Split-pseudopaths in split-prime extensions

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Abstract

Let G be a graph, a *split* in G is a bi-partition (X, Y) of its vertex set $V(G)$ such that $\mid X \mid, \mid Y \mid \geq 2$ and there are all possible edges between $X^+ = X \cap N(Y)$ and $Y^+ = Y \cap N(X)$ where $N(X)$ and $N(Y)$ are respectively neighborhood of X and Y in G. Let X^- and Y^- be respectively the sets $X \setminus X^+$ and $Y \setminus Y^+$. Whenever $X^- = \emptyset$ (resp. $Y^- = \emptyset$) the set X (resp. Y) is a *non-trivial module* of G. Let H be a graph without split containing G as induced subgraph. We show that in the graph induced by $V(H) \setminus V(G)$ and for any split (X, Y) of G there exists a particular kind of graph the (X, Y) -*split-pseudopath.* The structure of the split-pseudopath generalizes that of the W-pseudopath introduced by I. Zverovich in [8] where W is a non trivial module of G.

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1 Motivation and previous results

For terms not defined here the reader is referred to [2]. All considered graphs are finite, without loops nor multiple edges. Let G be a graph, the set of its vertices will be noted by $V(G)$ while the set of its edges will be noted by $E(G)$. The neighborhood of a vertex $v \in V(G)$ is denoted by $N(v)$ and the neighborhood of a set $S \subseteq V(G)$ is the set $N(S) = \cup_{v \in S} N(v) \setminus S$ while the subgraph of G induced by S will be denoted $[S]$. A vertex a will be *total*, *indifferent* or *partial*

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with respect to a set A if a is respectively adjacent to all, to none or to some but not all of the vertices of A. A *split* in G is a bi-partition (X, Y) of its vertex set $V(G)$ such that $\mid X \mid, \mid Y \mid \geq 2$ and there are all possible edges between $X^+ = X \cap N(Y)$ and $Y^+ = Y \cap N(X)$. Let X^- and Y^- be respectively the sets $X \setminus X^+$ and $Y \setminus Y^+$. Whenever $X^- = \emptyset$ (resp. $Y^- = \emptyset$) the set X (resp. Y) is a *non-trivial module* or a *homogeneous set* of G. Singletons, the empty set and $V(G)$ are trivial modules and whenever a graph G contains only trivial modules it is called *prime* otherwise G is called *decomposable*. Whenever a graph G does not contain any split is called *split-prime* otherwise is called *split-decomposable*.

Let (X, Y) be a split of G then we can decompose G in two graphs G_1 and G_2 such that $V(G_1) = X \cup \{m_1\}$ and $V(G_2) = Y \cup \{m_2\}$ where m_1 and m_2 are two new vertices (markers) such that the neighborhood of m_1 (respectively m_2) in G_1 (resp. G_2) is the set X^+ (resp. Y^+). The *split-composition* of two disjoint graphs G_1 and G_2 which is the inverse operation of split-decomposition is obtained by first removed two vertices m_1 of G_1 and m_2 of G_2 and then making every neighbor of m_1 in G_1 adjacent to any neighbor of m_2 in G_2. Whenever X (resp. Y) is an homogeneous set of G we can decompose G in two graphs $[X]$ and G_2 (resp. $[Y]$ and G_1). The *substitution composition* of two disjoint graphs G_1 and G_2 is obtained by first removing a vertex m_2 from G_2 and then making every vertex of G_1 adjacent to all neighbors of m_2 in G_2. Applying recursively the decomposition of G following its splits or its non-trivial modules we obtain a set Π of graphs that are split-prime or respectively prime. The set Π is unique up to isomorphism but the corresponding decomposition trees are not necessarily unique except if we consider maximal splits with respect to set-inclusion. We recall that the split decomposition has been originally introduced in [4] and in [5] it is proposed a linear time algorithm for it. Concerning the decomposition of G following its modules we obtain a unique modular decomposition by decomposing recursively G following its *strong* modules (M is a strong module if for any other module M' of G either $M \cap M' = \emptyset$ or one module is included into the other). There are three linear time algorithms for modular decomposition (see [2] for references). Split decomposition and modular decomposition are of basic importance for the design of efficient algorithms and an impressive amount of researching works uses as framework both forms of these decompositions. This is certainly due to the fact that split-composition and substitution-composition preserve many of the properties of the composed graphs as for example *perfection* (see [2] for references).

Let H be a split-prime (resp. prime) graph containing a split-decomposable (resp. decomposable) graph G, then H will be called a split-prime (resp. prime) *extension* of G. The graph H will be a *minimal* split-prime (resp. prime) extension if there is no proper subgraph of H which is split-prime (resp. prime) and contains a subgraph isomorphic to G.

Let $\mathcal{Z}$ be a set of graphs, a graph G will be called $\mathcal{Z}$-free if G does not contain any induced subgraph isomorphic to a graph of $\mathcal{Z}$. A set of graphs $\mathcal{F}$

will be called $\mathcal{Z}$-free if every graph of $\mathcal{F}$ is $\mathcal{Z}$-free. Let $\mathcal{F}$ be a family of graphs defined by a set $\mathcal{Z}$ of induced subgraphs and let $\mathcal{F}^*$ be the closure of $\mathcal{F}$ under substitution composition. Let $Ext(Z)$ be the set of minimal prime extensions of Z.

Problem 1 : Forbidden induced subgraph characterization of $\mathcal{F}^*$

In [6] it is proved:

1. The closure under substitution $\mathcal{F}^*$ of $\mathcal{F}$ is defined by the union of the sets $Ext(Z)$ where Z is a graph of $\mathcal{Z}$.
2. $Ext(Z)$ is not necessarily a finite set

Problem 2 : Find necessary and sufficient conditions for $\mathcal{Z}^*$ to be finite

Various researchers investigated the solution of the problem 2 and many sufficient conditions have been presented. It is worth noting that such characterizations are likely to lead to efficient solutions for graph optimization problems including the weighted stability number and the domination problem (see for example [1]). In a recent paper [7] it is presented a complete answer to Problem 2 by characterizing all classes of graphs whose minimal prime extensions is a finite set and by giving a simple method for generating an infinite number of extensions for all the other classes of graphs.

A powerful tool for the solution of Problem 2 below as well as for the study of several classes of graphs is the notion of reducing W-pseudopath (or W-pseudopath for shortly) introduced by I. Zverovich in [8].

Definition 1. Let G be an induced subgraph of a graph H and let W be a homogeneous set of G. We define a reducing W-pseudopath in H as a sequence $R = (u_1, u_2, ..., u_t)$, with $t \geq 1$, of pairwise distinct vertices of $V(H) \setminus V(G)$ satisfying the following conditions :

1. u_1 is partial with respect to W;
2. $\forall\ i = 2, ..., t$, either u_i is adjacent to u_{i-1} and indifferent with respect to $W \cup \{u_1, ..., u_{i-2}\}$ or u_i is total with respect to $W \cup \{u_1, ..., u_{i-2}\}$ and non-adjacent to u_{i-1} (when $i = 2$, $\{u_1, u_2, ..., u_{i-2}\} = \emptyset$);
3. $\forall\ i = 1, ..., t-1$, vertex u_i is total with respect to $N(W)$ in G and indifferent with respect to $V(G) \setminus \{N(W) \cup W\}$ and either u_t is non-adjacent to a vertex of $N(W)$ or u_t is adjacent to a vertex of $V(G) \setminus N_G(W)$.

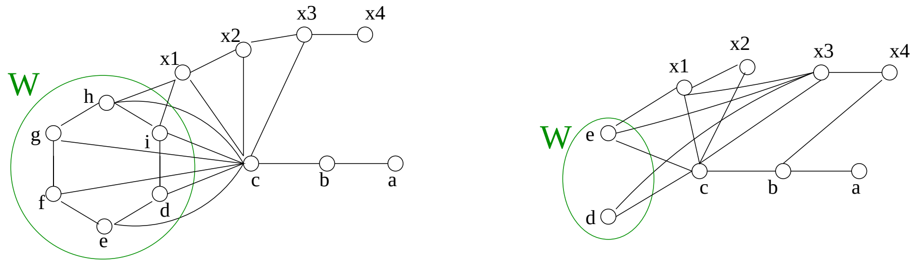

Figure 1: *Two illustrations of the structure of a W-pseudopath.*

We refer the reader to Figure 1 below for an illustration of a reducing W-pseudopath.

Theorem 2. *[8] Let H be an extension of its induced subgraph G and let W be a homogeneous set of G. Then there exists a reducing W-pseudopath with respect to any induced copy of G in H.*

In this paper we show that whenever H is a split-prime extension of a graph G there exists a special structure in H, the (X, Y)-split-pseudopath which generalize the structure of the W-pseudopath . In Section 2 we describe this new structure while in Section 3 we prove the existence of a (X, Y)-split-pseudopath in H by a constructional proof that makes in evidence how this new structure is formed in H. Finally in Section 4 we conclude by giving some problems for which we believe that the (X, Y)-split-pseudopath will play a crucial role for their solution.

2 (X,Y)-pseudopath: a generalization of the reducing W-pseudopath

Let G be a graph, for $X, Y \subseteq V(G)$ the notation $X \sim Y$ (respectively $X \nsim Y$) means that every vertex of X is adjacent (respectively non-adjacent) to every vertex of Y. When $X = \{x\}$ we shall write $x \sim Y$ and $x \nsim Y$ instead of $\{x\} \sim Y$ and $\{x\} \nsim Y$ respectively. Finally $x \sim y$ (respectively $x \nsim y$) means that the vertex x is adjacent (respectively non-adjacent) to the vertex y. Let G be a graph and (X, Y) be a split of G. Let G^* be a split-prime extension of G. For $A \subseteq V(G^*) \setminus V(G)$ and $Z \in \{X, Y\}$ we define the following sets : $Tot_Z(A) = \{x \in A/x$ is adjacent to every vertex of Z^+ and indifferent with respect to $Z^-\}$, $Ind_Z(A) = \{x \in A/x$ is indifferent with respect to $Z\}$ and $Par_Z(A) = \{x \in A/x$ is either adjacent to a vertex of Z^-or is partial with respect to $Z^+\}$.

2.1 The structure of X-split-pseudopath and Y-split-pseudopath

As we shall see in the next section, a (X, Y)-split-pseudopath is formed by a couple of two sequences of vertices of $V(G^*) \setminus V(G)$, the X-split-speudopath and the Y-split-pseudopath. Let us then first describe below their structure and then give in Figure 2 below an illustration.

Definition 3. Let G be a graph and (X, Y) be a split of G. Let G^* be a split-prime extension of G. For $Z \in \{X, Y\}$ let $\bar{Z} \in \{X, Y\}$ such that $\bar{Z} \neq Z$ then a *Z-split-pseudopath* $P = (z_1, ..., z_k)$ is a sequence of vertices of $V(G^*) \setminus V(G)$ satisfying the following conditions :

1. $z_1 \in Par_Z(P)$.
2. We have that:
   (a) If $k > 1$ then, for all $i = 1, ..., k$, $z_i \in Ind_{\bar{Z}}(P)$ or $z_i \in Tot_{\bar{Z}}(P)$.
   (b) If $k = 1$ then z_1 verify (2.a) or $z_1 \in Par_{\bar{Z}}(P)$.
3. If $k > 1$, then, for all $i = 2, ..., k$, $z_i \in V(G^*) \setminus V(G)$ and one of the following holds :
   (a) z_i is of type 1, i.e. $z_i \sim z_{i-1}$ and $z_i \nsim Z \cup \{z_1, ..., z_{i-2}\}$.
   (b) z_i is of type 2, i.e. $z_i \sim z_{i-1}$, $z_{i-1} \in Ind_{\bar{Z}}(P)$, $z_i \sim Z^+ \cup Tot_{\bar{Z}}(\{z_1, ..., z_{i-2}\})$ and $z_i \nsim Z^- \cup Ind_{\bar{Z}}(\{z_1, ..., z_{i-2}\})$.
   (c) z_i is of type 3, i.e. $z_i \nsim z_{i-1}$, $z_{i-1} \in Tot_{\bar{Z}}(P)$, $z_i \sim Z^+ \cup Tot_{\bar{Z}}(\{z_1, ..., z_{i-2}\})$ and $z_i \nsim Z^- \cup Ind_{\bar{Z}}(\{z_1, ..., z_{i-2}\})$.

We give in Figure 2 below an illustration of the three types of a X-split pseudopath

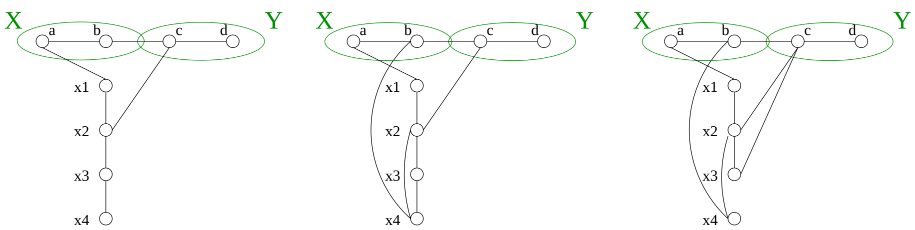

Figure 2: *Illustration of the three types of a X-split-pseudopath ($X^+ = \{b\}$, $X^- = \{a\}$, $Y^+ = \{c\}$,$Y^- = \{d\}$)*

2.2 The structure of a (X,Y)-split-pseudopath

We are now in position to describe the structure of a (X, Y)-split-pseudopath.

Definition 4. Let G be a graph and (X, Y) be a split of G. Let G^* be a split-prime extension of G. A (X, Y)-*split-pseudopath* of G^* is a couple (P, Q) where $P = (x_1, ..., x_k)$ and $Q = (y_1, ..., y_k)$ satisfying the following conditions:

1. If $k > 1$, then, for all $i = 1, ..., k-1$ and for all $j = 1, ..., k-1$, we have that $x_i \neq y_j$.
2. P is a X-split-pseudopath and Q is a Y-split-pseudopath.
3. The following conditions hold :
   (a) If $k > 1$, we have $Tot_Y(P \setminus \{x_k\}) \sim Tot_X(Q \setminus \{y_k\})$, $Ind_Y(P \setminus \{x_k\}) \nsim Q \setminus \{y_k\}$ and $Ind_X(Q \setminus \{y_k\}) \nsim P \setminus \{x_k\}$.
   (b) Either $x_k = y_k$ or $x_k \neq y_k$ and then :
      i. If $k > 1$ and $x_k \in Ind_Y(P)$ (respectively $y_k \in Ind_X(Q)$), then $x_k \nsim Q \setminus \{y_k\}$ (respectively $y_k \nsim P \setminus \{x_k\}$) .
      ii. If $k > 1$ and if $x_k \in Tot_Y(P)$ (respectively $y_k \in Tot_X(Q)$), then $x_k \sim Tot_X(Q) \setminus \{y_k\}$ and $x_k \nsim Ind_X(Q) \setminus \{y_k\}$ (respectively $y_k \sim Tot_Y(P) \setminus \{x_k\}$ and $y_k \nsim Ind_Y(P) \setminus \{x_k\}$).
      iii. If $x_k \in Tot_Y(P)$ and $y_k \in Tot_X(Q)$ then $x_k \nsim y_k$, else (i.e. $x_k \in Ind_Y(P)$ or $y_k \in Ind_X(Q)$) $x_k \sim y_k$.

We give in Figure 3 below an illustration of the structure of a (X, Y)-split-pseudopath.

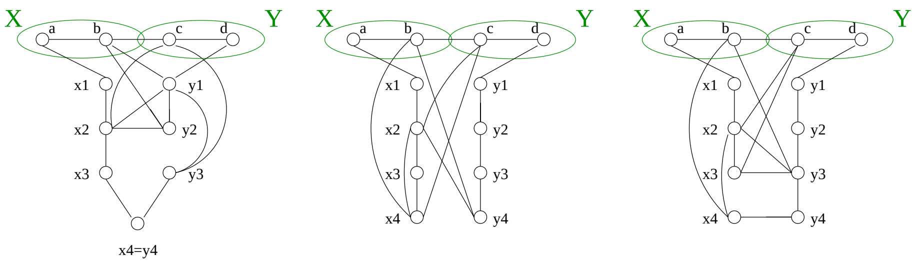

Figure 3: *The three kinds of a* (X, Y)*-split-pseudopath(* $X^+ = \{b\}$, $X^- = \{a\}$,$Y^+ = \{c\}$, $Y^- = \{d\}$*)*

3 The main theorem

We shall prove now that in any split-prime extension of a graph G having a split (X, Y) there exists a (X, Y)-split-psuedopath.

Theorem 5. *Let G be a graph and (X, Y) be a split of G. Let G^* be a split-prime extension of G, then there exists an (X, Y)-split-pseudopath in G^* for any induced copy of G.*

Proof. Let us denote $\Omega = V(G^*) \setminus V(G)$. Since the proof of this theorem is quite technical, we present first the steps that we shall use for it:

- In a first time, we assume the existence of two sequences of sets of vertices of Ω : $(X_1, ..., X_k)$ and $(Y_1, ..., Y_k)$, $k \geq 1$ that verify some conditions Φ (these sequences will be used for obtaining respectively a X-split-pseudopath, a Y-split-pseudopath as well as a (X, Y)-split-pseudopath).
- Secondly, we construct the sets X_1 and Y_1 and we prove the truth of conditions Φ for these sets .
- Then, we suppose that have construct two sequences of vertices $(X_1, ..., X_i)$ and $(Y_1, ..., Y_i)$, for some $i \geq 1$ which satisfy conditions Φ and we prove that we can construct X_{i+1} and Y_{i+1} which satisfy also these conditions.
- We then prove that $(X_1, ..., X_k)$ and $(Y_1, ..., Y_k)$, $k \geq 1$ exist.
- Finally, we show that we can construct a (X, Y)-split-pseudopath from these sets.

Definition of $(X_1, ..., X_k)$ and $(Y_1, ..., Y_k)$, $k \geq 1$

Let $(X_1, ..., X_k)$ and $(Y_1, ..., Y_k)$, $k \geq 1$ be two sequences of sets of vertices of Ω which satisfy the following conditions :

1. [*Sets are non empty and pairwise distinct*]
   For $1 \leq i \leq k$:
   (a) $\emptyset \neq X_i \subseteq \Omega \setminus ((X_1 \cup ... \cup X_{i-1}) \cup (Y_1 \cup ... \cup Y_{i-1}))$, and
   (b) $\emptyset \neq Y_i \subseteq \Omega \setminus ((X_1 \cup ... \cup X_{i-1}) \cup (Y_1 \cup ... \cup Y_{i-1}))$.
2. [*Existence of X- and Y-split-pseudopaths*]
   For $1 \leq i \leq k$:
   (a) for every vertex $x_i \in X_i$, there exists a X-split-pseudopath $P_i = (x_1, ..., x_i)$ in $X_1 \cup ... \cup X_i$ such that $x_j \in X_j$ for every $j \in \{1, ..., i\}$, and

(b) for every vertex $y_i \in Y_i$, there exists a Y-split-pseudopath $Q_i = (y_1, ..., y_i)$ in $Y_1 \cup ... \cup Y_i$ such that $y_j \in Y_j$ for every $j \in \{1, ..., i\}$.

3. [*Neighborhood between X- and Y-split-pseudopaths*]

   The following conditions hold :

   (a) When $k > 1$, for $1 \leq i < k$, we have : $Tot_Y(X_1 \cup ... \cup X_i) \sim Tot_X(Y_1 \cup ... \cup Y_i)$, $Ind_Y(X_1 \cup ... \cup X_i) \nsim Y_1 \cup ... \cup Y_i$, and $Ind_X(Y_1 \cup ... \cup Y_i) \nsim X_1 \cup ... \cup X_i$.

   (b) Either $X_k \cap Y_k \neq \emptyset$ or $X_k \cap Y_k = \emptyset$ and then :

      i. If $k > 1$, then $Ind_Y(X_k) \nsim Y_1 \cup ... \cup Y_{k-1}$ and $Ind_X(Y_k) \nsim X_1 \cup ... \cup X_{k-1}$.

      ii. If $k > 1$, then $Tot_Y(X_k) \sim Tot_X(Y_1 \cup ... \cup Y_{k-1})$, $Tot_Y(X_k) \nsim Ind_X(Y_1 \cup ... \cup Y_{k-1})$ and $Tot_X(Y_k) \sim Tot_Y(X_1 \cup ... \cup X_{k-1})$, $Tot_X(Y_k) \nsim Ind_Y(X_1 \cup ... \cup X_{k-1})$.

      iii. There exists $x_k \in X_k$ and $y_k \in Y_k$ such that one of the following holds :
      $x_k \in Tot_Y(X_k)$ and $y_k \in Tot_X(Y_k) \Rightarrow x_k \nsim y_k$
      $x_k \in Ind_Y(X_k)$ or $y_k \in Ind_X(Y_k) \Rightarrow x_k \sim y_k$.

We will prove that these two sequences exist by a constructional proof. We shall first construct X_1 and Y_1.

Construction of X_1 and Y_1

We define these two sets in the following manner :

$$X_1 = \Omega \setminus (Tot_X(\Omega) \cup Ind_X(\Omega)) \Leftrightarrow X_1 = Par_X(\Omega)$$

$$Y_1 = \Omega \setminus (Tot_Y(\Omega) \cup Ind_Y(\Omega)) \Leftrightarrow Y_1 = Par_Y(\Omega)$$

We denote :

$$\begin{array}{ll} X_1^+ = X_1 \cap Tot_Y(\Omega) & Y_1^+ = Y_1 \cap Tot_X(\Omega) \\ X_1^- = X_1 \cap Ind_Y(\Omega) & Y_1^- = Y_1 \cap Ind_X(\Omega) \end{array}$$

To verify the condition 1, we must prove that $\emptyset \neq X_1 \subseteq \Omega$ and $\emptyset \neq Y_1 \subseteq \Omega$.

We first prove that $X_1, Y_1 \neq \emptyset$. If $X_1 \cap Y_1 \neq \emptyset$ this is obvious. On the contrary case, assume that $X_1 = \emptyset$. Then, we have $\forall x \in \Omega,\ x \in Ind_Y(\Omega)\ or\ x \in Tot_Y(\Omega)$. This implies that $(X, Y \cup \Omega)$ is a split of G^* contradicting the fact that G^* is a split-prime graph. By a similar way, we can prove that $Y_1 \neq \emptyset$.

By the construction, we have $X_1 \subseteq \Omega$ and $Y_1 \subseteq \Omega$. The condition 1 is then verified. In order to check the other conditions, we must study two cases.

Case 1 : $X_1 \cap Y_1 \neq \emptyset$.
In this case, we can put $k = 1$ and consider a vertex x_1 such that $x_1 \in X_1 \cap Y_1$. As $x_1 \in Par_X(\Omega)$ and $x_1 \in Par_Y(\Omega)$, we can deduce that (x_1) is a X- and a Y-split-pseudopath (condition 2). Since $X_1 \cap Y_1 \neq \emptyset$ and $k = 1$, the condition 3 is obvious. Then, the construction is ended (see at the end of the proof for the construction of the (X, Y)-split-pseudopath).

Case 2 : $X_1 \cap Y_1 = \emptyset$.
If $x_1 \in X_1^+$, we have that $x_1 \in Tot_Y(\Omega)$ and if $x_1 \in X_1^-$, we have that $x_1 \in Ind_Y(\Omega)$. Consequently for each $x_1 \in X_1$, $P_1 = (x_1)$ is an X-split-pseudopath (since $x_1 \in Par_X(\Omega)$ by construction). In the same way we can prove that for each $y_1 \in Y_1$, $Q_1 = (y_1)$ is an Y-split-pseudopath, and hence the condition 2 above is verified.
If there exists $x_1 \in X_1$ and $y_1 \in Y_1$ which satisfies the condition 3.b.iii, we can put $k = 1$ since the condition 3 is verified. Then, our construction is finished (see at the end of the proof for the construction of the (X, Y)-split-pseudopath). In the contrary case, we must have:

(R) $\forall (x_1, y_1) \in Tot_Y(X_1) \times Tot_X(Y_1),\ x_1 \sim y_1$

(S) $\forall (x_1, y_1) \in Ind_Y(X_1) \times Y_1,\ x_1 \nsim y_1$

(T) $\forall (x_1, y_1) \in X_1 \times Ind_X(Y_1),\ x_1 \nsim y_1$

By (R), we can deduce that $Tot_Y(X_1) \sim Tot_X(Y_1)$, by (S), we can deduce that $Ind_Y(X_1) \nsim Y_1$ and by (T), we can deduce that $X_1 \nsim Ind_X(Y_1)$. The condition 3 is then verified with $k > 1$ and we can continue our construction.

<u>Construction of X_{i+1} and Y_{i+1}</u>

Assume now we have construct the sets $(X_1, ..., X_i)$ and $(Y_1, ..., Y_i)$ which satisfy the conditions 1, 2 and 3.a for $i \geq 1$. Then, we denote $\mathcal{X}_i = X \cup X_1 \cup ... \cup X_i$ and $\mathcal{Y}_i = Y \cup Y_1 \cup ... \cup Y_i$. Clearly $(\mathcal{X}_i, \mathcal{Y}_i)$ is a split in the subgraph of G^* induced by the vertices of $\mathcal{X}_i \cup \mathcal{Y}_i$.

We define X_{i+1} and Y_{i+1} as follows:

$$X_{i+1} = \Omega \setminus (\mathcal{X}_i \cup \mathcal{Y}_i \cup Tot_{\mathcal{X}_i}(\Omega) \cup Ind_{\mathcal{X}_i}(\Omega))$$

$$Y_{i+1} = \Omega \setminus (\mathcal{X}_i \cup \mathcal{Y}_i \cup Tot_{\mathcal{Y}_i}(\Omega) \cup Ind_{\mathcal{Y}_i}(\Omega))$$

We denote :

$$X_{i+1}^+ = X_{i+1} \cap Tot_{\mathcal{Y}_i}(\Omega) \qquad Y_{i+1}^+ = Y_{i+1} \cap Tot_{\mathcal{X}_i}(\Omega)$$
$$X_{i+1}^- = X_{i+1} \cap Ind_{\mathcal{Y}_i}(\Omega) \qquad Y_{i+1}^- = Y_{i+1} \cap Ind_{\mathcal{X}_i}(\Omega)$$

In a first time, we are going to check the condition 1.

We shall prove that $X_{i+1}, Y_{i+1} \neq \emptyset$. If $X_{i+1} \cap Y_{i+1} \neq \emptyset$ this is obvious. In the contrary case, assume that $X_{i+1} = \emptyset$. Then $\forall x \in \Omega \setminus ((X_1 \cup ... \cup X_i) \cup (Y_1 \cup ... \cup Y_i))$, $x \in Tot_{\mathcal{X}_i}(\Omega)$ or $x \in Ind_{\mathcal{X}_i}(\Omega))$. Consequently $(X \cup \mathcal{X}_i, Y \cup (\Omega \setminus \mathcal{X}_i)$ is a split of G^*, contradicting the fact that G^* is a split-prime graph. Hence $X_{i+1} \neq \emptyset$ and in a similar way we prove that $Y_{i+1} \neq \emptyset$. By construction, we have that $X_{i+1}, Y_{i+1} \subseteq \Omega \setminus (\mathcal{X}_i \cup \mathcal{Y}_i)$, which implies that $X_{i+1} \subseteq \Omega \setminus ((X_1 \cup ... \cup X_i) \cup (Y_1 \cup ... \cup Y_i))$ and that $Y_{i+1} \subseteq \Omega \setminus ((X_1 \cup ... \cup X_i) \cup (Y_1 \cup ... \cup Y_i))$. Thus, the condition 1 is true.

We now must verify the condition 2. We are going to prove only 2.a, since 2.b has a similar proof. We show now that every vertex x_{i+1} belongs to $Tot_Y(\Omega)$ or to $Ind_Y(\Omega)$. Indeed, if we assume that there exists $x_{i+1} \in X_{i+1}$ such that x_{i+1} is partial with respect to Y^+ or adjacent to at least one vertex of Y^- then $x_{i+1} \in Y_1$ and we obtain a contradiction with the fact that $X_{i+1} \subseteq \Omega \setminus ((X_1 \cup ... \cup X_i) \cup (Y_1 \cup ... \cup Y_i))$ with $i \geq 1$.

Remember that, by condition 2 at the rank i, there exists a X-split-pseudopath $P(x_i) = (x_1, ..., x_i)$ (with $x_j \in X_j$ for $1 \leq j \leq i$) for every vertex $x_i \in X_i$. It remains to prove that every $x_{i+1} \in X_{i+1}$ is of type 1, 2 , ou 3 for a $P(x_i) = x_1, ..., x_i$.

Note that $X_{i+1} \subseteq Tot_{\mathcal{X}_{i-1}}(\Omega) \cup Ind_{\mathcal{X}_{i-1}}(\Omega)$.

Let $x_{i+1} \in X_{i+1}$, then either

(I) $x_{i+1} \in Tot_{\mathcal{X}_{i-1}}(\Omega)$, or

(II) $x_{i+1} \in Ind_{\mathcal{X}_{i-1}}(\Omega)$.

Furthermore $x_{i+1} \in X_{i+1}$ if and only if one of the following holds :

(A) x_{i+1} is partial with respect to X_i^+ and is indifferent with respect to X_i^-

(B) x_{i+1} is indifferent with respect to X_i^+ and is adjacent to at least one vertex of X_i^-

(C) x_{i+1} is total with respect to X_i^+ and is adjacent to at least one vertex of X_i^-

(D) x_{i+1} is partial with respect to X_i^+ and is adjacent to at least one vertex of X_i^-

We have to study 8 different cases which are the combination of the conditions (I), (II) and (A), (B), (C), (D) above.

(A)(I) We have $x_{i+1} \in Tot_{\mathcal{X}_{i-1}}(\Omega) \Rightarrow x_{i+1} \sim X^+ \cup X_1^+ \cup ... \cup X_{i-1}^+$. Now, x_{i+1} is partial with respect to $\mathcal{X}_i^+ = X^+ \cup X_1^+ \cup ... \cup X_{i-1}^+ \cup X_i^+$. Thus, there exists $x_i \in X_i^+$ such that $x_{i+1} \nsim x_i$.

Then, we can verify that x_{i+1} is of type 3 with respect to $P(x_i)$ because : $x_i \in Tot_Y(\Omega)$ (since $x_i \in X_i^+$) and $x_{i+1} \in Tot_{\mathcal{X}_{i-1}}(\Omega)$ allow us to deduce that $x_{i+1} \sim X^+ \cup X_1^+ \cup ... \cup X_{i-1}^+$, $x_{i+1} \nsim X^- \cup X_1^- \cup ... \cup X_{i-1}^-$ and hence $x_{i+1} \sim X^+ \cup Tot_Y(\{x_1, ..., x_{i-1}\})$ and $x_{i+1} \nsim X^- \cup Ind_Y(\{x_1, ..., x_{i-1}\})$.

(A)(II) Since we have $x_{i+1} \in Ind_{\mathcal{X}_{i-1}}(\Omega)$, we can say that $x_{i+1} \nsim \mathcal{X}_{i-1} = X \cup X_1 \cup ... \cup X_{i-1}$ and hence $x_{i+1} \nsim X^+ \cup X_1^+ \cup ... \cup X_{i-1}^+$. Now, x_{i+1} is partial with respect to $\mathcal{X}_i^+ = X^+ \cup X_1^+ \cup ... \cup X_{i-1}^+ \cup X_i^+$. Thus, there exists $x_i \in X_i^+$ such that $x_{i+1} \sim x_i$.

Then, we can verify that x_{i+1} is of type 1 with respect to $P(x_i)$ because : $x_{i+1} \in Ind_{\mathcal{X}_{i-1}}(\Omega)$ then $x_{i+1} \nsim X \cup \{x_1, ..., x_{i-1}\}$.

(B)(I) This case contains a contradiction since $\mathcal{X}_{i-1}^+ \subseteq \mathcal{X}_i^+$ and $x_{i+1} \in Tot_{\mathcal{X}_{i-1}}(\Omega)$ implies that $x_{i+1} \sim \mathcal{X}_{i-1}^+$.

(B)(II) Since we have $x_{i+1} \in Ind_{\mathcal{X}_{i-1}}(\Omega)$, we can deduce that $x_{i+1} \nsim X \cup X_1 \cup ... \cup X_{i-1}$ and hence $x_{i+1} \nsim X^- \cup X_1^- \cup ... \cup X_{i-1}^-$. Now, x_{i+1} is adjacent at at least one vertex of $\mathcal{X}_i^- = X \cup X_1^- \cup ... \cup X_{i-1}^- \cup X_i^-$. Then, we can deduce that there exists $x \in X_i^- \subseteq X_i$ such that $x_{i+1} \sim x_i$.

Then, we can verify that x_{i+1} is of type 1 with respect to $P(x_i)$ because $x_{i+1} \in Ind_{\mathcal{X}_{i-1}}(\Omega)$ implies that $x_{i+1} \nsim X \cup \{x_1, ..., x_{i-1}\}$.

(C)(I) By the fact that $x_{i+1} \in Tot_{\mathcal{X}_{i-1}}(\Omega)$, we have $x_{i+1} \nsim X^- \cup X_1^- \cup ... \cup X_{i-1}^-$. Now, x_{i+1} is adjacent at at least one vertex of $\mathcal{X}_i^- = X \cup X_1^- \cup ... \cup X_{i-1}^- \cup X_i^-$. Then, we can deduce that there exists $x_i \in X_i^- \subseteq X_i$ such that $x_{i+1} \sim x_i$.

Then, we can verify that x_{i+1} is of type 2 with respect to $P(x_i)$ because : $x_i \in Ind_Y(\Omega)$ (since $x_i \in X_i^-$) and $x_{i+1} \in Tot_{\mathcal{X}_{i-1}}(\Omega)$ allow us to deduce $x_{i+1} \sim X^+ \cup X_1^+ \cup ... \cup X_{i-1}^+$, $x_{i+1} \nsim X^- \cup X_1^- \cup ... \cup X_{i-1}^-$ and hence $x_{i+1} \sim X^+ \cup Tot_Y(\{x_1, ..., x_{i-1}\})$ and $x_{i+1} \nsim X^- \cup Ind_Y(\{x_1, ..., x_{i-1}\})$.

(C)(II) This case contains a contradiction since $\mathcal{X}_{i-1} \subseteq \mathcal{X}_i$ and $x_{i+1} \in Ind_{\mathcal{X}_{i-1}}(\Omega)$ implies that $x_{i+1} \nsim \mathcal{X}_{i-1}$.

(D)(I) The proofs of (AI) or (CI) can be applied to this case.

(D)(II) The proofs of (AII) or (BII) can be applied to this case.

In each case, we proved that x_{i+1} is of type 1, 2 ou 3 with respect to a X-split-pseudopath $P(x_i)$. So, we can deduce that $P(x_i) \cup \{x_{i+1}\}$ is a X-split-pseudopath and hence we proved the condition 2.

It remains to prove the condition (3). We need to distinguish two cases.
Case 1 : $X_{i+1} \cap Y_{i+1} \neq \emptyset$.
If we put $k = i+1$, the condition 3 becomes true and the construction is ended (see at the end of the proof for the construction of the (X, Y)-split-pseudopath).

Case 2 : $X_{i+1} \cap Y_{i+1} = \emptyset$.
Assume first that there exists $x_{i+1} \in X_{i+1}$ and $y_{i+1} \in Y_{i+1}$ which satisfies one of the conditions of 3.b.iii. Then we put $k = i+1$ and since $k > 1$ we must check the conditions 3.b.i and 3.b.ii (note that we have not to verify 3.a since $i+1 = k$ and by assumption $\mathcal{X}_i$ and $\mathcal{Y}_i$ verify this condition).

By construction, we have that $(Y_k, Tot_{\mathcal{Y}_{k-1}}(\Omega), Ind_{\mathcal{Y}_{k-1}}(\Omega))$ is a partition of $\Omega \setminus ((X_1 \cup ... \cup X_{k-1}) \cup (Y_1 \cup ... \cup Y_{k-1}))$. Then $\forall x_k \in X_k$, $x_k \in X_k \cap Y_k = \emptyset$ or $x_k \in X_k \cap Tot_{\mathcal{Y}_{k-1}}(\Omega) = X_k^+$ or $x_k \in X_k \cap Ind_{\mathcal{Y}_{k-1}}(\Omega) = X_k^-$. If $x_k \in X_k^+$, we have by definition $x_k \in Tot_{\mathcal{Y}_{k-1}}(\Omega)$ and hence $x_k \sim Tot_X(Y_1 \cup ... \cup Y_{k-1})$ and $x_k \nsim Ind_X(Y_1 \cup ... \cup Y_{k-1})$. If $x_k \in X_k^-$, we have by definition $x_k \in Ind_{\mathcal{Y}_{k-1}}(\Omega)$, so $x_k \nsim Y_1 \cup ... \cup Y_{k-1}$. Thus, conditions 3.b.i and 3.b.ii are verified for X_k (we can prove these results for Y_k by a similar way).
Then, the construction is ended (see at the end of the proof for the construction of the (X, Y)-split-pseudopath).

Assume now that the condition 3.b.iii is not verified, we want to prove that $k > i+1$ (i.e. that the construction process is not ended). We shall prove then the truth of condition 3.a for $\mathcal{X}_{i+1}$ and $\mathcal{Y}_{i+1}$. We can easily obtain that : $Tot_Y(X_{i+1}) \sim Tot_X(Y_1 \cup ... \cup Y_i)$, $Tot_Y(X_{i+1}) \nsim Ind_X(Y_1 \cup ... \cup Y_i)$ and $Ind_Y(X_{i+1}) \nsim Y_1 \cup ... \cup Y_i$. Hence for Y_{i+1}, we have that $Tot_X(Y_{i+1}) \sim Tot_Y(X_1 \cup ... \cup X_i)$, $Tot_X(Y_{i+1}) \nsim Ind_Y(X_1 \cup ... \cup X_i)$ and $Ind_X(Y_{i+1}) \nsim X_1 \cup ... \cup X_i$.
Since 3.b.iii is not verified the following conditions hold:

(a) $\forall (x_{i+1}, y_{i+1}) \in Tot_Y(X_{i+1}) \times Tot_X(Y_{i+1}),\ x_{i+1} \sim y_{i+1}$

(b) $\forall (x_{i+1}, y_{i+1}) \in Ind_Y(X_{i+1}) \times Y_{i+1},\ x_{i+1} \nsim y_{i+1}$

(c) $\forall (x_{i+1}, y_{i+1}) \in X_{i+1} \times Ind_X(Y_{i+1}),\ x_{i+1} \nsim y_{i+1}$

By (a), we can deduce that $Tot_Y(X_{i+1}) \sim Tot_X(Y_{i+1})$. By (b), we can deduce that $Ind_Y(X_{i+1}) \nsim Y_{i+1}$. By (c), we can deduce that $X_{i+1} \nsim Ind_X(Y_{i+1})$.
Since $\mathcal{X}_i$ and $\mathcal{Y}_i$ verify the condition 3.a, we have that $Tot_Y(X_1 \cup ... \cup X_i) \sim Tot_X(Y_1 \cup ... \cup Y_i)$, $Ind_Y(X_1 \cup ... \cup X_i) \nsim Y_1 \cup ... \cup Y_i$ and $Ind_X(Y_1 \cup ... \cup Y_i) \nsim X_1 \cup ... \cup X_i$. It follows that $Tot_Y(X_1 \cup ... \cup X_{i+1}) \sim Tot_X(Y_1 \cup ... \cup Y_{i+1})$, $Ind_Y(X_1 \cup ... \cup X_{i+1}) \nsim Y_1 \cup ... \cup Y_{i+1}$ and $Ind_X(Y_1 \cup ... \cup Y_{i+1}) \nsim X_1 \cup ... \cup X_{i+1}$ which allows us to conclude that the condition 3.a holds for $\mathcal{X}_{i+1}$ and $\mathcal{Y}_{i+1}$ as claimed.

$(X_1, ..., X_k)$ and $(Y_1, ..., Y_k)$, $k \geq 1$ exist

Observe that if $X_{i+1} \cap Y_{i+1} = \emptyset$ and the condition 3.b.iii is not verified, then $(\mathcal{X}_{i+1}, \mathcal{Y}_{i+1})$ is a split in the subgraph of G^* induced by the vertices of $\mathcal{X}_{i+1} \cup \mathcal{Y}_{i+1}$. But if $X_{i+1} \cap Y_{i+1} \neq \emptyset$ or respectively $X_{i+1} \cap Y_{i+1} = \emptyset$ and there exists $x_{i+1} \in X_{i+1}$ and $y_{i+1} \in Y_{i+1}$ which satisfy the condition of 3.b then a vertex $x \in X_{i+1} \cap Y_{i+1}$ or respectively (x_{i+1}, y_{i+1}) "breaks" the split in the subgraph of G^* induced by the vertices of $\mathcal{X}_i \cup \mathcal{Y}_i$. Since all sets of $\mathcal{X}_{i+1} \cup \mathcal{Y}_{i+1}$ are non empty and pairwise disjoint and G^* is finite and split-prime we can easily deduce that the two sequences $(X_1, ..., X_k)$ and $(Y_1, ..., Y_k)$ exist and verify the conditions 1, 2 and 3 defined at the beginning of this proof.

Existence of a (X, Y)-split-pseudopath

It remains to determine a (X, Y)-split-pseudopath (P, Q) where $P = (x_1, ..., x_k)$ and $Q = (y_1, ..., y_k)$ which satisfy all the properties of the Definition 4. For this, we take for x_k and y_k the two vertices which allow us to stop the construction (either $x_k = y_k$ or x_k and y_k are distinct and verify the condition 3.b). Then, the condition 2 allows us to choose $x_j \in X_j$ and $y_j \in Y_j$ for every $1 \leq j \leq k-1$ such as P and Q are respectively a X- and a Y-split-pseudopath. The fact that $x_j \in X_j$ and $y_j \in Y_j$ for every $1 \leq j \leq k$ allows us to deduce respectively from the conditions 1 and 3 that P and Q satisfy the properties 1 and 3 of the Definition 4. So, we can conclude that (P, Q) is a (X, Y)-split-pseudopath in G^*, as claimed.

□

4 Conclusion

We believe that the structure of a split-pseudopath will play a crucial role for the resolution of the problems presented in Section 1 as well as for studying the structural properties of various classes of graphs. For example, we could apply the ideas developed in [5] for the study of minimal prime extensions of graphs in order to find classes of graphs having an infinite number of minimal split-prime extensions. More precisely, given a split-decomposable graph G we could first search how to add a minimal number of new vertices in order to "break" all splits of G except exactly one split (X, Y). Then, in the resulting graph G' we could try to obtain an infinite number of split-prime graphs containing G' by adding to G' an (X, Y)-split-pseudopath of arbitrary length. We could also study which kind of (X, Y)-split-speudopath produces a finite number of minimal prime extensions for a given class of graphs. It would be also interesting to study the relations of minimal split-prime extensions of a graph with the monadic second-order logic formulas introduced in [3] for different decomposition of graphs included split decomposition . All these directions are for us an exciting area for further work.

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