Variant Real Quantifier Elimination: Algorithm and Application

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ABSTRACT

We study a *variant* of the real quantifier elimination problem (QE). The variant problem requires the input to satisfy a certain *extra condition*, and allows the ouput to be *almost* equivalent to the input. In a sense, we are strengthening the pre-condition and weakening the post-condition of the standard QE problem.

The motivation/rationale for studying such a variant QE problem is that many quantified formulas arising in applications do satisfy the extra conditions. Furthermore, in most applications, it is sufficient that the ouput formula is almost equivalent to the input formula. Thus, we propose to solve a *variant* of the initial quantifier elimination problem.

We present an algorithm (VQE) , that exploits the strengthened pre-condition and the weakened post-condition. The main idea underlying the algorithm is to substitute the repeated projection step of CAD by a single projection without carrying out a parametric existential decision over the reals.

We find that the algorithm VQE can tackle important and challenging problems, such as numerical stability analysis of the widely-used MacCormack's scheme. The problem has been practically out of reach for standard QE algorithms in spite of many attempts to tackle it. However the current implementation of VQE can solve it in about 1 day.

1. INTRODUCTION

Real quantifier elimination (QE) is a fundamental problem in mathematical logic and computational real algebraic geometry. Furthermore, it naturally arises in many challenging problems in diverse application areas. Thus, there have been extensive research on developing mathematical theories, efficient algorithms, software systems, and applications (to cite only a few: [32, 7, 12, 20, 2, 1, 5, 16, 17, 18, 29, 31]).

In this paper, we study a *variant* of the QE problem, obtained by strengthening the pre-condition and weakening the post-condition of the standard QE problem. Roughly speaking, we strengthen the pre-condition by requiring that the input quantified formula has a certain logical (boolean and quantification) structure and that some polynomials satisfy certain geometric conditions (such as equidimentionality, smoothness, compactness, etc). We weaken the post-condition by allowing that the input and the output are "almost" equivalent, unlike the standard QE where the input and the output are required to be exactly equivalent.

The motivation/rationale for studying a variant QE problem is that currently many important and challenging application problems are still practically out of reach for standard QE algorithms/software systems, in spite of tremendous progress made in their efficiency during last 30 years. We choose to strengthen the pre-condition because many important quantified formulas arising in real-life applications (for example, numerical stability analysis, control system design, etc) naturally satisfy the extra conditions. Furthermore, in most real-life applications, it is sufficient that the ouput formula is almost equivalent to the input formula.

We present an algorithm (VQE), that exploits the strengthened pre-condition and the weakened post-condition. The main idea underlying the algorithm is to substitute the repeated projection step of CAD by a single projection without carrying out a parametric existential decision over the reals.

We find that the algorithm VQE can tackle challenging problems such as stability analysis of the renowned MacCormack's scheme. The problem has been practically out of reach for standard QE algorithms implemented in MATHEMATICA, REDLOG or QEPCAD. However the current implementation of the algorithm VQE solves it in about 1 day.

Structure of the paper: Section 2 provides a precise statement of the variant QE problem. Section 3 presents an algorithm VQE for the problem. Section 4 gives a proof of the algorithm's correctness. Section 5 describes a case study where the algorithm is successfully applied to a challenging problem arising from numerical stability anaysis.

2. PROBLEM

In this section, we state the variant quantifier elimination problem precisely and illustrate it by a simple (toy) example. As stated in the abstract, the variant QE problem strengthens the pre-condition and weakens the post-condition of the standard QE problem. First, we introduce a notion that will be used in strengthening the pre-condition.

DEFINITION 1 (NATURAL SYSTEM). *We say that a polynomial system* $\mathcal{G} = \{g_1, \ldots, g_k\} \subset \mathbb{Q}[\mathbf{X}]$ *is a* natural system *iff the following two conditions are met*

- $\mathbf{H_1}$: $\langle \mathcal{G} \rangle$ *is radical and the complex variety defined by* $\mathcal{G}$ *is equidimensional, smooth, and of co-dimension* k
- $\mathbf{H_2}$: *the real variety defined by* $\mathcal{G}$ *in the* $\mathbf{X}$*-space is compact.*

We chose the above two conditions because they are naturally satisfied by many QE problems arising from applications, especially in stability analysis. Next, we introduce a notion that will be used in weakening the post-condition.

DEFINITION 2 (ALMOST EQUIVALENT). *We say that two formulas* Ψ *and* Φ *are* almost equivalent *iff the closure of the interior of the solution set of* Ψ *is the same as the closure of the interior of the solution set of* Φ.

Considering the closure of the interior of a semi-algebraic set is classically referred to as the *regular closure* of the considered semi-algebraic set (see [30]).

Now we are ready to state the problem.

Problem: Variant Quantifier Elimination (VQE)

Input: Ψ, a quantified formula of the form

$$\forall \mathbf{X} \qquad \mathcal{G}(\mathbf{X}) = 0 \quad \Longrightarrow \quad f(\mathbf{X}, \mathbf{Y}) \leq 0$$

where $\mathbf{X}$ and $\mathbf{Y}$ are lists of variables, $f \in \mathbb{Q}[\mathbf{X}, \mathbf{Y}]$, and $\mathcal{G} \subset \mathbb{Q}[\mathbf{X}]$ is a natural system.

Output: Φ, a quantifier-free formula almost equivalent to Ψ.

Example: We will illustrate the problem by a simple (toy) example. A non-trivial example will be given later in the application section. We claim that the input and the output in the following example satisfy the conditions in the above problem statement.

Input: Ψ, the quantified formula

$$\forall \mathbf{X} \qquad \mathcal{G}(\mathbf{X}) = 0 \quad \Longrightarrow \quad f(\mathbf{X}, \mathbf{Y}) \leq 0$$

where

$$\begin{aligned} \mathbf{X} &= \{X_1, X_2\} \\ \mathbf{Y} &= \{Y\} \\ \mathcal{G} &= \{X_1^2 + X_2^2 - 1\} \\ f &= X_1^2 Y - (X_2 - 1)^2 \end{aligned}$$

Output: Φ, the quantifier-free formula

$$Y < 0$$

To check the claim, let us take a look at the surfaces defined by the vanishing of $\mathcal{G}$ and f as shown in Figure 1. The cylinder (in red) is the vanishing set of $\mathcal{G}$ and the Whitney umbrella (in blue) is that of f.

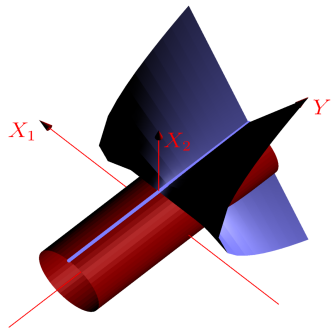

Figure 1: Simple example

It is immediate that $\langle \mathcal{G} \rangle$ is radical, that the complex variety defined by $\mathcal{G}$ is equidimensional, smooth, and of co-dimension 1, and that the real variety defined by $\mathcal{G}$ in the $\{X_1, X_2\}$-space is compact. Thus $\mathcal{G}$ is a natural system, and the input satisfies the condition in the problem statement.

It is also immediate from the drawings that the solution set of Ψ is given by $Y \leq 0$. Thus, the closure of the interior of the solution set of Ψ is $\{Y \mid Y \leq 0\}$. Likewise the closure of the interior of the solution set of Φ is $\{Y \mid Y \leq 0\}$. Therefore Ψ and Φ are almost equivalent. Hence the ouput satisfies the condition in the problem statement. □

3. ALGORITHM

In this section, we present an algorithm for the variant quantifier elimination problem stated in the previous section. We will also illustrate the algorithm by tracing it on the same simple (toy) example used for illustrating the problem. We will use the notations introduced in the previous section, such as $\Psi, \Phi, \mathbf{X}, \mathbf{Y}, \mathcal{G}, f$ and k.

Algorithm: $\Phi \leftarrow \text{VQE}(\Psi)$

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1. $J_1 \leftarrow$ the jacobian of $\mathcal{G} \cup \{f\}$ with respect to $\mathbf{X}$
2. $\Delta_1 \leftarrow$ the set of all minors of J_1 of size $k+1$
3. $J \leftarrow$ the jacobian of $\mathcal{G} \cup \{f\}$ with respect to $\mathbf{X} \cup \mathbf{Y}$
4. $\Delta \leftarrow$ the set of all minors of J of size $k+1$
5. $G \leftarrow$ a set of generators of $\langle \mathcal{G} \cup \Delta_1 \rangle : \langle \mathcal{G} \cup \Delta \rangle^\infty$
6. $E \leftarrow$ a set of generators of $\langle G \cup \{f\} \rangle \cap \mathbb{Q}[\mathbf{Y}]$
7. $P \leftarrow$ the set of all squarefree parts of E
8. $\mathscr{C} \leftarrow \mathsf{SemiAlgebraicDescription}(P)$
9. $\Phi \leftarrow \bigvee \{C \mid (C, S) \in \mathscr{C} \text{ and } \Psi(S) \text{ is true}\}$

Subalgorithms: Steps 1 and 7 can be carried out using standard algorithms available in computer algebra systems. Step 9 can be carried out by using any real decision algorithms such as CAD method, critical point method, etc. For the moment, the critical point method seems to be the best practical choice.

In Step 8, the subalgorithm SemiAlgebraicDescription takes as input a list of polynomials P in $\mathbb{Q}[\mathbf{Y}]$ defining an algebraic set V and outputs a set of couples

$$\mathscr{C} = \{(C_1, S_1) \ldots, (C_N, S_N)\}$$

such that C_i is a quantifier-free formula, S_i is a sample point of the semi-algebraic set defined by C_i and the closure

of each semi-algebraically connected component of $\mathbb{R}^p \setminus V$ equals the closure of the semi-algebraic sets defined by some of the C_i's. It can be carried out by using open-CAD algorithm. One can also use critical point methods [24, 11] and roadmap algorithms [6, 3, 27] and their parameterized versions. For the moment, CAD seems to be the best practical choice. □

Example continued: We will illustrate the algorithm VQE by tracing it on the simple (toy) example that we used for illustrating the variant quantifier elimination problem. Recall that

$$\begin{aligned} \mathbf{X} &= \{X_1, X_2\} \\ \mathbf{Y} &= \{Y\} \\ \mathcal{G} &= \{X_1^2 + X_2^2 - 1\} \\ f &= X_1^2 Y - (X_2 - 1)^2 \end{aligned}$$

1. We compute the jacobian of $\mathcal{G} \cup \{f\}$ with respect to $\mathbf{X}$, obtaining
$$J_1 = \begin{bmatrix} 2\,X_1 & 2\,X_2 \\ 2\,X_1 Y & -2\,X_2 + 2 \end{bmatrix}$$
2. We compute the set of all the minors of J_1 of size $1+1$, obtaining
$$\Delta_1 = \{-4\,X_1\,(X_2 - 1 + X_2 Y)\}$$
3. We compute the jacobian of $\mathcal{G} \cup \{f\}$ with respect to $\mathbf{X} \cup \mathbf{Y}$, obtaining
$$J = \begin{bmatrix} 2\,X_1 & 2\,X_2 & 0 \\ 2\,X_1 Y & -2\,X_2 + 2 & {X_1}^2 \end{bmatrix}$$
4. We compute the set of all minors of J of size $1+1$, obtaining
$$\Delta = \left\{-4\,X_1\,(X_2 - 1 + X_2 Y)\,, \quad 2\,{X_1}^3, \quad 2\,X_2 {X_1}^2\right\}$$
5. We compute a set of generators of $\langle \mathcal{G} \cup \Delta_1 \rangle : \langle \mathcal{G} \cup \Delta \rangle^\infty$, obtaining
$$G = \left\{{X_1}^2 + {X_2}^2 - 1, \quad X_2 - 1 + X_2 Y\right\}$$
6. We compute a set of generators of $\langle G \cup \{f\} \rangle \cap \mathbb{Q}[\mathbf{Y}]$, obtaining
$$E = \left\{Y^2\right\}$$
7. We compute a set of squarefree parts of E, obtaining
$$P = \{Y\}$$
8. We call SemiAlgebraicDescription(P), obtaining
$$\mathscr{C} = \{ \quad [y < 0, \quad -1]\,, \quad [y > 0, \quad +1] \quad \}$$
9. We compute $\bigvee\{C \mid (C, S) \in \mathscr{C} \text{ and } \Psi(S) \text{ is true}\}$, obtaining
$$\Phi \equiv y < 0$$

Comparison to CAD: It is instructive to observe how CAD would handle the problem. Note that $\mathcal{G} = 0$ can be viewed as "equational constraint". Hence we use the improved version of CAD that utilizes equational constraints [8, 18]. The first projection with respect to X_2 produces the polynomials:

$$\{\, X_1 \left(-4\,Y + {X_1}^2 (Y+1)^2\right), \quad 4\,(X_1 - 1)\,(X_1 + 1)\,\}$$

which are the square-free part of the resultant of f and g and the square-free part of the resultant of g and $\frac{\partial g}{\partial X_2}$. The second projection with respect to X_1 produces the polynomials

$$\{\, Y, \quad Y - 1, \quad Y + 1\,\}$$

The lifting phase will eventually produce, using the projection polynomials and sample point checks, a quantifier-free formula $Y \leq 0$.

It is *crucial* note that the projection polynomials

$$Y - 1, \quad Y + 1$$

are *irrelevant* to the quantifier elimination problem. They induce *useless* cells, causing inefficiency. In comparision, the VQE algorithm does not produce the irrelevant polynomials.

We will explain why this happens geometrically. See Figure 2 below. The CAD algorithm, among others, projects

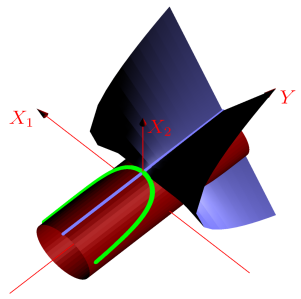

Figure 2: Simple example continued

the intersection of the red cylinder ($\mathcal{G}$) and the blue Whitney umbrella (f), which is complicated. On the other hand, the VQE algorithm projects the intersection of the green curve (G in Step 5) and the blue Whitney umbrella (f), which is much simpler.

This kind of advantage becomes much more pronounced for larger problems, yielding significant improvement in computing time. Indeed, on the one hand, the Bézout theorem of [13, Prop. 2.3] shows that the degree of the variety defined by G, which is computed at Step 5 (and therefore the degree of the variety defined by E which is computed at Step E) is bounded by $D^p((p+1)D)^{n-p}$ where $p = \sharp\mathcal{G}$, $n = \sharp\mathbf{X}$, and D bounds the degrees of f and the polynomials in $\mathcal{G}$. On the other hand, the set of the Y-projected polynomials computed by CAD has a degree which is still polynomial in D but doubly exponential in n. □

4. CORRECTNESS

In this section, we prove the correctness of the algorithm. Before plunging into the details, we provide a quick overview. Algorithm VQE has substituted the recursive projection step of CAD by the computation of the Zariski-closure of the projection of an algebraic set onto the $\mathbf{Y}$'s space (Step 6). Thus, we consider the projection $\pi : (\mathbf{x}_1, \ldots, \mathbf{x}_n, \mathbf{y}_1, \ldots, \mathbf{y}_p) \rightarrow (\mathbf{y}_1, \ldots, \mathbf{y}_p)$. The algebraic variety defined by $g_1 = \cdots = g_k = 0$ in $\mathbb{C}^{n+p}$ is denoted by $\mathscr{X}$. Denote by R a real closed field containing $\mathbb{R}$ and by C the algebraic closure of R. Given $(\mathbf{e}, \mathbf{y}) \in \mathsf{R}^{p+1}$, we denote by $\mathcal{V}_\mathbf{e} \subset \mathsf{C}^{n+p}$ the real algebraic set defined by $f(\mathbf{X}, \mathbf{Y}) - \mathbf{e} = g_1(\mathbf{X}) = \cdots = g_k(\mathbf{X}) = 0$. Denote by $\mathscr{S}$ the semi-algebraic set defined by the input of Algorithm VQE. We are looking at the points $\mathbf{y} \in \mathbb{R}^p$ such

that there exists $\mathbf{e} > 0$ such that $\pi^{-1}(\mathbf{y}) \cap \mathcal{V}_\mathbf{e} \cap \mathbb{R}^{n+p}$ is not empty. To this end, we try to characterize the frontier of $\mathscr{S}$. Roughly speaking, we prove hereafter that the vanishing set V associated to the polynomial system computed at Steps 6 and 7 of the algorithm contains the frontier of $\mathscr{S}$. Then, by studying the semi-algebraically connected components of $\mathbb{R}^p \setminus V$, one gets a description of the closure of $\text{int}(\mathscr{S})$. To this end, we prove that V is the projection of the *critical points* of the restriction π to $\mathcal{V}_\mathbf{e}$ when "$\mathbf{e} \to 0$".

Now we elaborate the technical details of the proof. Below, we introduce formally the notions of critical points and critical values and give some results that are useful for the proof of the algorithm. Assumption $\mathbf{H_1}$ here is technical but fundamental: it allows us to generate polynomial systems defining critical points of the restriction of π to $\mathcal{V}_\mathbf{e}$ for some "generic" $\mathbf{e}$. Then, we use assumption $\mathbf{H_2}$ to study the properties of the images of the semi-algebraically connected components of $\mathcal{V}_\mathbf{e}$ by π for some generic $\mathbf{e}$. Before that, we introduce some notions about infinitesimals, which will allow us to study formally the limits of the aforementionned critical points when $\mathbf{e} \to 0$.

Preliminaries: Let $\mathbb{K}$ be a field containing $\mathbb{Q}$. We consider in the sequel an infinitesimal ε and the Puiseux series field $\mathbb{K}\langle\varepsilon\rangle$. If $z = \sum_{i \geq i_0} a_i \varepsilon^{i/q} \in \mathbb{K}\langle\varepsilon\rangle$ for $q \in \mathbb{Q}^\star$ and $i_0 \in \mathbb{Z}$, z is *bounded* if and only if $i_0 \geq 0$. Note that, if $\mathbb{K} = \mathbb{R}$, this implies that there exists $\eta \in \mathbb{R}$ such that $z < \eta$.

Given $z = (z_1, \ldots, z_n) \in \mathbb{K}\langle\varepsilon\rangle^n$ we say that z is *bounded* if for $1 \leq i \leq n$, z_i is bounded by an element of $\mathbb{K}$. Given $q \in \mathbb{Q}^\star$, $i_0 \in \mathbb{Z}$ and a bounded element $z = \sum_{i \geq i_0} a_i \varepsilon^{i/q} \in \mathbb{K}\langle\varepsilon\rangle$ we denote by $\lim_{\varepsilon \to 0} z$, the real a_0. Given a bounded element $z \in \mathbb{K}\langle\varepsilon\rangle^n$, we denote by $\lim_{\varepsilon \to 0} z$ the point $(\lim_{\varepsilon \to 0}(z_1), \ldots, \lim_{\varepsilon \to 0}(z_n)) \in \mathbb{K}^n$.

Given a subset $A \subset \mathbb{K}\langle\varepsilon\rangle^n$, we denote by $\lim_{\varepsilon \to 0}(A)$ the set $\{\lim_{\varepsilon \to 0}(z) \mid z \in A \text{ and } z \text{ is bounded}\}$. Given a semi-algebraic (resp. constructible) set $A \subset \mathbb{R}^n$ (resp. $A \subset \mathbb{C}^n$) defined by a quantifier-free formula Φ with polynomials in $\mathbb{R}[X_1, \ldots, X_n]$, we denote by $\text{Ext}(A, \mathbb{R}\langle\varepsilon\rangle^n)$ (resp. $\text{Ext}(A, \mathbb{C}\langle\varepsilon\rangle^n)$) the set of solutions of Φ in $\mathbb{R}\langle\varepsilon\rangle^n$ (resp. $\mathbb{C}\langle\varepsilon\rangle^n$).

Denote by $\bar{\mathbb{K}}$ the algebraic closure of $\mathbb{K}$. Let $V \subset \bar{\mathbb{K}}^n$ be an algebraic variety and φ be a polynomial mapping $V \to \bar{\mathbb{K}}^m$. The set of regular points of V is denoted by $\text{reg}(V)$. Given $\mathbf{x} \in V$, the tangent space to V at $\mathbf{x}$ is denoted by $T_\mathbf{x} V$. The differential of φ at $\mathbf{x}$ is denoted by $d_\mathbf{x}\varphi$.

A point $\mathbf{x} \in \text{reg}(V)$ is a *critical point* of φ if $d_\mathbf{x}\varphi(T_\mathbf{x} V) \neq \bar{\mathbb{K}}^m$; we denote by $\text{crit}(\varphi, V)$ the set of all critical points of φ. A *critical value* of φ is the image by φ of a critical point. We denote by $\mathscr{D}(\varphi, V)$ the set of critical values of φ. A *regular value* is a point of $\bar{\mathbb{K}}^m$ which is a not a critical value. The algebraic version of Sard's theorem (see [28, Chapter 6, Theorem 2, pp 141]) asserts that $\mathscr{D}(\varphi, V)$ is enclosed in a proper Zariski-closed subset of $\bar{\mathbb{K}}^m$. Given a polynomial family $H = (h_1, \ldots, h_r)$ and a set of variables $\mathbf{L} = [\ell_1, \ldots, \ell_s]$, $\text{jac}(H, \mathbf{L})$ denotes the jacobian matrix $(\frac{\partial h_i}{\partial \ell_j})_{(i,j) \in \{1,\ldots,r\} \times \{1,\ldots,s\}}$

LEMMA 1. *Let $\{h_1, \ldots, h_r, h\} \subset \mathbb{K}[X_1, \ldots, X_n]$ such that the ideal $\langle h_1, \ldots, h_r\rangle$ is equidimensional, radical and that its associated algebraic variety V is smooth of co-dimension r.*

There exists a non-empty Zariski-open subset $\mathscr{E} \subset \bar{\mathbb{K}}$ such that for all $\mathbf{e} \in \mathscr{E}$, the algebraic variety $V_\mathbf{e}$ defined by $h_1 = \cdots = h_r = h - \mathbf{e} = 0$ is smooth and the ideal $\langle h_1, \ldots, h_r, h - \mathbf{e}\rangle \subset \mathbb{K}[\mathbf{X}, \mathbf{Y}]$ is radical and equidimensional. Moreover, either $V_\mathbf{e}$ is empty or it has co-dimension $r + 1$.

PROOF. Consider the polynomial mapping $\tilde{h} : x \in V \to h(x)$. Suppose that $\dim(\tilde{f}(V)) = 1$. From the algebraic version of Sard's Theorem [28, Chapter 6, Theorem 2, pp 141], the set E of critical values of $\tilde{h}$ is Zariski-closed and for all $\mathbf{e} \in \bar{\mathbb{K}} \setminus E$, the intersection $V_\mathbf{e}$ of V with the hypersurface defined by $h - \mathbf{e} = 0$ is smooth and $d_z\tilde{h}$ is surjective for all $z \in V_\mathbf{e}$. Since $\langle h_1, \ldots, h_r\rangle$ is radical and equidimensional of co-dimension r, for all $z \in V_\mathbf{e} \subset V$, $\text{jac}_z(h_1, \ldots, h_r)$ has rank r. Since $d_z\tilde{f}$ is surjective, $\text{jac}_z(h_1, \ldots, h_r, h)$ has rank $r + 1$. Thus, from the implicit function theorem, $\text{codim}_z(V_\mathbf{e}) = r + 1$. Thus, $V_\mathbf{e}$ is smooth and equidimensional. Moreover, from [9, Theorem 2.1], $\langle h_1, \ldots, h_r, h - \mathbf{e}\rangle$ is radical.

Suppose now that $\dim(\tilde{h}(V)) = 0$. This implies that there exists a Zariski-closed subset $E \subsetneq \bar{\mathbb{K}}$ such that for all $\mathbf{e} \in \bar{\mathbb{K}} \setminus E$, $V_\mathbf{e}$ is empty. □

LEMMA 2. *Let $\mathbb{K}$ be a field containing $\mathbb{Q}$ and $\{h_1, \ldots, h_r\} \subset \mathbb{K}[X_1, \ldots, X_n]$, V be the algebraic variety defined by $h_1 = \cdots = h_r = 0$ and $\varphi : \mathbf{x} \in V \to (\varphi_1(\mathbf{x}), \ldots, \varphi_s(\mathbf{x}))$ be a polynomial mapping. Suppose that $\langle h_1, \ldots, h_r\rangle$ is radical and equidimensional and that V is smooth of co-dimension r and that $s \leq n - r$. Let Δ be the set of $(r+s, r+s)$-minors of $\text{jac}(h_1, \ldots, h_r, \varphi_1, \ldots, \varphi_s)$. Then $\text{crit}(\varphi, V)$ is the algebraic variety associated to $\langle h_1, \ldots, h_r\rangle + \langle\Delta\rangle$.*

Exploiting compactness: Our goal now is to relate the frontier of $\mathscr{S}$ with $\pi(\lim_{\varepsilon \to 0}(\text{crit}(\mathcal{V}_\varepsilon)))$. To this end, we study the images by π of the semi-algebraically connected components of $\mathcal{V}_\mathbf{e} \cap \mathbb{R}^n$ (for $\mathbf{e}$ "generic enough").

LEMMA 3. *Let $\mathbf{e} \in \mathbb{R}$. Suppose that the assumptions $\mathbf{H_1}$ and $\mathbf{H_2}$ are satisfied and that $\mathcal{V}_\mathbf{e}$ is smooth and equidimensional. Let $C_\mathbf{e}$ be a semi-algebraically connected component of $\mathcal{V}_\mathbf{e} \cap \mathbb{R}^{n+p}$ and Z be a semi-algebraically connected component of $\mathbb{R}^p \setminus \mathscr{D}(\pi, \mathcal{V}_\mathbf{e})$. Then, $Z \cap \pi(C_\mathbf{e}) \neq 0$ implies that $Z \subset \pi(C_\mathbf{e})$.*

PROOF. We claim that the frontier of $\pi(C_\mathbf{e})$ is contained in $\mathscr{D}(\pi, \mathcal{V}_\mathbf{e})$. Let $\mathbf{y} \in \mathbb{R}^p$ be a point of the frontier of $\pi(C_\mathbf{e})$. Since the real algebraic set defined by $g_1 = \cdots = g_k = 0$ in $\mathbb{R}^n$ is compact (Assumption $\mathbf{H_1}$), there exists a closed ball $B(\mathbf{y}, r)$ of center $\mathbf{y}$ and radius r such that $\pi^{-1}(B(\mathbf{y}, r)) \cap C_\mathbf{e}$ is compact since $C_\mathbf{e} \subset (\mathcal{V}_\mathbf{e} \cap \mathbb{R}^{n+p}) \subset (\mathscr{X} \cap \mathbb{R}^{n+p})$. Consider a sequence of points $(\mathbf{y}_\ell)_{\ell \in \mathbb{N}} \subset B(\mathbf{y}, r) \cap \pi(C_\mathbf{e})$ converging to $\mathbf{y}$. Then, there exists a sequence of points $(\mathbf{x}_\ell)_{\ell \in \mathbb{N}} \subset \mathbb{R}^n$ such that $(\mathbf{x}_\ell, \mathbf{y}_\ell) \in \pi^{-1}(B(\mathbf{y}, r)) \cap C_\mathbf{e}$. Since $\pi^{-1}(B(\mathbf{y}, r)) \cap C_\mathbf{e}$ is compact, once can extract a converging subsequence from $(\mathbf{x}_\ell, \mathbf{y}_\ell)$. The projection of the limit of this subsequence is $\mathbf{y}$ since $\mathbf{y}_\ell$ tends to $\mathbf{y}$. Thus, $\mathbf{y} \in \pi(C_\mathbf{e})$ which means that there exists $\mathbf{x} \in \mathbb{R}^n$ such that $(\mathbf{x}, \mathbf{y}) \in C_\mathbf{e}$. This implies that $d_{(\mathbf{x},\mathbf{y})}\pi$ is not surjective, which means that $\mathbf{y} \in \mathscr{D}(\pi, \mathcal{V}_\mathbf{e})$. Our claim follows.

Consider now a semi-algebraically connected component Z of $\mathbb{R}^p \setminus \mathscr{D}(\pi, \mathcal{V}_\mathbf{e})$ such that $Z \cap C_\mathbf{e} \neq \emptyset$. Now, we prove that this implies $Z \subset \pi(C_\mathbf{e})$.

Since $Z \cap C_\mathbf{e} \neq \emptyset$, there exists $z \in Z \cap \pi(C_\mathbf{e})$. Suppose, by contradiction, that there exists $z' \in Z$ such that $z' \notin \pi(C_\mathbf{e})$. Since $Z \subset \mathbb{R}^p$ is semi-algebraically connected, there exists a semi-algebraic continuous path $\gamma : t \in [0, 1] \to \gamma(t) \subset Z$ such that $\gamma(0) = z$ and $\gamma(1) = z'$. Since $z \in \pi(C_\mathbf{e})$, $z' \notin \pi(C_\mathbf{e})$ and γ is continuous, there exists $\vartheta \in [0, 1]$ such that $\gamma(\vartheta) \in Z$ belongs to the boundary of $\pi(C_\mathbf{e})$. This implies that $\gamma(\vartheta) \in \mathscr{D}(\pi, \mathcal{V}_\mathbf{e}) \cap Z$ which contradicts the fact that Z is a semi-algebraically connected component of $\mathbb{R}^p \setminus \mathscr{D}(\pi, \mathcal{V}_\mathbf{e})$. □

We denote by $\mathscr{S}'$ the complementary of $\mathscr{S}$ in $\mathbb{R}^p$.

LEMMA 4. *Suppose that the assumptions* $\mathbf{H_1}$ *and* $\mathbf{H_2}$ *are satisfied. Let* S' *be a semi-algebraically connected component of* $\mathscr{S}'$. *Given* $\mathbf{e} \in \mathbb{R}$, *denote by* $Z_\mathbf{e}$ *a semi-algebraically connected component of* $\mathbb{R}^p \setminus \mathscr{D}(\pi, \mathscr{V}_\mathbf{e})$. *Then, there exists* $\mathbf{e}_0 \in]0, +\infty[$ *such that, for all* $\mathbf{e} \in]0, \mathbf{e}_0[$, $Z_\mathbf{e} \cap S' \neq \emptyset$ *implies* $Z_\mathbf{e} \subset S'$.

PROOF. Suppose that there exists a semi-algebraically connected component C of $\mathscr{X} \cap \mathbb{R}^{n+p}$ such that for all $(\mathbf{x}, \mathbf{y}) \in C$, $f(\mathbf{x}, \mathbf{y}) > 0$. Since C is a semi-algebraically connected component of the real real algebraic set defined by $g_1 = \cdots = g_k = 0$ and since $\{g_1, \ldots, g_k\} \subset \mathbb{Q}[\mathbf{X}]$, $\pi(C) = \mathbb{R}^p$. This implies that $\mathscr{S}'$ is semi-algebraically connected and equalled to $\mathbb{R}^p$. Since, from the algebraic Sard's Theorem, for all $\mathbf{e} > 0$, $\mathscr{D}(\pi, \mathcal{V}_\mathbf{e})$ is contained in a strict Zariski-closed subset of $\mathbb{R}^p$ (see Lemma 1), each semi-algebraically connected component of its complementary in $\mathbb{R}^p$ is non-empty and contained in $\mathscr{S}'$.

Suppose now that each semi-algebraically connected component of $\mathscr{X} \cap \mathbb{R}^{n+p}$, contains a point at which f is not positive. Consider $\mathbf{y}' \in S'$. Then, there exists $\mathbf{x}' \in \mathbb{R}^n$ such that $f(\mathbf{x}', \mathbf{y}') > 0$. Consider the connected component C of $\mathscr{X} \cap \mathbb{R}^{n+p}$ containing $(\mathbf{x}', \mathbf{y}')$ and let $(\mathbf{x}, \mathbf{y}) \in C$ such that $f(\mathbf{x}, \mathbf{y}) \leq 0$. Since C is semi-algebraically connected, there exists a semi-algebraic continuous path $\gamma : [0, 1] \to C$ such that $\gamma(0) = (\mathbf{x}', \mathbf{y}')$ and $\gamma(1) = (\mathbf{x}, \mathbf{y})$. Note that $f(\gamma(0)) > 0$ and $f(\gamma(1)) \leq 0$.

Thus, the set $A = \{t \in [0, 1] \mid f(\gamma(t)) > 0\}$ is semi-algebraic and non-empty. Denote by A_0 the semi-algebraically connected component of A containing 0. Since f and γ are continuous and semi-algebraic, $f \circ \gamma : [0, 1] \to \mathbb{R}$ is continuous and semi-algebraic. Therefore, A_0 is an open semi-algebraically connected interval which is bounded since it is contained in $[0, 1]$. Thus, $t_0 = \sup_{t \in A_0} t$ lies in $[0, 1]$ and $t_0 \notin A_0$. By continuity of $f \circ \gamma$, $f(\gamma(t_0)) = 0$ and choosing $\mathbf{e}_0 = \sup_{t \in A_0} f(\gamma(t))$, the intermediate value theorem implies that for all $0 < \mathbf{e} < \mathbf{e}_0$, there exists $t_\mathbf{e} \in A_0$ such that $f(\gamma(t_\mathbf{e})) = \mathbf{e}$ and $\gamma(t) \in C$. Note also that for all $t \in A_0$, $\pi(\gamma(t)) \in S'$ (since $f(\gamma(t)) > 0$ for all $t \in A_0$) and that $\{\pi(\gamma(t)) \mid t \in A_0\}$ is semi-algebraically connected.

For all $0 < \mathbf{e} < \mathbf{e}_0$ and $t_\mathbf{e} \in A_0$ such that $f(\gamma(t_\mathbf{e})) = \mathbf{e}$, denote by $C_\mathbf{e}$ the semi-algebraically connected component of $\mathcal{V}_\mathbf{e} \cap \mathbb{R}^{n+p}$ containing $\gamma(t_\mathbf{e})$. Suppose that $\pi(C_\mathbf{e}) \subset S'$.

Then, applying Lemma 1 with $\{h_1, \ldots, h_k\} = \{g_1, \ldots, g_k\}$ and $h = f$, there exists a non-empty Zariski-open set $\mathscr{E} \subset \mathbb{R}$ such that for all $\mathbf{e} \in \mathscr{E}$, $\mathcal{V}_\mathbf{e}$ is smooth and equidimensional. Thus, for all $\mathbf{e} \in]0, \mathbf{e}_0[\cap \mathscr{E}$, there exists a semi-algebraically connected component $C_\mathbf{e}$ of $\mathcal{V}_\mathbf{e} \cap \mathbb{R}^{n+p}$ such that $\pi(C_\mathbf{e}) \subset S'$ and $\mathcal{V}_\mathbf{e}$ is smooth and equidimensional. Applying Lemma 3 implies that there exists a semi-algebraically connected component Z of $\mathbb{R}^p \setminus \mathscr{D}(\pi, \mathcal{V}_e)$ such that $Z \subset \pi(C_\mathbf{e})$. Since $\pi(C_\mathbf{e}) \subset S'$, the result follows.

We prove now that $\pi(C_\mathbf{e}) \subset S'$. Since there exists $t_\mathbf{e} \in A_0$ such that $C_\mathbf{e}$ contains $\gamma(t_\mathbf{e})$ and since $\{\pi(\gamma(t)) \mid t \in [0, t_\mathbf{e}]\}$ is semi-algebraically connected, there exists a continuous semi-algebraic path γ_1 linking $\mathbf{y}' = \pi(\gamma(0))$ to $\pi(\gamma(t_\mathbf{e}))$. Consider now an arbitrary point $(\mathbf{x}'', \mathbf{y}'')$ in $C_\mathbf{e}$. Remark that this implies that $\mathbf{y}'' \in \mathscr{S}'$ (since, by definition, $f(\mathbf{x}'', \mathbf{y}'') = \mathbf{e} > 0$). Since $C_\mathbf{e}$ is semi-algebraically connected and since it contains $\gamma(t_\mathbf{e})$, there exists a continuous semi-algebraic path $\gamma_\mathbf{e}$ linking $\gamma(t_\mathbf{e})$ to $(\mathbf{x}'', \mathbf{y}'')$. Denote by γ_2 the continuous semi-algebraic path $\pi(\gamma_\mathbf{e})$. Note that γ_2 links $\pi(\gamma(t_\mathbf{e}))$ to $\mathbf{y}''$. Now, remark that $\gamma_1 \cup \gamma_2$ is a continuous semi-algebraic path linking $\mathbf{y}'$ to $\mathbf{y}''$. Since S' is a semi-algebraically connected component of $\mathscr{S}'$, $\mathbf{y}' \in S'$, and $\mathbf{y}'' \in \mathscr{S}'$, this implies that $\mathbf{y}'' \in S'$. Our claim follows. □

Geometric results: The result below allows us to characterize the frontier of $\mathscr{S}'$ (and hence the one of $\mathscr{S}$).

LEMMA 5. *Suppose that the assumptions* $\mathbf{H_1}$ *and* $\mathbf{H_2}$ *are satisfied. Let* S' *be a semi-algebraically connected component of* $\mathscr{S}'$. *Denote by* $\mathrm{fr}(\mathscr{S}')$ *its frontier (for the euclidean topology). Then* $\mathrm{fr}(\mathscr{S}')$ *is contained in* $\pi(\lim_{\varepsilon \to 0}(\mathrm{crit}(\pi, \mathcal{V}_\varepsilon)))$.

PROOF. Let S' be a semi-algebraically connected component of $\mathscr{S}'$ and $\mathbf{y}$ be a point in $\mathrm{fr}(S')$. We first prove that $\mathbf{y} \in \lim_{\varepsilon \to 0}(\mathscr{D}(\pi, \mathcal{V}_\varepsilon))$.

In the sequel, given $r > 0$, $B(\mathbf{y}, r) \subset \mathbb{R}^p$ denotes the ball of $\mathbb{R}^p$ of center $\mathbf{y}$ and radius r. From the Transfer Principle, proving that $\mathbf{y} \in \lim_{\varepsilon \to 0}(\mathscr{D}(\pi, \mathcal{V}_\varepsilon))$ is equivalent to prove that for all $r > 0$, there exists $\mathbf{e}_0 > 0$ such that for all $\mathbf{e} \in]0, \mathbf{e}_0[$, $B(\mathbf{y}, r) \cap \mathscr{D}(\pi, \mathcal{V}_\mathbf{e}) \neq \emptyset$.

Suppose on the contrary that there exists $r_0 > 0$ such that for all $\mathbf{e}_0 > 0$ there exists $\mathbf{e} \in]0, \mathbf{e}_0[$ such that $B(\mathbf{y}, r_0) \cap \mathscr{D}(\pi, \mathcal{V}_\mathbf{e}) = \emptyset$. This implies that there exists a semi-algebraically connected component $Z_\mathbf{e}$ of $\mathbb{R}^p \setminus \mathscr{D}(\pi, \mathcal{V}_\mathbf{e})$ such that $B(\mathbf{y}, r_0) \subset Z_\mathbf{e}$. Since $\mathbf{y}$ belongs to the frontier of S', $B(\mathbf{y}, r_0) \cap S' \neq \emptyset$ which implies that $Z_\mathbf{e} \cap S' \neq \emptyset$. From Lemma 4, this implies that $Z_\mathbf{e} \subset \pi(S')$. One deduces that $B(\mathbf{y}, r_0) \subset S'$ since $B(\mathbf{y}, r_0) \subset Z_\mathbf{e}$. This contradicts the fact that $\mathbf{y}$ belongs to the frontier of S'.

Now, we prove that $\mathbf{y} \in \pi(\lim_{\varepsilon \to 0}(\mathrm{crit}(\pi, \mathcal{V}_\varepsilon)))$. Since the restriction of π to $\mathscr{X}$ is supposed to be proper, there exists $r_0 > 0$ such that $\mathscr{B} = \pi^{-1}(B(\mathbf{y}, r_0)) \cap \mathscr{X}$ is compact. Since $\mathbf{y} \in \lim_{\varepsilon \to 0}(\mathscr{D}(\pi, \mathcal{V}_\varepsilon))$, there exists a semi-algebraically connected component A of $\mathrm{crit}(\pi, \mathcal{V}_\varepsilon) \cap \mathscr{B}$ such that $\mathbf{y}$ belongs to $\lim_{\varepsilon \to 0} \pi(A)$. Since A is semi-algebraically connected and bounded, [4, Proposition 12.43] implies that $\lim_{\varepsilon \to 0}(A)$ exists and is semi-algebraically connected, closed and bounded. Thus $\pi(\lim_{\varepsilon \to 0}(A))$ is closed (see [4, Theorem 3.20]) and contains $\mathbf{y}$ since $\mathbf{y} \in \lim_{\varepsilon \to 0}(\pi(A))$ (see [4, Lemma 3.21]). Now, remark that $A \subset \mathrm{crit}(\pi, \mathcal{V}_\varepsilon)$ implies $\lim_{\varepsilon \to 0}(A) \subset \lim_{\varepsilon \to 0}(\mathrm{crit}(\pi, \mathcal{V}_\varepsilon))$. Thus, $\mathbf{y} \in \lim_{\varepsilon \to 0}(\pi(A)) = \pi(\lim_{\varepsilon \to 0}(A)) \subset \pi(\lim_{\varepsilon \to 0}(\mathrm{crit}(\pi, \mathcal{V}_\varepsilon)))$. □

The result below allows us to compute the Zariski-closure of $\pi(\lim_{\varepsilon \to 0}(\mathrm{crit}(\pi, \mathcal{V}_\varepsilon)))$ without introducing explicitely infinitesimals in the computations. This is crucial for reaching practical efficiency. A similar approach is developped in [22] for grabbing sampling points in real singular hypersurfaces.

LEMMA 6. *Suppose that the assumption* $\mathbf{H_1}$ *is satisfied. Given* $1 \leq i \leq p$, *let* Δ_1 *be the set of* $(k+1, k+1)$*-minors of* $\mathrm{jac}([g_1, \ldots, g_k, f], [\mathbf{X}])$ *and* Δ *be the set of* $(k+1, k+1)$*-minors of* $\mathrm{jac}([g_1, \ldots, g_k, f], [\mathbf{X}, \mathbf{Y}])$. *Denote by* I *the ideal* $\langle \{g_1, \ldots, g_k\} \cup \Delta_1 \rangle$ *and by* J *the ideal* $\langle \{g_1, \ldots, g_k\} \cup \Delta_1 \rangle : \langle \{g_1, \ldots, g_k\} \cup \Delta \rangle^\infty$.
⋆ *the algebraic variety associated to* $(J + \langle f \rangle) \cap \mathbb{Q}[Y_1, \ldots, Y_p]$ *equals the Zariski-closure of* $\pi(\lim_{\varepsilon \to 0}(\mathrm{crit}(\pi, \mathcal{V}_\varepsilon)))$.
⋆ *the algebraic variety associated to* $(J + \langle f \rangle) \cap \mathbb{Q}[Y_1, \ldots, Y_p]$ *has dimension less than* p.

PROOF. Proving that the algebraic variety associated to $J + \langle f \rangle$ equals $\lim_{\varepsilon \to 0}(\mathrm{crit}(\pi, \mathcal{V}_\varepsilon))$ is sufficient.

We first prove that the algebraic variety associated to $J + \langle f \rangle$ is contained in $\lim_{\varepsilon \to 0}(\mathrm{crit}(\pi, \mathcal{V}_\varepsilon))$. Consider an element $\mathbf{z}$ of the algebraic variety associated to $J + \langle f \rangle$ and

denote by Z the irreducible component of this variety containing $\mathbf{z}$. Given $r > 0$, denote by $B(\mathbf{z}, r) \subset \mathbb{C}^{n+p}$ the ball centered at $\mathbf{z}$ of radius r. We prove that for all $r > 0$, $\text{Ext}(Z \cap B(\mathbf{z}, r), \mathbb{C}\langle\varepsilon\rangle^{n+p})$ has a non-empty intersection with $\text{crit}(\pi, \mathcal{V}_\varepsilon)$ which implies that $\lim_{\varepsilon\to 0}(\text{crit}(\pi, \mathcal{V}_\varepsilon))$ contains $\mathbf{z}$. Since $J = \langle\{g_1, \ldots, g_k\} \cup \Delta_1\rangle : \langle\{g_1, \ldots, g_k\} \cup \Delta\rangle^\infty$, Z contains points such $\text{jac}([g_1, \ldots, g_k, f], [\mathbf{X}, \mathbf{Y}])$ has maximal rank. Denote by $\mathfrak{S}$ the algebraic variety associated to $\langle\{g_1, \ldots, g_k\} \cup \Delta\rangle$. Then, $Z \setminus \mathfrak{S}$ is not empty. Moreover, $\{t \in \mathbb{C} \mid \exists \mathbf{z}' \in Z,\ f(\mathbf{z}') = t\}$ has dimension 1 which implies that $Z \setminus \mathfrak{S}$ can not have dimension 0. Thus, for all $r > 0$, $B(\mathbf{z}, r) \cap Z \setminus \mathfrak{S}$ is positive dimensional and then, it is connected. Remark now that $f(\mathbf{z}) = 0$. Thus, from the intermediate value theorem, there exists $\mathbf{z}' \in \text{Ext}((Z \setminus \mathfrak{S}) \cap B(\mathbf{y}, r), \mathbb{C}\langle\varepsilon\rangle^{n+p})$ such that $f(\mathbf{z}') = \varepsilon$. From Lemma 2 applied with $\{h_1, \ldots, h_r\} = \{g_1, \ldots, g_k, f - \varepsilon\}$ and $\varphi = p$, $\mathbf{z}' \in \text{crit}(\pi, \mathcal{V}_\varepsilon)$.

Now, we prove that $\lim_{\varepsilon\to 0}(\text{crit}(\pi, \mathcal{V}_\varepsilon))$ is contained in the intersection of the algebraic variety associated to J and the hypersurface defined by $f = 0$.

Let $\mathbf{z} \in \lim_{\varepsilon\to 0}(\text{crit}(\pi, \mathcal{V}_\varepsilon))$. By continuity of f, this implies that that $f(z) = 0$. Thus, it remains to prove that $\mathbf{z}$ belongs to V_J. Since $\mathbf{z} \in \lim_{\varepsilon\to 0}(\text{crit}(\pi, \mathcal{V}_\varepsilon))$, from the Transfer Principle, for all $r > 0$, there exists an open set $U \in \mathbb{C}\setminus\{0\}$ whose closure contains 0 such that for all $\mathbf{e} \in U$, $B(\mathbf{z}, r)$ has a non-empty intersection with $\text{crit}(\pi, \mathcal{V}_\mathbf{e})$. Since $\mathbf{H_1}$ is satisfied, one can apply Lemma 1. Then, there exists a non-empty Zariski-open subset $\mathscr{E}$ of $\mathbb{C}$ such that for all $\mathbf{e} \in \mathscr{E}$, $\mathcal{V}_\mathbf{e}$ is smooth and equidimensional. Note also that one can apply Lemma 2 in order to characterize the critical points of $\mathcal{V}_\mathbf{e}$ for $\mathbf{e} \in \mathscr{E}$.

Thus, without loss of generality, one can suppose that $U \subset \mathscr{E}$. This implies that for all $\mathbf{z_e} \in \mathcal{V}_\mathbf{e}$, $\text{rank}(\text{jac}([g_1, \ldots, g_k, f], [\mathbf{X}, \mathbf{Y}]))$ is maximal which implies that one of the minors in Δ does not vanish. This implies that $\mathbf{z_e}$ does not belong to the algebraic variety associated to $\langle\{g_1, \ldots, g_k\} \cup \Delta\rangle$. If $\mathbf{z_e} \in \text{crit}(\pi, \mathcal{V}_\mathbf{e})$ then $\mathbf{z_e}$ belongs to the algebraic variety associated to I since the set of polynomials generating I is contained in the one obtained applying Lemma 2 with $\{h_1, \ldots, h_r\} = \{g_1, \ldots, g_k, f - \mathbf{e}\}$ and $\varphi = \pi$. Thus $\mathbf{z_e}$ belongs to V_J. Thus, for all $r > 0$, there exists an open set $U \in \mathbb{C}\setminus\{0\}$ whose closure contains 0 such that for all $\mathbf{e} \in U$, $\mathcal{V}_\mathbf{e}$ is contained in V_J and it has a non-empty intersection with $B(\mathbf{z}, r)$. Since V_J, as an algebraic set, is closed, this implies that $\mathbf{z}$ belongs to V_J.

Now, we prove that the algebraic variety associated to $(J + \langle f\rangle) \cap \mathbb{Q}[Y_1, \ldots, Y_p]$ has dimension less than p. To this end, it is sufficient to prove that $\lim_{\varepsilon\to 0}(\mathscr{D}(\pi, \mathcal{V}_\varepsilon))$ has dimension less than p since $\pi(\lim_{\varepsilon\to 0}(\text{crit}(\pi, \mathcal{V}_\varepsilon))) \subset \lim_{\varepsilon\to 0}(\mathscr{D}(\pi, \mathcal{V}_\varepsilon))$.

Denote by $\mathfrak{e}$ a formal parameter and consider the ideal $J + \langle f - \mathfrak{e}\rangle \subset \mathbb{Q}(\mathfrak{e})[\mathbf{X}, \mathbf{Y}]$. Given $h \in \mathbb{Q}(\mathfrak{e})[\mathbf{X}, \mathbf{Y}]$ and $\mathbf{e} \in \mathbb{K}$ where $\mathbb{K}$ is a field containing $\mathbb{Q}$, denote by $\varphi_\mathbf{e}(h)$ the polynomial obtained by subsituting $\mathfrak{e}$ by $\mathbf{e}$ in p. From Lemma 1, there exists a non-empty Zariski open set $\mathscr{E}$ such that for all $\mathbf{e} \in \mathscr{E}$, $\varphi_\mathbf{e}(J + \langle f - \mathbf{e}\rangle) \cap \mathbb{Q}[\mathbf{Y}]$ is non-empty. This implies that $(J + \langle f - \mathfrak{e}\rangle) \cap \mathbb{Q}(\mathfrak{e})[\mathbf{Y}]$ is non-empty. Remark now that $\varphi_\varepsilon(J + \langle f - \mathfrak{e}\rangle) = J + \langle f - \varepsilon\rangle$. Thus $(J + \langle f - \varepsilon\rangle) \cap \mathbb{Q}[\mathbf{Y}] = \varphi_\varepsilon((J + \langle f - \mathfrak{e}\rangle) \cap \mathbb{Q}(\mathfrak{e})[\mathbf{Y}])$ is non-empty. Consider now $h \in (J + \langle f - \mathfrak{e}\rangle) \cap \mathbb{Q}(\mathfrak{e})[\mathbf{Y}]$ and $\bar{h}$ the primitive part of polynomial obtained by multiplying h by the ppcm of its coefficients. Remark now that $\varphi_\varepsilon(\bar{h}) \in (J + \langle f - \varepsilon\rangle)\mathbb{Q}(\varepsilon)[\mathbf{Y}]$ since $(J + \langle f - \varepsilon\rangle) \cap \mathbb{Q}(\varepsilon)[\mathbf{Y}] = \varphi_\varepsilon((J + \langle f - \mathfrak{e}\rangle) \cap \mathbb{Q}(\mathfrak{e})[\mathbf{Y}])$. Note also that $\varphi_\varepsilon(\bar{h}) \in \mathbb{Q}[\varepsilon][\mathbf{Y}]$ since $\bar{h}$ has no content and that the set of solutions of $\varphi_\varepsilon(\bar{h})$ in $\mathbb{C}\langle\varepsilon\rangle^p$ contains $\mathscr{D}(\pi, \mathcal{V}_\varepsilon)$ since $\varphi_\varepsilon(\bar{h}) \in J + \langle f - \varepsilon\rangle$. Denote by $h_0 \in \mathbb{Q}[\mathbf{Y}]$ the polynomial $\varphi_0(\varphi_\varepsilon(\bar{h}))$ and note that $h_0 \neq 0$ (since $\varphi_\varepsilon(\bar{h}) \in \mathbb{Q}[\varepsilon][\mathbf{Y}]$ has no content). The set of solutions of h_0 has dimension less than p since $h_0 \neq 0$ and it contains obviously $\lim_{\varepsilon\to 0}(\{\mathbf{z} \in \mathbb{C}\langle\varepsilon\rangle^p \mid h_0(\mathbf{z}) = 0\}$. Since $\lim_{\varepsilon\to 0}(\mathscr{D}(\pi, \mathcal{V}_\varepsilon)) \subset \lim_{\varepsilon\to 0}(\{\mathbf{z} \in \mathbb{C}\langle\varepsilon\rangle^p \mid \varphi_\varepsilon(\bar{h})(\mathbf{z}) = 0\})$, we are done. □

Proof of the algorithm: The following lemma states that $\mathscr{S}$ is a closed semi-algebraic set.

LEMMA 7. *Suppose that* $\mathbf{H_1}$ *is satisfied. The semi-algebraic set* $\mathscr{S}$ *is closed for the euclidean topology.*

PROOF. It is sufficient to prove that $\mathscr{S}'$ which is the complementary of $\mathscr{S}$ is open for the euclidean topology. Consider $\mathbf{y} \in \mathscr{S}$. Then, there exists $\mathbf{x} \in \mathbb{R}^n$ such that $g_1(\mathbf{x}) = \cdots = g_k(\mathbf{x}) = 0$ and $f(\mathbf{x}, \mathbf{y}) > 0$. Then, there exists an open neighbourhood U of $(\mathbf{x}, \mathbf{y})$ such that for all $\mathbf{z} \in U$, $f(\mathbf{z}) > 0$. Since $g_1, \ldots, g_k$ is a reduced regular sequence defining a smooth algebraic variety $\mathscr{X}$, from Lemma 2, $\mathbf{z} \in \text{crit}(\pi, \mathscr{X}) \Rightarrow \text{rank}(\text{jac}_\mathbf{z}(g_1, \ldots, g_k), [\mathbf{X}]) < k$. This contradicts the fact that $\mathscr{X}$ is smooth (since $\{g_1, \ldots, g_k\} \subset \mathbb{Q}[\mathbf{X}]$) and $g_1, \ldots, g_k$ is a reduced regular sequence (assumption $\mathbf{H_1}$). Hence, $\text{crit}(\pi, \mathscr{X}) = \emptyset$. In particular, $(\mathbf{x}, \mathbf{y})$ is neither a critical point of the restriction of π to $\mathscr{X}$ nor a singular point of $\mathscr{X}$. Hence, the differential of π at $(\mathbf{x}, \mathbf{y})$ is surjectve. This implies that $\pi(U \cap \mathscr{X})$ has dimension p, it contains $\mathbf{y}$ and it is open since π is a projection. □

LEMMA 8. *Let* $V \subset \mathbb{R}^p$ *be a real algebraic set containing the frontier of* $\mathscr{S}$*,* Z *be a semi-algebraically connected component* Z *of* $\mathbb{R}^p \setminus V$ *and* p *be an arbitrarily chosen point in* Z*. The semi-algebraic set defined by* $g_1 = \cdots = g_k = 0, f(\mathbf{X}, p) > 0$ *is empty if and only if* Z *has a non-empty intersection with the interior of a semi-algebraically connected component of* $\mathscr{S}$*.*

PROOF. Consider a semi-algebraically connected component S of $\mathscr{S}$ such that $\text{int}(S) \neq \emptyset$. Then, there exists a semi-algebraically connected component Z of $\mathbb{R}^p \setminus V$ such that $Z \cap \text{int}(S) \neq \emptyset$. Suppose that Z is not contained in S. This means that there exists a couple of points $(\mathbf{y}, \mathbf{y}') \in Z \times Z$ such that $\mathbf{y} \in Z \cap S$ and $\mathbf{y}' \notin S$. Since Z is semi-algebraically connected, there exists a continuous semi-algebraic path $\gamma : [0, 1] \to Z$ such that $\gamma(0) = \mathbf{y}$ and $\gamma(1) = \mathbf{y}'$. Consider $\{t \in [0, 1] \mid \gamma(t) \in S\}$. This is a closed semi-algebraic set since S is closed (see Lemma 7) and γ is continuous. Moreover this semi-algebraic set does not equal $[0, 1]$ (since $\gamma(1) \notin S$). Then, its frontier in $[0, 1]$ is non-empty and there exists $t \in [0, 1]$ such that $\gamma(t)$ belongs to the frontier of S. This implies that there exists a semi-algebraically connected component S' of $\mathbb{R}^p \setminus \mathscr{S}$ such that $\gamma(t)$ belongs to the frontier of S'. From Lemma 5, this implies that $\mathbf{y}' \in \pi(\lim_{\varepsilon\to 0}(\text{crit}(\pi, \mathcal{V}_\varepsilon))) \subset V$ which contradicts the fact that $\gamma(t)$ belongs to a semi-algebraically connected component of $\mathbb{R}^p \setminus V$. Thus, for all $p \in Z$, the semi-algebraic set defined by $g_1 = \cdots = g_k = 0, f(\mathbf{X}, p) > 0$ is empty.

Conversely, consider a semi-algebraically connected component Z of $\mathbb{R}^p \setminus V$ such that $Z \cap \text{int}(\mathscr{S}) = \emptyset$. If $Z \cap \mathscr{S} \neq \emptyset$, $Z \cap \mathscr{S} \subset \mathscr{S} \setminus \text{int}(\mathscr{S})$ and the points of $Z \cap \mathscr{S}$ lie in the frontier of a semi-algebraically connected component of $\mathbb{R}^p \setminus \mathscr{S}$. From Lemma 5, these points lie in V. This implies that these points do not belong to Z which is a contradiction.

Suppose now that $Z \cap \mathscr{S} = \emptyset$. Then, Z is contained in $\mathbb{R}^p \setminus \mathscr{S}$. Since the frontier of $\mathscr{S}' = \mathbb{R}^p \setminus \mathscr{S}$ is contained in V (see Lemma 5) and Z is a semi-algebraically connected component of $\mathbb{R}^p \setminus V$, Z is contained in a semi-algebraically connected component of $\mathbb{R}^p \setminus \mathscr{S}$. This implies that for all $p \in Z$, the semi-algebraic set defined by $g_1 = \cdots = g_k = 0, f(\mathbf{X}, p) > 0$ is non-empty. Our claim follows. □

THEOREM 9. *Algorithm VQE is correct.*

PROOF. Denote by V the real algebraic set defined by P which is computed at Step 7. From Lemma 6, $V \neq \mathbb{R}^p$ (since the algebraic set defined by P has co-dimension greater than or equalled to 1) and it contains $\pi(\lim_{\varepsilon \to 0}(\mathrm{crit}(\pi, \mathcal{V}_\varepsilon))) \cap \mathbb{R}^p$.

Remark that any semi-algebraically connected component S of $\mathscr{S}$ such that $\mathrm{int}(S) \neq \emptyset$ has a non-empty intersection with a semi-algebraically connected component of $\mathbb{R}^p \setminus V$.

Let S be a semi-algebraically connected component of $\mathscr{S}$, $Z_1, \ldots, Z_r$ be the set of semi-algebraically connected components of $\mathbb{R}^p \setminus V$ such that $Z_i \cap \mathrm{int}(S) \neq \emptyset$ (for $1 \leq i \leq r$). Denote by Z the union $Z_1 \cup \cdots \cup Z_r$. Lemma 8 implies $Z \subset \mathrm{int}(S)$. Moreover, since the frontier of $\mathrm{int}(S)$ is contained in V and since Z is the union of semi-algebraically connected components of $\mathbb{R}^p \setminus V$, $\mathrm{int}(S)$ and Z have the same closure. From the specification of SemiAlgebraicDescription, the union U_i of some semi-algebraic sets defined by its output (computed at Step 8) equals the closure of Z_i. This ends the proof. □

5. APPLICATION

Stability analysis is one of the most important tasks in numerically solving differential equations. It is essentially a quantifier elimination problem [16, 15]. In this section, we consider the MacCormack's scheme which is widely used in solving hyperbolic partial differential equations, especially in aerodynamics. The stability analysis of MacCormack's scheme can be reduced to eliminating quantifiers from

$$\forall \mathbf{X} \qquad \mathcal{G}(\mathbf{X}) = 0 \quad \Longrightarrow \quad f(\mathbf{X}, \mathbf{Y}) \leq 0$$

where $\mathbf{X} = \{c_1, s_1, c_2, s_2\}$ and $\mathbf{Y} = \{a, b\}$, and where $\mathcal{G} = \{\, c_1^2 + s_1^2 - 1,\ c_2^2 + s_2^2 - 1 \,\}$ and f is a polynomial of degree 14 and 163 terms:

$$\begin{aligned}
&4\,a^6b^2c_1{}^4c_2{}^2 - 8\,a^5b^3s_1s_2c_1{}^3c_2 - 8\,a^5b^3s_1s_2c_1{}^2c_2{}^2 + 4\,a^4b^4c_1{}^4c_2{}^2 + \\
&16\,a^4b^4c_1{}^3c_2{}^3 + 4\,a^4b^4c_1{}^2c_2{}^4 - 8\,a^3b^5s_1s_2c_1{}^2c_2{}^2 - 8\,a^3b^5s_1s_2c_1c_2{}^3 + \\
&4\,a^2b^6c_1{}^2c_2{}^4 - 4\,a^7bs_1s_2c_1{}^3 + 4\,a^6b^2c_1{}^4c_2 - 4\,a^6b^2c_1{}^3c_2{}^2 + 8\,a^5b^3s_1s_2c_1{}^3 + \\
&12\,a^5b^3s_1s_2c_1{}^2c_2 + 16\,a^5b^3s_1s_2c_1c_2{}^2 - 8\,a^4b^4c_1{}^4c_2 - 24\,a^4b^4c_1{}^3c_2{}^2 - \\
&24\,a^4b^4c_1{}^2c_2{}^3 - 8\,a^4b^4c_1c_2{}^4 + 16\,a^3b^5s_1s_2c_1{}^2c_2 + 12\,a^3b^5s_1s_2c_1c_2{}^2 + \\
&8\,a^3b^5s_1s_2c_2{}^3 - 4\,a^2b^6c_1{}^2c_2{}^3 + 4\,a^2b^6c_1c_2{}^4 - 4\,ab^7s_1s_2c_2{}^3 + a^8c_1{}^4 + \\
&12\,a^7bs_1s_2c_1{}^2 - 8\,a^6b^2c_1{}^4 - 12\,a^6b^2c_1{}^3c_2 - 12\,a^6b^2c_1{}^2c_2{}^2 - 4\,a^5b^3s_1s_2c_1{}^2 - \\
&8\,a^5b^3s_1s_2c_2{}^2 + 4\,a^4b^4c_1{}^4 + 22\,a^4b^4c_1{}^2c_2{}^2 + 4\,a^4b^4c_2{}^4 - 4\,a^4b^2c_1{}^4c_2{}^2 - \\
&8\,a^3b^5s_1s_2c_1{}^2 - 4\,a^3b^5s_1s_2c_2{}^2 + 8\,a^3b^3s_1s_2c_1{}^2c_2{}^2 - 12\,a^2b^6c_1{}^2c_2{}^2 - \\
&12\,a^2b^6c_1c_2{}^3 - 8\,a^2b^6c_2{}^4 - 4\,a^2b^4c_1{}^2c_2{}^4 + 12\,ab^7s_1s_2c_2{}^2 + b^8c_2{}^4 - \\
&4\,a^8c_1{}^3 - 12\,a^7bs_1s_2c_1 + 16\,a^6b^2c_1{}^3 + 12\,a^6b^2c_1{}^2c_2 + 20\,a^6b^2c_1c_2{}^2 - \\
&16\,a^5b^3s_1s_2c_1 - 4\,a^5b^3s_1s_2c_2 + 4\,a^5bs_1s_2c_1{}^3 + 8\,a^4b^4c_1{}^3 + 12\,a^4b^4c_1{}^2c_2 + \\
&12\,a^4b^4c_1c_2{}^2 + 8\,a^4b^4c_2{}^3 + 4\,a^4b^2c_1{}^4c_2 + 4\,a^4b^2c_1{}^3c_2{}^2 - 4\,a^3b^5s_1s_2c_1 - \\
&16\,a^3b^5s_1s_2c_2 - 12\,a^3b^3s_1s_2c_1{}^2c_2 - 12\,a^3b^3s_1s_2c_1c_2{}^2 + 20\,a^2b^6c_1{}^2c_2 + \\
&12\,a^2b^6c_1c_2{}^2 + 16\,a^2b^6c_2{}^3 + 4\,a^2b^4c_1{}^2c_2{}^3 + 4\,a^2b^4c_1c_2{}^4 - 12\,ab^7s_1s_2c_2 + \\
&4\,ab^5s_1s_2c_2{}^3 - 4\,b^8c_2{}^3 + 6\,a^8c_1{}^2 + 4\,a^7bs_1s_2 - 4\,a^6b^2c_1c_2 - 8\,a^6b^2c_2{}^2 - \\
&2\,a^6c_1{}^4 + 12\,a^5b^3s_1s_2 - 12\,a^5bs_1s_2c_1{}^2 - 14\,a^4b^4c_1{}^2 + 8\,a^4b^4c_1c_2 - \\
&14\,a^4b^4c_2{}^2 - 4\,a^4b^2c_1{}^3c_2 + 10\,a^4b^2c_1{}^2c_2{}^2 + 12\,a^3b^5s_1s_2 + 4\,a^3b^3s_1s_2c_1{}^2 + \\
&16\,a^3b^3s_1s_2c_1c_2 + 4\,a^3b^3s_1s_2c_2{}^2 - 8\,a^2b^6c_1{}^2 - 4\,a^2b^6c_1c_2 + 10\,a^2b^4c_1{}^2c_2{}^2 - \\
&4\,a^2b^4c_1c_2{}^3 + 4\,ab^7s_1s_2 - 12\,ab^5s_1s_2c_2{}^2 + 6\,b^8c_2{}^2 - 2\,b^6c_2{}^4 - 4\,a^8c_1 - \\
&16\,a^6b^2c_1 + 8\,a^6c_1{}^3 + 12\,a^5bs_1s_2c_1 - 12\,a^4b^4c_1 - 12\,a^4b^4c_2 - 8\,a^4b^2c_1{}^2c_2 - \\
&16\,a^4b^2c_1c_2{}^2 - 4\,a^3b^3s_1s_2c_1 - 4\,a^3b^3s_1s_2c_2 - 16\,a^2b^6c_2 - 16\,a^2b^4c_1{}^2c_2 - \\
&8\,a^2b^4c_1c_2{}^2 + 12\,ab^5s_1s_2c_2 - 4\,b^8c_2 + 8\,b^6c_2{}^3 + a^8 + 8\,a^6b^2 - 12\,a^6c_1{}^2 - \\
&4\,a^5bs_1s_2 + 14\,a^4b^4 - 2\,a^4b^2c_1{}^2 + 12\,a^4b^2c_1c_2 + 6\,a^4b^2c_2{}^2 + a^4c_1{}^4 + 8\,a^2b^6 + \\
&6\,a^2b^4c_1{}^2 + 12\,a^2b^4c_1c_2 - 2\,a^2b^4c_2{}^2 + 2\,a^2b^2c_1{}^2c_2{}^2 - 4\,ab^5s_1s_2 + b^8 - \\
&12\,b^6c_2{}^2 + b^4c_2{}^4 + 8\,a^6c_1 + 4\,a^4b^2c_1 - 4\,a^4b^2c_2 - 4\,a^4c_1{}^3 - 4\,a^3bs_1s_2c_1 - \\
&4\,a^2b^4c_1 + 4\,a^2b^4c_2 - 4\,ab^3s_1s_2c_2 + 8\,b^6c_2 - 4\,b^4c_2{}^3 - 2\,a^6 - 2\,a^4b^2 + 8\,a^4c_1{}^2 + \\
&4\,a^3bs_1s_2 - 2\,a^2b^4 - 2\,a^2b^2c_1{}^2 + 4\,a^2b^2c_1c_2 - 2\,a^2b^2c_2{}^2 + 4\,ab^3s_1s_2 - 2\,b^6 + \\
&8\,b^4c_2{}^2 - 8\,a^4c_1 - 4\,a^2b^2c_1 - 4\,a^2b^2c_2 - 8\,b^4c_2 + 3\,a^4 + 6\,a^2b^2 - 2\,a^2c_1{}^2 + \\
&3\,b^4 - 2\,b^2c_2{}^2 + 4\,a^2c_1 + 4\,b^2c_2 - 2\,a^2 - 2\,b^2
\end{aligned}$$

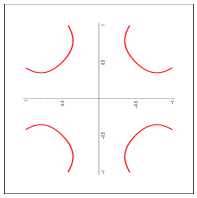

Figure 3: h_1

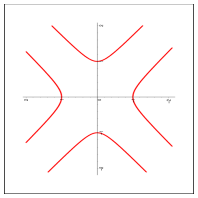

Figure 4: h_2

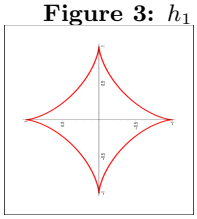

Figure 5: h_3

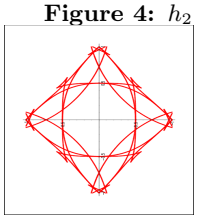

Figure 6: h_4

All the previous attempts to algorithmically solve the QE problem on a computer failed (aborted after several weeks of computation). Some conservative estimate indicates that it would take more than million years. After months' effort, an ad-hoc pen-paper solution has been found in [14]. However, it seems that the underlying idea cannot be turned into an algorithm. We applied the VQE algorithm to this problem. It is trivial to check that $\mathcal{G}$ is a natural system. Thus, the VQE algorithm can be used for the problem. We implemented the VQE on top of the following packages:

- **FGb**[10]: in C, by J.C. Faugère, for Gröbner bases computations for Steps 1 through 7
- **RS**[21]: in C, by F. Rouillier, for isolating the real solutions of zero-dimensional ideals for Step 8
- **OpenCAD**[19]: in Maple, by G. Moroz and F. Rouillier, for Step 8.
- **RAGlib**[23]: in Maple, by M. Safey El Din, for Step 9.

The computations have been performed on a PC Intel(R) Xeon(R) 2.50GHz with 6144 KB of Cache and 20 GB of RAM. The following table gives the computing times for each step:

Step	1	2	3	4	5	6	7	8	9
Time	<1s	<1s	<1s	<1s	1m	4h	2m	1h	15h

Thus the problem is solved in about 1 day. Note that Steps 1–5 takes short time as expected. Step 6–8 ("projection and lifting") takes 5 hours. It is significantly faster than the standard QE softwares, (where they had to be aborted after several weeks). Step 9 is most time-consuming because it applies a decision algorithm on each of the 7652 sample points produced by Step 8. The number 7652 is large, but it estimated to be much smaller than what the standard QE would produce (estimated to be at least millions). We also believe that further improvements of VQE could reduce 7652 to a smaller number.

Step 7 produces 9 polynomials. The five of them are trivial: $a+1, a, b, a-1, a^4 - a^2 + 1/2$. The remaining four are non-trivial:

$$\begin{aligned}
h_1 &= a^4 - a^2 + 1/2 - 2a^2b^2 - b^2 + b^4 \\
h_2 &= a^4 - a^2 - 2a^2b^2 - b^2 + b^4
\end{aligned}$$

$$
\begin{aligned}
h_3 &= a^6 - 1 + 3b^2a^4 + 3a^2b^4 + b^6 - 3a^4 \\
&\quad +21a^2b^2 - 3b^4 + 3a^2 + 3b^2 \\
h_4 &= 4627325525704704\, b^{80}a^{18} \\
&\quad + \cdots + \textbf{1199 terms} + \cdots + \\
&\quad 850320000000000\, a^2.
\end{aligned}
$$

The polynomial h_4 has degree 98. Figures 3, 4, 5 and 6 show the real curves defined by the polynomials. As expected, Figure 6 gives the most complicated curve.
Step 9 produced the output formula:

$$h_3 < 0$$

which corresponds to the stability region obtained by the pen-paper proof in [14].

Future works: We expect to make further reduction in the size of the set in Step 7. For instance, the complicated polynomial h_4 in the above example is "useless". Steps 8 and 9 takes long, mainly because of the presence of this useless polynomial. Not generating such a polynomial will provide even further improvement of the efficiency of the VQE algorithm. We also expect to weaken or remove assumption $\mathbf{H_2}$ either by using properness assumptions [26] or by computing *generalized critical values* [25]. Due to the lack of space, the complexity analysis of the algorithm VQE will be carried out in a future paper.

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