



Verifying zeros of interval polynomials

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Received: 24 December 2025 / Revised: 18 August 2026 / Accepted: 22 September 2026

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Abstract

This paper studies the problem of characterizing and verifying the zeros of interval polynomials, i.e. families of polynomials whose coefficients vary independently within prescribed real or complex intervals. Such polynomials naturally arise in applications involving uncertainty, including numerical analysis, control theory, and robust computation. The central question addressed is: given a complex number z , how can one decide whether it is a zero of at least one polynomial in the interval family?

Keywords Polynomial root · Pseudozero set · Uncertainty · Perturbation · Interval arithmetic · Interval polynomial

Mathematics Subject Classification 65G30 · 65H04

1 Introduction

Polynomials whose coefficients are not known exactly but only within prescribed bounds arise naturally in many areas of applied mathematics, engineering, and computer science. Such uncertainty may reflect measurement errors, modeling approximations, or rounding effects inherent to numerical computation [1, 2]. A convenient and widely used formalism to capture this uncertainty is provided by *interval polynomials*, that is, families of polynomials whose coefficients vary independently within given real or complex intervals [3–5].

The study of the zeros of interval polynomials is a classical and challenging problem. Given an interval polynomial, one is typically interested in characterizing the set of all complex (or real) numbers that can occur as zeros of at least one polynomial in the family. This question is closely related to robust stability analysis in control theory [6, 7], sensitivity analysis in numerical computation [1, 2], and validated methods in computer algebra [8]. Despite its apparent simplicity, the geometry of the zero sets of interval polynomials can be highly nontrivial, even for low degrees.

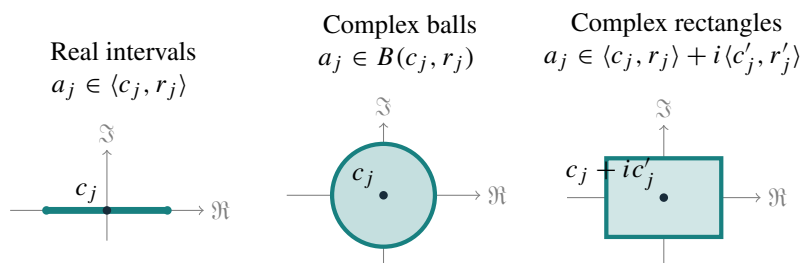
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Early work on interval polynomials focused primarily on real roots and stability questions, notably through results such as Kharitonov's theorem [9] and its extensions. In the complex plane, however, the behavior of zeros is considerably more intricate. The set of possible zeros is generally neither convex nor easily describable, and its computation poses significant theoretical and algorithmic difficulties. As a result, much of the existing literature has concentrated on obtaining enclosures, exclusion regions, or sufficient conditions for the absence of zeros in prescribed domains. The notion of zeros of interval polynomials is closely related to the notion of pseudozero sets introduced by Mosier [10]. It has been studied in numerous papers [1, 11, 12] and is linked to the notion of pseudospectra [6, 11, 13].

In recent years, renewed interest in this topic has been driven by advances in symbolic–numeric algorithms and validated numerics. Techniques combining interval arithmetic, complex analysis, and algebraic methods have led to more precise descriptions of zero sets and to effective algorithms for their approximation. These developments open new perspectives for understanding the structure of the solution space and for designing reliable computational tools [14–17].

The goal of this article is to contribute to the study of zeros of interval polynomials by giving computable formulas to decide whether a complex number is a zero of an interval polynomials. We will deal with polynomials whose coefficients are either real intervals, circular complex intervals and rectangular complex intervals.



The paper is organized as follows. In Sect. 2, we study the zeros of real interval polynomials. Section 3 is devoted to the study of the zeros of complex interval polynomials. We provide some results on the localization of zeros of interval polynomials in Sect. 4. In Sect. 5, we discuss algorithmic aspects and illustrate our approach with numerical examples. Finally, Sect. 6 concludes the paper and outlines directions for future work.

2 Zeros of real interval polynomials

In this section, we will deal with real intervals. We will use the center-radius form that is to say we will define an interval $\langle c, r \rangle$, with $c, r \in \mathbb{R}$ and $r \geq 0$ as

$$\langle c, r \rangle = \{x \in \mathbb{R} : |x - c| \leq r\}.$$

Definition 2.1 A *real interval polynomial* is a polynomial whose coefficients are real intervals. Let $\mathbf{p}(x)$ be such an interval polynomial. It can be written as

$$\mathbf{p}(x) = \sum_{j=0}^n \langle c_j, r_j \rangle x^j.$$

We can generalize the notion of zero for polynomials to interval polynomials.

Definition 2.2 A number $z \in \mathbb{C}$ is said to be a zero of the interval polynomial $\mathbf{p}(x) = \sum_{j=0}^n \langle c_j, r_j \rangle x^j$ if there exist $d_j \in \langle c_j, r_j \rangle$, $j = 0, \dots, n$ such that

$$\sum_{j=0}^n d_j z^j = 0.$$

2.1 Real zeros of real interval polynomials

We propose a simple necessary and sufficient condition to test if a real number is a zero of an interval polynomial.

Theorem 2.1 Let $\mathbf{p}(x) = \sum_{j=0}^n \langle c_j, r_j \rangle x^j$ be a real interval polynomial. A real number $z \in \mathbb{R}$ is a zero of $\mathbf{p}(x)$ if and only if

$$\left| \sum_{j=0}^n c_j z^j \right| \leq \sum_{j=0}^n r_j |z|^j.$$

Proof Let us consider $z \in \mathbb{R}$ and let us check if z is a real zero of an interval polynomial $\mathbf{p}(x) = \sum_{j=0}^n \langle c_j, r_j \rangle x^j$. We can evaluate $\mathbf{p}(z)$ using interval arithmetic. Indeed, it is clear that for $c_1, c_2, r_1, r_2, a \in \mathbb{R}$ with $r_1, r_2 \geq 0$, we have

$$a + \langle c_1, r_1 \rangle = \langle a + c_1, r_1 \rangle, \quad a \langle c_1, r_1 \rangle = \langle ac_1, |a|r_1 \rangle$$

and

$$\langle c_1, r_1 \rangle + \langle c_2, r_2 \rangle = \langle c_1 + c_2, r_1 + r_2 \rangle.$$

Using those operations for evaluating $\mathbf{p}(z)$ with interval arithmetic, we obtain

$$\mathbf{p}(z) = \sum_{j=0}^n \langle c_j, r_j \rangle z^j = \left\langle \sum_{j=0}^n c_j z^j, \sum_{j=0}^n r_j |z|^j \right\rangle.$$

It follows that z is a zero of $\mathbf{p}(x)$ if and only if $\mathbf{p}(z)$ is an interval containing zero. This is the case if and only if the radius is bigger than the absolute value of the center. $\square$

This result was already known [14] but was proved in the context of pseudozero set. Here we provide a simpler proof that only relies on interval arithmetic. The problem is more complex when one want to check if a complex number z is a zero of a real interval polynomial.

2.2 Complex zeros of real interval polynomials

We propose a simple necessary and sufficient condition to test if a complex number is a zero of a real interval polynomial. In the rest of the paper, the indices of the elements of vectors begin at 0 to match the coefficients of polynomials.

Theorem 2.2 *Let $\mathbf{p}(x) = \sum_{j=0}^n \langle c_j, r_j \rangle x^j$ be a real interval polynomial. A complex number $z \in \mathbb{C}$ is a zero of $\mathbf{p}(x)$ if and only if*

$$d(G_R(z), \mathbb{R}G_I(z)) \geq 1,$$

where $G_R(z)$ and $G_I(z)$ are the real and imaginary parts of $G(z) = \frac{1}{p_c(z)}(1, z, \dots, z^n)^T$.

Proof Let $\mathbf{p}(x) = \sum_{j=0}^n \langle c_j, r_j \rangle x^j$ be a real interval polynomial and $z \in \mathbb{C}$. One want to check if z is a zero of $\mathbf{p}(x)$. Let us define the center polynomial of $\mathbf{p}(x)$ as $p_c(x) = \sum_{j=0}^n c_j x^j$. Let us note that $p_c(x)$ is a polynomial and not an interval polynomial. A complex number z is a complex zero of $\mathbf{p}(x)$ if and only if there exist $(\alpha_j) \in \mathbb{R}$, $j = 0, \dots, n$ with $|\alpha_j| \leq r_j$ such that

$$p_c(z) + \sum_{j=0}^n \alpha_j z^j = 0.$$

This can be written as

$$p_c(z) + (\alpha_0, \alpha_1, \dots, \alpha_n) \cdot (1, z, \dots, z^n)^T = 0.$$

If $p_c(z) \neq 0$ then we have

$$1 + \frac{1}{p_c(z)} (\alpha_0, \alpha_1, \dots, \alpha_n) \cdot (1, z, \dots, z^n)^T = 0.$$

Let us denote

$$G(z) = \frac{1}{p_c(z)}(1, z, \dots, z^n)^T, \quad z \in \mathbb{C},$$

with $p_c(z) \neq 0$. Then we obtain that $(\alpha_0, \alpha_1, \dots, \alpha_n) \cdot G(z) = -1$.

We now define $G_R(z)$ and $G_I(z)$ as the real and imaginary parts of $G(z)$. This means that $G(z) = G_R(z) + iG_I(z)$ with $G_R(z), G_I(z) \in \mathbb{R}^{n+1}$. Let us also denote $\alpha = -(\alpha_0, \alpha_1, \dots, \alpha_n)^T$ and $R = (r_0, r_1, \dots, r_n)^T$. It follows that

$$\begin{cases} \alpha^T \cdot G_R(z) = 1, \\ \alpha^T \cdot G_I(z) = 0, \end{cases}$$

and that $|\alpha| \leq |R|$, this inequality being componentwise. Let us write $U = G_R(z) \in \mathbb{R}^{n+1}$ and $V = G_I(z) \in \mathbb{R}^{n+1}$. It follows that z is a zero of the interval polynomial

$p(x)$ if and only if there exists $\alpha \in \mathbb{R}^{n+1}$ such that $|\alpha| \leq R$ and

$$\begin{cases} \alpha^T \cdot U = 1, \\ \alpha^T \cdot V = 0. \end{cases}$$

So it follows that for all $t \in \mathbb{R}$, we have $\alpha^T(U - tV) = 1$. We now consider the weighted norm, denoted $\|\cdot\|_{\infty,R}$, defined by

$$\|\alpha\|_{\infty,R} = \max_{i=0:n} \{|\alpha_i|/r_i\}.$$

The norm $\|\cdot\|_{1,R}$ is defined for $x \in \mathbb{R}^n$ by

$$\|x\|_{1,R} = \sum_{i=0}^n r_i |x_i|.$$

We define for $x, y \in \mathbb{R}^n$,

$$d(x, \mathbb{R}y) = \inf_{t \in \mathbb{R}} \|x - ty\|_{1,R},$$

the distance of a point $x \in \mathbb{R}^{n+1}$ from the linear subspace $\mathbb{R}y = \{ty, t \in \mathbb{R}\}$. It is shown in [13, Prop. 7.7.2] that

$$d(x, \mathbb{R}y) = \begin{cases} \min_{\substack{i=0:n \\ y_i \neq 0}} \|x - (x_i/y_i)y\|_{1,R} & \text{if } y \neq 0, \\ \|x\|_{1,R} & \text{if } y = 0. \end{cases}$$

A $\alpha^T(U - tV) = 1$, it follows that $1 \leq \|\alpha\|_{\infty,R} \cdot \|U - tV\|_{1,R}$ and so $1 \leq \|\alpha\|_{\infty,R} \cdot d(U, \mathbb{R}V)$. As a consequence if there exists such an α satisfying $\|\alpha\|_{\infty,R} \leq 1$ then $d(U, \mathbb{R}V) \geq 1$.

Conversely, if $d(U, \mathbb{R}V) \geq 1$ then, from a duality theorem (see [18, p.119]), there exists a vector $\alpha \in \mathbb{R}^{n+1}$ with $\|\alpha\|_{\infty,R} = 1$ satisfying

$$\alpha^T U = d(U, \mathbb{R}V) \quad \text{and} \quad \alpha^T V = 0.$$

Let us consider

$$\alpha' = \frac{1}{d(U, \mathbb{R}V)} \alpha.$$

It satisfies $\|\alpha'\|_{\infty,R} \leq 1$ and

$$\begin{cases} \alpha'^T \cdot U = 1, \\ \alpha'^T \cdot V = 0. \end{cases}$$

As a consequence it follows that z is a zero of the polynomial $p_c(x) = \sum_{j=0}^n (c_j - \alpha'_j)x^j$ with $|\alpha'_j| \leq r_j$ for $j = 0, \dots, n$. $\square$

This result was already known [13, 14] but was proved in the context of pseudozero set and spectral value sets. Here the proof is done directly with polynomials whose coefficients are described as real midpoint-center intervals.

3 Zeros of complex interval polynomials

We now assume that the interval polynomials have coefficients that are either circular or rectangular. By circular, we mean complex balls $B(c, r) = \{x \in \mathbb{C} : |x - c| \leq r\}$ with $c \in \mathbb{C}$ and $r \in \mathbb{R}, r \geq 0$. By rectangular, we mean that both real part and imaginary part are real intervals so it can be written as $\langle c, r \rangle + i \langle c', r' \rangle$, with $c, c', r, r' \in \mathbb{R}$ and $r, r' \geq 0$.

3.1 Zeros of complex circular interval polynomials

We propose a simple necessary and sufficient condition to test if a complex number is a zero of a complex circular interval polynomial.

Theorem 3.1 *Let $\mathbf{p}(x) = \sum_{j=0}^n B(c_j, r_j)x^j$ be a complex circular interval polynomial. A complex number $z \in \mathbb{C}$ is a zero of $\mathbf{p}(x)$ if and only if*

$$\left| \sum_{j=0}^n c_j z^j \right| \leq \sum_{j=0}^n r_j |z|^j.$$

Proof Let us consider $z \in \mathbb{C}$ and let us check if z is a zero of an interval polynomial $\mathbf{p}(x) = \sum_{j=0}^n B(c_j, r_j)x^j$. We can evaluate $\mathbf{p}(z)$ using circular (or ball) arithmetic. Indeed, it is clear that for $c_1, c_2, r_1, r_2, a \in \mathbb{R}$ with $r_1, r_2 \geq 0$, we have

$$a + B(c_1, r_1) = B(a + c_1, r_1), \quad aB(c_1, r_1) = B(ac_1, |a|r_1)$$

and

$$B(c_1, r_1) + B(c_2, r_2) = B(c_1 + c_2, r_1 + r_2).$$

Using those operations for evaluating $\mathbf{p}(z)$ with circular arithmetic, we obtain

$$\mathbf{p}(z) = \sum_{j=0}^n B(c_j, r_j)z^j = B\left(\sum_{j=0}^n c_j z^j, \sum_{j=0}^n r_j |z|^j\right).$$

It follows that z is a zero of $\mathbf{p}(x)$ if and only if $\mathbf{p}(z)$ is a ball containing zero. This is the case if and only if the radius is bigger than the module of the center. $\square$

This result was already known [10] but was proved in the context of pseudozero set. Here we provide a simpler proof that only relies on circular arithmetic [19].

3.2 Zeros of complex rectangular interval polynomials

We propose a simple necessary and sufficient condition to test if a complex number is a zero of a complex rectangular interval polynomial.

Theorem 3.2 Let $\mathbf{p}(x) = \sum_{j=0}^n (\langle c_j, r_j \rangle + i \langle c'_j, r'_j \rangle) x^j$ be an interval polynomial. A complex number $z \in \mathbb{R}$ is a zero of $\mathbf{p}(x)$ if and only if

$$d(G_R(z), \mathbb{R}G_I(z)) \geq 1,$$

where $G_R(z)$ and $G_I(z)$ are the real and imaginary parts of $G(z) = \frac{1}{p_c(z)}(1, z, \dots, z^n, i, iz, \dots, iz^n)^T$.

Proof Let $\mathbf{p}(x) = \sum_{j=0}^n (\langle c_j, r_j \rangle + i \langle c'_j, r'_j \rangle) x^j$ be an interval polynomial and $z \in \mathbb{C}$. One want to check if z is a zero of $\mathbf{p}(x)$. Let us define the center polynomial of $\mathbf{p}(x)$ as $p_c(x) = \sum_{j=0}^n (c_j + i c'_j) x^j$. Let us note that $p_c(x)$ is a polynomial and not an interval polynomial. A complex number z is a complex zero of $\mathbf{p}(x)$ if and only if there exist $(\alpha_j) \in \mathbb{R}$ and $(\beta_j) \in \mathbb{R}$, $j = 0, \dots, n$ with $|\alpha_j| \leq r_j$ and $|\beta_j| \leq r'_j$ such that

$$p_c(z) + \sum_{j=0}^n (\alpha_j + i \beta_j) z^j = 0.$$

This can be written as

$$p_c(z) + (\alpha_0, \alpha_1, \dots, \alpha_n, \beta_0, \beta_1, \dots, \beta_n) \cdot (1, z, \dots, z^n, i, iz, \dots, iz^n)^T = 0.$$

If $p_c(z) \neq 0$ then we have

$$1 + \frac{1}{p_c(z)} (\alpha_0, \alpha_1, \dots, \alpha_n, \beta_0, \beta_1, \dots, \beta_n) \cdot (1, z, \dots, z^n, i, iz, \dots, iz^n)^T = 0.$$

Let us denote

$$G(z) = \frac{1}{p_c(z)} (1, z, \dots, z^n, i, iz, \dots, iz^n)^T, \quad z \in \mathbb{C},$$

with $p_c(z) \neq 0$. Then we obtain that $(\alpha_0, \alpha_1, \dots, \alpha_n, \beta_0, \beta_1, \dots, \beta_n) \cdot G(z) = -1$.

We now define $G_R(z)$ and $G_I(z)$ as the real and imaginary parts of $G(z)$. This means that $G(z) = G_R(z) + i G_I(z)$ with $G_R(z), G_I(z) \in \mathbb{R}^{2(n+1)}$. Let us also denote $\gamma = -(\alpha_0, \alpha_1, \dots, \alpha_n, \beta_0, \beta_1, \dots, \beta_n)^T$ and $R = (r_0, r_1, \dots, r_n, r'_0, r'_1, \dots, r'_n)^T$. It follows that

$$\begin{cases} \gamma^T \cdot G_R(z) = 1, \\ \gamma^T \cdot G_I(z) = 0, \end{cases}$$

and that $|\gamma| \leq |R|$, this inequality being componentwise. Let use write $U = G_R(z) \in \mathbb{R}^{2(n+1)}$ and $V = G_I(z) \in \mathbb{R}^{2(n+1)}$. It follows that z is a zero of the interval polynomial

$p(x)$ if and only if there exists $\gamma \in \mathbb{R}^{2(n+1)}$ such that $|\gamma| \leq R$ and

$$\begin{cases} \gamma^T \cdot U = 1, \\ \gamma^T \cdot V = 0. \end{cases}$$

So it follows that for all $t \in \mathbb{R}$, we have $\gamma^T(U - tV) = 1$. We now consider the weighted norm, denoted $\|\cdot\|_{\infty, R}$, defined by

$$\|\gamma\|_{\infty, R} = \max_{i=0:2n+1} \{|\alpha_i|/R_i\}.$$

The norm $\|\cdot\|_{1, R}$ is defined for $x \in \mathbb{R}^{2(n+1)}$ by

$$\|x\|_{1, R} = \sum_{i=0}^{2n+1} R_i |x_i|.$$

We define for $x, y \in \mathbb{R}^{n+1}$,

$$d(x, \mathbb{R}y) = \inf_{t \in \mathbb{R}} \|x - ty\|_{1, R},$$

the distance of a point $x \in \mathbb{R}^{2(n+1)}$ from the linear subspace $\mathbb{R}y = \{ty, t \in \mathbb{R}\}$. It is shown in [13, Prop. 7.7.2] that

$$d(x, \mathbb{R}y) = \begin{cases} \min_{\substack{i=0:2n+1 \\ y_i \neq 0}} \|x - (x_i/y_i)y\|_{1, R} & \text{if } y \neq 0, \\ \|x\|_{1, R} & \text{if } y = 0. \end{cases}$$

A $\gamma^T(U - tV) = 1$, it follows that $1 \leq \|\gamma\|_{\infty, R} \cdot \|U - tV\|_{1, R}$ and so $1 \leq \|\gamma\|_{\infty, R} \cdot d(U, \mathbb{R}V)$. As a consequence if there exists such an γ satisfying $\|\gamma\|_{\infty, R} \leq 1$ then $d(U, \mathbb{R}V) \geq 1$.

Conversely, if $d(U, \mathbb{R}V) \geq 1$ then, from a duality theorem (see [18, p.119]), there exists a vector $\gamma \in \mathbb{R}^{2(n+1)}$ with $\|\gamma\|_{\infty, R} = 1$ satisfying

$$\gamma^T U = d(U, \mathbb{R}V) \quad \text{and} \quad \gamma^T V = 0.$$

Let us consider

$$\gamma' = \frac{1}{d(U, \mathbb{R}V)} \gamma.$$

It satisfies $\|\gamma'\|_{\infty, R} \leq 1$ and

$$\begin{cases} \gamma'^T \cdot U = 1, \\ \gamma'^T \cdot V = 0. \end{cases}$$

As a consequence it follows that z is a zero of the polynomial $p_c(x) = \sum_{j=0}^n [(c_j - \gamma'_j) + i(c'_j - \gamma'_{n+j})]x^j$ with $|\gamma'_j| \leq r_j$ and $|\gamma'_{n+j}| \leq r'_j$ for $j = 0, \dots, n$. $\square$

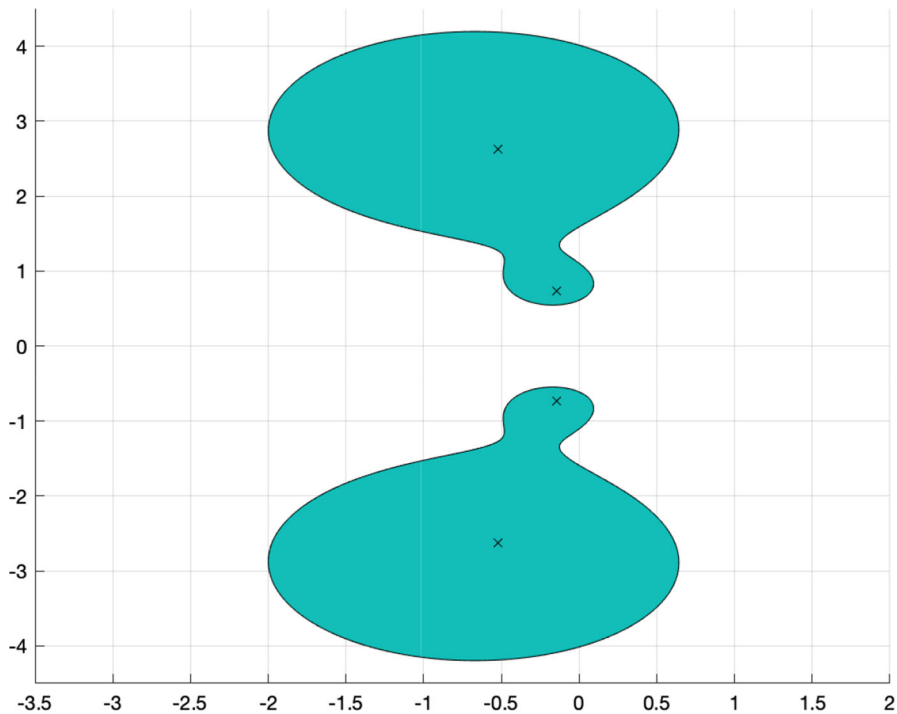


Fig. 1 Zeros of $p(x)$

Up to our best knowledge, this is the first time that one provides a computable formulas for deciding if a complex number is a zero of an interval polynomials whose coefficients are complex rectangles.

4 Localization of zeros of interval polynomials

Now we have a computable formula to compute the zeros of an interval polynomial. The only problem staying to deal with is to find a grid that contains the whole set of zeros. This is the subject of the following lemma.

Lemma 4.1 *Let $p(x) = \sum_{j=0}^n \langle c_j, r_j \rangle x^j$ be a real interval polynomial then define*

$$R := 1 + \frac{\max_{j=0:n} \{\max\{|c_j - r_j|, |c_j + r_j|\}\}}{\min\{|c_n - r_n|, |c_n + r_n|\}}.$$

Let $p(x) = \sum_{j=0}^n B(c_j, r_j)x^j$ be a complex interval polynomial then define

$$R := 1 + \frac{\max_{j=0:n} \{\max\{|c_j - r_j(1+i)|, |c_j + r_j(1+i)|\}\}}{\min\{|c_n - r_n(1+i)|, |c_n + r_n(1+i)|\}}.$$

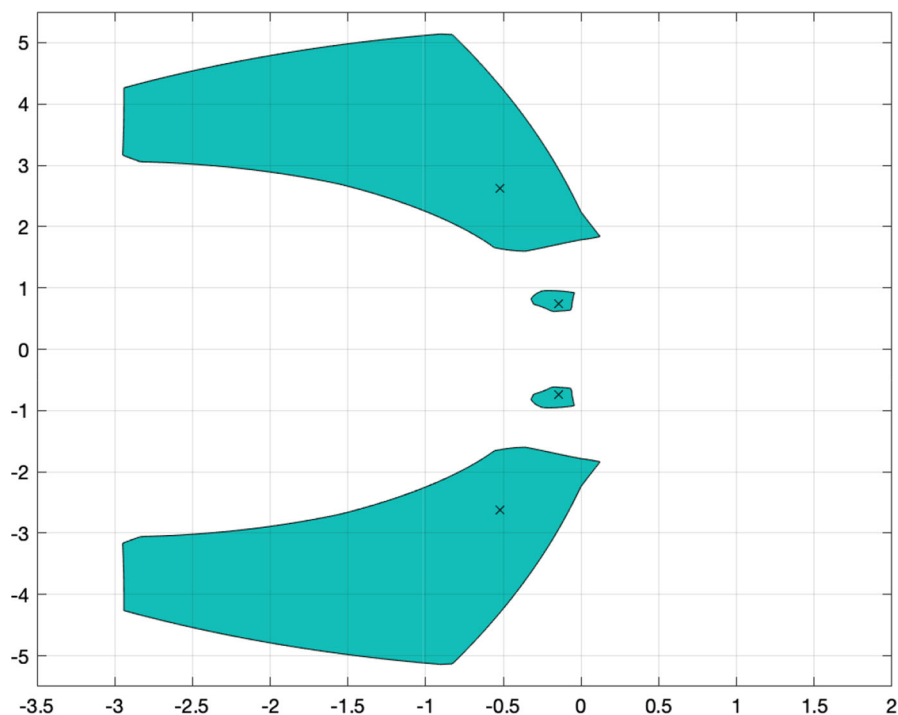


Fig. 2 Zeros of $q(x)$

Let $\mathbf{p}(x) = \sum_{j=0}^n (\langle c_j, r_j \rangle + \langle c'_j, r'_j \rangle) x^j$ be a rectangular complex interval polynomial then define

$$R := 1 + \frac{\max_{j=0:n} \{ \max \{ |(c_j + r_j) + i(c'_j + r'_j)|, |(c_j - r_j) + i(c'_j - r'_j)| \} \}}{\min \{ |(c_n + r_n) + i(c'_n + r'_n)|, |(c_n - r_n) + i(c'_n - r'_n)| \}}.$$

Then we have the zeros of the interval polynomial $\mathbf{p}(x)$ is included in $B(O, R)$ that denotes the ball of the complex plane $\mathbb{C}$ of center O and radius R .

Proof Let us prove the result for $\mathbf{p}(x) = \sum_{j=0}^n \langle c_j, r_j \rangle x^j$ being a real interval polynomial. The proof is similar for the other cases. Let us denote by $\{z_j\}_{j=1:n}$ the zeros of a polynomial q counting with their multiplicities and $r = \max_j |z_j|$. It is well known (see [20, p.154]) that

$$r \leq 1 + \frac{\max \{|q_{n-1}|, |q_{n-2}|, \dots, |q_0|\}}{|q_n|}. \quad (4.1)$$

Let z being a zero of $\mathbf{p}(x)$. Then there exists a polynomial $m(z) = \sum_{j=0}^n m_j z^j$ with $m_j \in \langle c_j, r_j \rangle$ for $j = 0, \dots, n$. It is easy to see that $|m_j| \leq \max \{|c_j - r_j|, |c_j + r_j|\}$

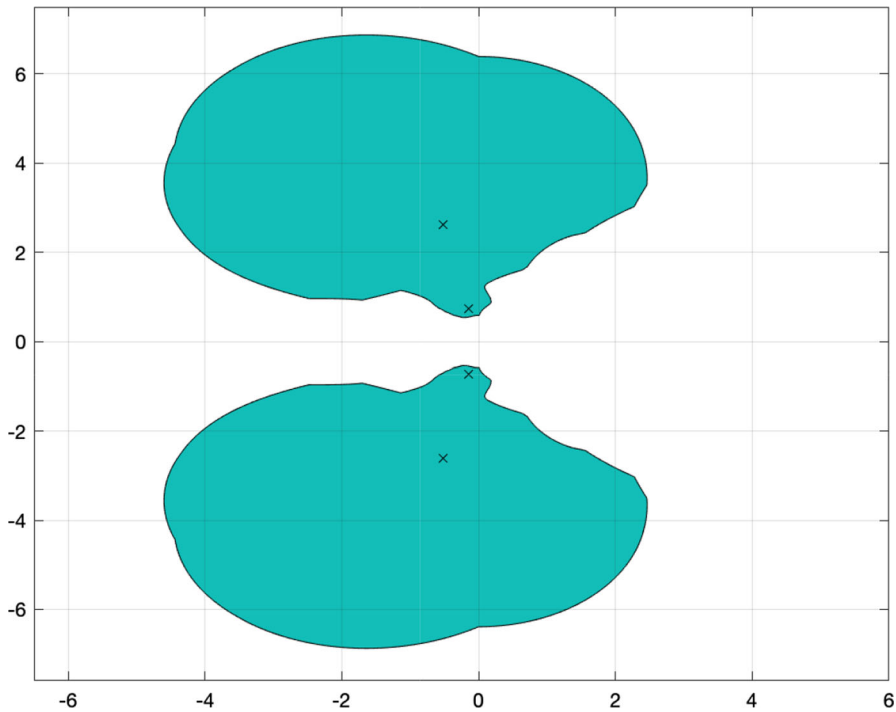


Fig. 3 Zeros of $\mathbf{r}(x)$

and $|m_n| \geq \min\{|c_n - r_n|, |c_n + r_n|\}$. By applying (4.1), it follows that

$$|z| \leq 1 + \frac{\max_{j=0:n}\{\max\{|c_j - r_j|, |c_j + r_j|\}\}}{\min\{|c_n - r_n|, |c_n + r_n|\}}.$$

□

5 Numerical experiments

We can now derive an algorithm to approximate the whole set of zeros of an interval polynomial. We have seen in the previous sections that z is a zero of an interval polynomial if and only if $g(z) \geq 1$ where

$$g(z) = \frac{\sum_{j=0}^n r_j |z|^j}{\left| \sum_{j=0}^n c_j z^j \right|}$$

or

$$g(z) = d(G_R(z), \mathbb{R}G_I(z)),$$

depending on the form of the interval coefficients.

Algorithm 1 Computation of zero of interval polynomials (MATLAB version)**Require:** interval polynomial $\mathbf{p}$ **Ensure:** zeros of $\mathbf{p}$ layout in the complex plane1: We grid a square containing all the zeros of $\mathbf{p}$ with the MATLAB command `meshgrid`.2: We compute $g(z)$ at the grid nodes z .3: We draw the level line $g(z) = 1$ with the MATLAB command `contourf`.

Let us take an example with $\mathbf{p}(x) = B(1.5, 0.5)x^4 + B(2, 1)x^3 + B(12, 2)x^2 + B(4, 1)x + B(6, 1)$. Figure 1 shows the complex zeros of this complex interval polynomial.

Let us take an example with $\mathbf{q}(x) = \langle 1.5, 0.5 \rangle x^4 + \langle 2, 1 \rangle x^3 + \langle 12, 2 \rangle x^2 + \langle 4, 1 \rangle x + \langle 6, 1 \rangle$. Figure 2 shows the complex zeros of this real interval polynomial.

Let us take an example with $\mathbf{r}(x) = (\langle 1.5, 0.5 \rangle + i \langle 0, 0.5 \rangle)x^4 + (\langle 2, 1 \rangle + i \langle 0, 1 \rangle)x^3 + (\langle 12, 2 \rangle + i \langle 0, 2 \rangle)x^2 + (\langle 4, 1 \rangle + i \langle 0, 1 \rangle)x + (\langle 6, 1 \rangle + i \langle 0, 1 \rangle)$. Figure 3 shows the complex zeros of this rectangular interval polynomial.

6 Conclusion

In this paper, we studied the problem of verifying whether a given real or complex number is a zero of an interval polynomial, that is, a polynomial whose coefficients vary independently within prescribed bounds. By relying on interval arithmetic, circular arithmetic, and geometric arguments, we derived necessary and sufficient conditions for zero membership that are fully computable.

For real interval polynomials, we provided a simple and exact characterization of real zeros based on the comparison between the center polynomial and the radii of the coefficients. For complex zeros, both in the real-coefficient and complex-coefficient settings, the problem was reduced to a geometric condition involving weighted distances between vectors associated with the evaluation point. This approach leads to concise criteria expressed in terms of norms and distances, avoiding overly conservative enclosures.

We also extended these results to complex interval polynomials with circular and rectangular coefficient uncertainties. In particular, for rectangular complex intervals, we obtained—up to our knowledge—the first explicit and computable criterion allowing one to decide whether a complex number belongs to the zero set of such an interval polynomial. This fills a gap in the existing literature, where most results were limited to sufficient conditions or to special classes of uncertainties.

Finally, we discussed algorithmic aspects and proposed a simple numerical procedure to approximate and visualize the zero sets of interval polynomials in the complex plane. The numerical experiments illustrate how the theoretical conditions can be effectively used to compute meaningful and reliable enclosures of the zero regions.

Future work includes improving the efficiency of the proposed algorithms, refining the visualization of zero sets, and extending the approach to multivariate polynomials or structured coefficient dependencies. Moreover we plan to provide a verified inner and outer approximation using bisection algorithms and interval arithmetic. These

directions are promising for the development of robust symbolic–numeric tools for polynomial problems under uncertainty.

Acknowledgements This work was partly supported by the FPT-4 project (ANR-24-CE46-7572) of the French National Agency for Research (ANR).

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Data Availability No datasets were generated or analysed during the current study.

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