^2 + a^4d^8 + 4b^6c^2d^2 + 6a^4b^6c^2 - 8a^4b^2d^2 + \\
&4a^4b^2c^2 - 2a^8b^2d^2 + 6a^4c^2d^2 + 4a^2b^4d^2 - 6a^6b^2d^4 + 6a^4b^4c^4 - 2a^6c^2d^4 + \\
&2b^4c^6d^2 + 2a^2b^2c^6 - 6a^4c^2d^4 + b^8c^4 + 2a^4b^2 - 4a^4d^2 + a^4 - 2b^6 - 2a^6 + a^8 + \\
&b^8 + b^4 + 2a^2b^4 + 2b^6c^6 - 2b^8c^2 - 6b^6c^4 + 2a^6b^4 - 2a^2b^2 - 2a^6b^6 + 2a^4b^6 - \\
&2a^2b^8 - 6a^4b^2c^4d^2 + 2a^2b^4c^4d^2 + 2a^4b^2c^2d^4 - 6a^2b^4c^2d^4 - 6a^4b^2c^2d^2 - 6a^2b^4c^2d^2 + \\
&4a^2b^2c^4d^4 + 2a^2b^2c^2d^6 + 2a^2b^2c^4d^2 + 2a^2b^2c^2d^4 - 10a^2b^2c^2d^2 + 6a^2b^6c^2d^2 - \\
&6a^4b^4 + 2a^2b^6 - 2a^8b^2 + 2a^6b^2 + 6a^6b^2c^2d^2 - 10a^4b^4c^2d^2 - 4b^4c^6 + 6b^4c^4 + 6b^6c^2 - \\
&2a^6c^2 + 2a^2b^2c^6d^2 + a^4c^4d^4 - 2a^4c^2 - 2b^6d^2 - 4a^4d^6 + 2a^6d^6 - \\
&2a^8d^2 - 6a^6d^4 + 6a^6d^2 + b^4c^4d^4 - 4b^4c^2 + 6a^4d^4 - 2b^4d^2
\end{aligned}$$

Below, in the column **JK** we give the timings for computing generalized critical values by using the algorithm of [28]. We obviously use the same Gröbner engine `FGb` than ours for this algorithm. The column **AlgoHyp** corresponds to the maximum of the timings obtained by

- our algorithm computing at least one point in each connected component of the semi-algebraic set defined by the positivity of our input;
- our algorithm computing at least one point in each connected component of the semi-algebraic set defined by the negativity of our input.

The column **CAD** contains the timings of an implementation of the open CAD algorithm in Maple which is due to G. Moroz and F. Rouillier.
The algorithms provided in [7] never end on these examples.

Pbm	♯**vars**	**Degree**	**JK**	**AlgoHyp**	**CAD**
Lazard I	6	8	∞	60 sec.	∞
Lazard II	5	18	∞	10 hours.	∞
Ellips-Circle	4	12	∞	90 sec.	5 min.

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